Normal radicals

by

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Abstract. The paper gives a number of characterizations of normal radicals, i.e., radicals of associative rings which satisfy a natural condition on Morita contexts (cf. [1], [5]). It is proved that for normal radicals the lattices of radical ideals of Morita equivalent rings are isomorphic. That section of the paper is "a radical counterpart" of the results of A. D. Sands [6]. In Section 2 lower and upper normal radicals are constructed.

Introduction. A radical (a radical class, a radical property) of associative rings is a normal radical if for every Morita context (R, V, W, S), where R, S are rings, V is an R-S-bimodule and W is an S-R-bimodule, we have

$$VN(S)W \subseteq N(R), \quad \text{or equivalently} \quad WN(R)V \subseteq N(S).$$

It is not assumed that the rings have identities or that the modules are unitary. (For the definition and notations of Morita contexts see [1] and [6]. For radical-theoretic terms and properties we refer the reader to [3].)

In [1] Amitsur proved that the radicals of Baer, Levitzki and Jacobson are normal. Sands [6] and the present author [4] generalized this result to certain axiomatically defined classes of radicals. The notion of a normal radical was introduced in [5].

In what follows we shall characterize normal radicals in several different ways, in particular in terms of rings with idempotents. We shall describe the connections between radical ideals of rings in some distinguished Morita contexts (Theorem 7), and we shall prove that the lattices of radical ideals of Morita equivalent rings are isomorphic for normal radicals. Moreover, we shall prove that all one-sided hereditary and one-sided strong radicals are normal (Theorem 4 — cf. [4], [6] and [7]). In Section 2 we shall construct lower and upper normal radicals for every class of associative rings.

Section 1 is "a radical counterpart" of Sands [6] and often the ideas of the proofs presented should be traced back to that paper. Following Sands, we shall apply in our considerations a Morita context (R, V, W, S) in matrix form, i.e., in the form of the ring $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ of all 2×2 matrices $\begin{pmatrix} r & v \\ w & s \end{pmatrix}$, $r \in R$, $v \in V$, $w \in W$, $s \in S$, with obvious definitions of addition and multiplication. One can observe that every such generalized matrix ring determines a Morita context, and so we shall use both forms interchangeably. If A, B, C, D are subsets of R, V, W, S, respectively, we shall denote by $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ the subset of $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ consisting of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \in A$, $b \in B$, $c \in C$, $d \in D$.

An ideal always means a two-sided ideal, and if I is an ideal of a ring R, we denote this by writing $I \lhd R$. A radical of an ideal of a ring R is a radical ideal of R (see [2] and [3] Theorem 47). This fact is one of our basic tools and we shall refer to it as the ADS-theorem (Anderson–Divinsky–Suliński).

1. Characterization of normal radicals. Applying the ADS-theorem to Morita contexts, we obtain the following lemma.

LEMMA 1. *If $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ is a Morita context, and N is any radical, then $N\begin{pmatrix} R & V \\ W & S \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ for certain ideals A, D of R, S and sub-bimodules B, C of ${}_RV_S$, ${}_SW_R$, respectively.*

Proof. Let $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ be a Morita context and let $R^\#$, $S^\#$ denote the usual over-rings with identity of the rings R and S, respectively. V and W are unitary $R^\#$- and $S^\#$-modules in an obvious way and $\begin{pmatrix} R^\# & V \\ W & S^\# \end{pmatrix}$ is a Morita context which contains as an ideal the context $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$. By the ADS-theorem, for every radical N we have $N\begin{pmatrix} R & V \\ W & S \end{pmatrix} \lhd \begin{pmatrix} R^\# & V \\ W & S^\# \end{pmatrix}$. Now, multiplying $N\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ by idempotents $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ from the left and right in all four possible ways, one can easily show that the radical of the context has the form stated in the lemma. ■

The following is a basic theorem on characterization of normal radicals.

THEOREM 2. *If N is a radical in a class of associative rings, then the following conditions are equivalent.*

(1) N is a normal radical.

(2) (a) If a ring R is an ideal of a ring $\overline{R}$ such that the factor ring $\overline{R}/R$ is isomorphic with the ring of integers, then

$$N(R) = N(\overline{R}) \cap R$$

and (b) if $e = e^2$ is an idempotent of some ring R, then

$$N(eRe) = eN(R)e.$$

(3) For every Morita context $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ we have

$$N\begin{pmatrix} R & V \\ W & S \end{pmatrix} = \begin{pmatrix} N(R) & B \\ C & N(S) \end{pmatrix},$$

where B and C are subbimodules of ${}_RV_S$ and ${}_SW_R$, respectively.

Proof. (1) ⇒ (2). This was proved in fact in [5], Lemma 1.5 and Theorem 1.9.

(2) ⇒ (3). Let $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ be a Morita context and let $R^\#$, $S^\#$ be as in Lemma 1. Since $\begin{pmatrix} R^\# & V \\ W & S^\# \end{pmatrix} \rhd \begin{pmatrix} R^\# & V \\ W & S \end{pmatrix} \rhd \begin{pmatrix} R & V \\ W & S \end{pmatrix}$, and both the factor rings are isomorphic with the integers, application of Lemma 1 and twofold application of (2a) give us

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = N\begin{pmatrix} R & V \\ W & S \end{pmatrix} = N\begin{pmatrix} R^\# & V \\ W & S^\# \end{pmatrix} \cap \begin{pmatrix} R & V \\ W & S \end{pmatrix},$$

where A, B, C and D are as in the lemma.

Let $e = e^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be a matrix from $\begin{pmatrix} R^\# & V \\ W & S^\# \end{pmatrix}$. Hence by conditions (2b) and (2a)

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = eN\begin{pmatrix} R & V \\ W & S \end{pmatrix}e = e\left(N\begin{pmatrix} R^\# & V \\ W & S^\# \end{pmatrix} \cap \begin{pmatrix} R & V \\ W & S \end{pmatrix}\right)e$$

$$= eN\begin{pmatrix} R^\# & V \\ W & S^\# \end{pmatrix}e \cap \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} = N\begin{pmatrix} R^\# & 0 \\ 0 & 0 \end{pmatrix} \cap \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} N(R^\#) \cap R & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} N(R) & 0 \\ 0 & 0 \end{pmatrix},$$

whence $A = N(R)$. Similarly $D = N(S)$.

(3) ⇒ (1). Let N be a radical such that for every Morita context (R, V, W, S) we have

$$N\begin{pmatrix} R & V \\ W & S \end{pmatrix} = \begin{pmatrix} N(R) & * \\ * & N(S) \end{pmatrix},$$

where the stars are for suitable bimodules. Since $N\begin{pmatrix} R & V \\ W & S \end{pmatrix} \lhd \begin{pmatrix} R & V \\ W & S \end{pmatrix}$, we have

$$\begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}\begin{pmatrix} N(R) & * \\ * & N(S) \end{pmatrix}\begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix} = \begin{pmatrix} VN(S)W & 0 \\ 0 & 0 \end{pmatrix} \subseteq \begin{pmatrix} N(R) & * \\ * & N(S) \end{pmatrix}.$$

Thus

$$VN(S)W \subseteq N(R),$$

so that the radical N is normal. ■

THEOREM 3. *Let N be a radical. Then N is normal and contains all nilpotent rings if and only if for every Morita context (R, V, W, S)*

$$N\begin{pmatrix} R & V \\ W & S \end{pmatrix} = \begin{pmatrix} N(R) & B \\ C & N(S) \end{pmatrix},$$

where either

$$B = \{v \in V \mid Wv \subseteq N(S)\}, \quad \text{or} \quad B = \{v \in V \mid vW \subseteq N(R)\},$$

$$\text{or} \quad C = \{w \in W \mid wV \subseteq N(S)\}, \quad \text{or} \quad C = \{w \in W \mid Vw \subseteq N(R)\}.$$

Moreover, if one of these equalities for B and C holds, then all of them hold.

Proof. Let N be a radical such that the N-radical of every Morita context (R,V,W,S) has the form

$$\begin{pmatrix} N(R) & B \\ C & N(S) \end{pmatrix}.$$

By the previous theorem N is normal. Furthermore, if $B = \{v \in V \mid Wv \subseteq N(S)\}$ for every Morita context (R,V,W,S), and if T is a ring with $T^2 = 0$, then let us consider the context $\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$ isomorphic as a ring with T. Our assumption gives us

$$N(T) \simeq N\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} \simeq T.$$

Thus all zero-rings are radical, whence all nilpotent rings are also radical. Analogously, we can prove this fact by assuming one of the other equalities.

Conversely, let us assume that N is a normal radical containing all nilpotent rings. By [4] Theorem 2, it follows that N is an N-radical in the sense of Sands [6]. Then [6] Theorem 8, provides us with the remaining part of our theorem. ■

A radical N is left hereditary if every left ideal of a radical ring is an N-ideal. A radical N is left strong if every left N-ideal of a ring R is contained in $N(R)$. Analogously we define a right hereditary radical and a right strong radical.

The following theorem generalizes both Theorem 1 of [4] and Theorem 2 of [6]. The proof is a slight modification of the proof of Theorem 1 of [6], due to A. D. Sands.

THEOREM 4. *If a radical N is either left or right strong and either left or right hereditary, then N is a normal radical.*

Proof. By Lemma 1 the N-radical of some context $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ is of the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. We shall prove that A, which is an ideal of R, equals $N(R)$. Since by a symmetrical argument one can prove $D = N(S)$, then this suffices for the normality of N by Theorem 2(3)

$$\begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} \subseteq N\begin{pmatrix} R & V \\ W & S \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \subseteq N\begin{pmatrix} R & V \\ W & S \end{pmatrix}.$$

Since N is either left or right hereditary, one of these one-sided ideals is an N-ring. So A, which is a homomorphic image of both of them, is also an N-ring. But $A \lhd R$, and so $A \subseteq N(R)$.

Now we shall prove the opposite inclusion. Accordingly, we shall show that the left ideal $\begin{pmatrix} N(R) & 0 \\ WN(R) & 0 \end{pmatrix}$ of the ring $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ is a radical ring. Krempa's Lemma [4] implies that the zero ring $N(R)^+$ on the additive group of the ring $N(R)$ is a radical ring. Hence $\begin{pmatrix} 0 & 0 \\ WN(R) & 0 \end{pmatrix}$ is also an N-ring, as the sum of all its ideals of the form $\begin{pmatrix} 0 & 0 \\ wN(R) & 0 \end{pmatrix}$, $w \in W$, which are N-rings as homomorphic images of $N(R)^+$. Thus

$\begin{pmatrix} N(R) & 0 \\ WN(R) & 0 \end{pmatrix}$ is an extension of a radical ring by a radical ring and so is a radical ring. If N is a left strong radical, then we have

$$\begin{pmatrix} N(R) & 0 \\ WN(R) & 0 \end{pmatrix} \subseteq N\begin{pmatrix} R & V \\ W & S \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

whence $N(R) \subseteq A$.

If the radical N is right strong, then we obtain the required inclusion by considering the right N-ideal $\begin{pmatrix} N(R) & N(R)V \\ 0 & 0 \end{pmatrix}$. Therefore in every case $A = N(R)$. ■

Remark 1. Let x and y denote either "left" or "right", and let x', y' be the opposite sides to x, y, respectively. If a radical N contains all nilpotent rings and if N is an x-hereditary and y-strong radical, then N is x'-hereditary and y'-strong too. This follows immediately by the above theorem and [4] Theorem 2.

Remark 2. Recently A. D. Sands proved in [7] that a radical N is normal if and only if N is left strong and all left ideals of the form Ra, $a \in R \in N$, also belong to N. But this result, so far, does not have a left-right form.

THEOREM 5. *Let N be a normal radical and let (R,V,W,S) be a Morita context. If A is an N-ideal of R, then the ideal*

$$\begin{pmatrix} A & AV \\ WA & WAV \end{pmatrix}$$

is an N-ideal of $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$.

Proof. Theorem 2 implies that $K = N\begin{pmatrix} A & AV \\ WA & WAV \end{pmatrix}$ contains

$$\begin{pmatrix} N(A) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

Since

$$\begin{pmatrix} A & AV \\ WA & WAV \end{pmatrix} \lhd \begin{pmatrix} R & V \\ W & S \end{pmatrix},$$

we have by the ADS-theorem $K \lhd \begin{pmatrix} R & V \\ W & S \end{pmatrix}$. So K contains the ideal of $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ generated by $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, which equals $\begin{pmatrix} A & AV \\ WA & WAV \end{pmatrix}$. Therefore

$$\begin{pmatrix} A & AV \\ WA & WAV \end{pmatrix} = N\begin{pmatrix} A & AV \\ WA & WAV \end{pmatrix}. \ ■$$

COROLLARY 6. *If N is a normal radical, then for every Morita context (R,V,W,S) and every N-ideal A of the ring R the ideal WAV is an N-ideal of the ring S.*

Proof. By Theorems 5 and 2 we have

$$\begin{pmatrix} A & \begin{pmatrix} AV \\ WAV \end{pmatrix} \\ WA & \end{pmatrix} = N\begin{pmatrix} A & AV \\ WA & WAV \end{pmatrix} = \begin{pmatrix} N(A) & * \\ * & N(WAV) \end{pmatrix}.$$

Hence $N(WAV) = WAV$. ∎

In the whole following text α and β will denote maps (between sets of ideals ordered by inclusion) and will be defined for every Morita context (R, V, W, S) as follows: $\alpha(A) = WAV \lhd S$, for every $A \lhd R$, and $\beta(B) = VBW$, for every $B \lhd S$. Furthermore, for X and Y, ideals of some ring, and a fixed radical N we will put $X \vee Y = X + Y$ (the sum of ideals) and $X \wedge Y = N(X \cap Y)$.

THEOREM 7. *Let N be a normal radical and let (R, V, W, S) be a Morita context. Then α and β, defined above, map N-ideals into N-ideals and preserve the order and the operation $\vee$. If $WV = S$ and S is a ring with a unity, then the composite $\alpha\beta$ is an identity map; so the restrictions of α and β to the sets of N-ideals of R and S are an epimorphism and a monomorphism, respectively, of ordered sets. If, moreover, $VW = R$ and R is a ring with a unity, i.e., if R and S are Morita equivalent, then α and β are mutually inverse isomorphisms of the lattices of N-ideals of R and S with operations $\vee$ and $\wedge$ as l.u.b. and g.l.b., respectively.*

Proof. Nearly all the statements of the theorem can be obtained by Corollary 6 and by easy checking. We shall only show that $\alpha(A \wedge A_1) = \alpha A \wedge \alpha A_1$ for Morita equivalent rings R and S and A, A_1 ideals of R. By the ADS-theorem, $A \wedge A_1 = N(A \cap A_1)$ is an N-ideal of R. Obviously $A \wedge A_1 \subseteq A$ and $A \wedge A_1 \subseteq A_1$. Since α preserves order, we have $\alpha(A \wedge A_1) \subseteq \alpha A \cap \alpha A_1$. But, by Corollary 6, $\alpha(A \wedge A_1)$ is an N-ideal of S, and so $\alpha(A \wedge A_1) \subseteq N(\alpha A \cap \alpha A_1) = \alpha A \wedge \alpha A_1$. Applying the same procedure to the ideals $\alpha A, \alpha A_1$ of S and the mapping β, we obtain

$$\beta(\alpha A \wedge \alpha A_1) \subseteq \beta\alpha A \wedge \beta\alpha A_1 = A \wedge A_1.$$

Thus

$$\alpha A \wedge \alpha A_1 = \alpha\beta(\alpha A \wedge \alpha A_1) \subseteq \alpha(A \wedge A_1),$$

and we have the required equality. ∎

THEOREM 8. *For every radical N the following conditions are equivalent.*

(1) *N is a normal radical.*

(2) *If (R, V, W, S) is a Morita context, $WV = S$ and S is a ring with a unity, then for every ideal A of R*

$$N(WAV) = WN(A)V.$$

(3) *For every ideal A of a ring R with an idempotent e*

$$N(eAe) = eN(A)e.$$

Proof. (1) ⇒ (2). Let us observe that, by the ADS-theorem, $N(A)$ and $N(\beta\alpha A)$ are N-ideals of R for $A \lhd R$. Since α preserves inclusion, we have $\alpha N(A) \subseteq \alpha A$. By Theorem 7 $\alpha N(A)$ is an N-ideal, and so $\alpha N(A) \subseteq N(\alpha A)$. Similarly $\beta N(\alpha A) \subseteq N(\beta\alpha A)$. Furthermore, $N(\beta\alpha A) \subseteq N(A)$ since $\beta\alpha A \lhd A$.

If (R, V, W, S) is such a context that $WV = S$ is a ring with a unity, then $\alpha\beta$ is an identity map and we have

$$\alpha N(A) \subseteq N(\alpha A) = \alpha\beta N(\alpha A) \subseteq \alpha N(\beta\alpha A) \subseteq \alpha N(A).$$

This means that

$$N(WAV) = N(\alpha A) = \alpha N(A) = WN(A)V.$$

(2) ⇒ (3). In the context (R, Re, eR, eRe), with multiplications as products, $eRe = eRe \cdot e = eR \cdot Re$ is a ring with a unity e, and $eAe = eR \cdot A \cdot Re$, for $A \lhd R$. Thus, if N satisfies (2), we have

$$N(eAe) = eRN(A)Re = eN(A)e.$$

(3) ⇒ (1). A Morita context $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ is an ideal of $\begin{pmatrix} R^{\#} & V \\ W & S \end{pmatrix}$ where $R^{\#}$ is the usual over-ring with identity of R. Thus for

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} R^{\#} & V \\ W & S \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} R & V \\ W & S \end{pmatrix} \lhd \begin{pmatrix} R^{\#} & V \\ W & S \end{pmatrix}$$

we obtain

$$eN\begin{pmatrix} R & V \\ W & S \end{pmatrix}e = N\left(e\begin{pmatrix} R & V \\ W & S \end{pmatrix}e\right) = \begin{pmatrix} N(R) & 0 \\ 0 & 0 \end{pmatrix};$$

hence $N\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ has the form $\begin{pmatrix} N(R) & * \\ * & * \end{pmatrix}$. Similar arguments show that in the bottom-right corner we have $N(S)$. Theorem 2(3) implies then that N is a normal radical. ∎

Condition (3) of this theorem gives us an easy method of finding normal radicals of matrix rings. By means of it one can essentially simplify the proofs of [5].

COROLLARY 9. *If N is a normal radical, (R, V, W, S) is a context with $WV = S$ and S has a unity, then*

$$N(S) = WN(R)V.$$

Proof. Since $VW \subseteq R$ and N is normal, we have

$$N(S) = N(WVWV) = WN(VW)V \subseteq WN(R)V \subseteq N(S).$$

Thus $N(S) = WN(R)V$. ∎

The following description of normal radicals by means of semisimple rings will be useful in the construction of an upper normal radical.

For a Morita context (R, V, W, S) by $\partial(S)$ we shall denote the ideal $\{s \in S \mid VsW = 0\}$ of S.

THEOREM 10. *A radical N is a normal radical if and only if for every Morita context (R, V, W, S) with $N(R) = 0$ we have $N(S/\partial(S)) = 0$.*

Proof. The "only if" part of the theorem was in fact proved by A. D. Sands [6], Theorem 4, and so we omit the proof.

Conversely, suppose that N satisfies the stated condition. If $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ is a Morita context, and

$B = \{v \in V \mid vV \subseteq N(R)\}$, $C = \{w \in W \mid Vw \subseteq N(R)\}$, $T = \{\bar{s} \in S \mid VsW \subseteq N(R)\}$,

then for the Morita context

$$(R/N(R), V/B, W/C, S/T) \simeq \begin{pmatrix} R & V \\ W & S \end{pmatrix} \Big/ \begin{pmatrix} N(R) & B \\ C & T \end{pmatrix}$$

we have, by our assumption, $N(S/T/\partial(S/T)) = 0$. If $\bar{s}$ denotes the coset of $s \in S$ in the factor ring S/T, then

$$\partial(S/T) = \{\bar{s} \in S/T \mid V/B \cdot \bar{s} \cdot W/C = \bar{0}\}$$
$$= \{\bar{s} \in S/T \mid VsW \subseteq N(R)\} = \{\bar{s} \in S/T \mid s \in T\} = \bar{0}.$$

This implies $N(S/T) = \bar{0}$, whence $N(S) \subseteq T$. This means $VN(S)W \subseteq N(R)$, and so N is a normal radical. ■

2. Lower and upper normal radicals. Let A be a class of associative rings. A normal radical N is the lower normal radical defined by A if $A \subseteq N$ and N is contained in every normal radical M which contains A. A normal radical N is the upper normal radical defined by A if all rings from A are N-semisimple and all normal radicals with this property are contained in N. In this section we shall construct a lower normal radical and an upper normal radical for every class of associative rings.

A subring A of a ring R is accessible if these exists a finite chain $A = A_0 \lhd A_1 \lhd \ldots \ldots \lhd A_n \lhd R$.

For every class of rings C classes $l(C)$ and $u(C)$ are defined as follows. A ring X belongs to $l(C)$ if there exists a Morita context (R, V, W, S) and an ideal A of R belonging to C and such that X is a homomorphic image of the ring WAV. A ring X belongs to $u(C)$ if X is an accessible subring of a ring T which either belongs to C, or is of the form $T = S/\partial(S)$, for Morita context (R, V, W, S) and $S \in C$. Obviously $C \subseteq l(C)$ and $C \subseteq u(C)$.

Now, for a class of rings A, we define by transfinite induction two families of classes. We put

$$A_0 = A.$$

If for every ordinal $\varkappa$ less than γ the class $A_\varkappa$ has been defined, then A_γ is the class of all rings for which every non-zero homomorphic image contains a non-zero accessible subring from $l(A_\varkappa)$ for some $\varkappa < \gamma$

$$LN\text{-}A = \bigcup_\gamma A_\gamma,$$

where the union is taken over all ordinal numbers. Next we put

$$A^0 = A.$$

If for every ordinal $\varkappa$ less than γ the class $A^\varkappa$ has been defined, then A^γ is the class of all rings for which every non-zero accessible subring has a non-zero homomorphic image in $u(A^\varkappa)$ for some $\varkappa < \gamma$. A ring R belongs to $UN\text{-}A$ if and only if R cannot be homomorphically mapped onto some non-zero ring from $\bigcup_\gamma A^\gamma$, where the union is taken over all ordinal numbers.

THEOREM 11. *The class $LN\text{-}A$ is lower normal radical and the class $UN\text{-}A$ is the upper normal radical defined by a given class A.*

Proof. By a standard procedure one can prove that $LN\text{-}A$ and $UN\text{-}A$ are radicals (cf. for example [3], constructions of lower and upper radicals). Moreover is easy to observe that $\bigcup A^\gamma$ is a class of $UN\text{-}A$-semisimple rings. We shall prove that these radicals are normal. Let $L(R)$ and $U(R)$ denote the corresponding radicals (i.e., maximal radical ideals) of a ring R. Let (R, V, W, S) be a Morita context. There exists an ordinal γ such that $L(R) \in A_\gamma$. By definition the ideal $WL(R)V$ of S belongs to $A_{\gamma+1} \subseteq LN\text{-}A$, whence

$$WL(R)V \subseteq L(S),$$

i.e., $LN\text{-}A$ is a normal radical.

If a ring R is $UN\text{-}A$-semisimple, then $R \in A^\gamma$ for some γ. Hence $S/\partial(S) \in A^{\gamma+1}$, and so is also $UN\text{-}A$-semisimple. The normality of $UN\text{-}A$ now follows from Theorem 10.

As immediate consequences of the ADS-theorem we infer that a ring is a radical ring if and only if every non-zero homomorphic image contains a non-zero radical accessible subring and that accessible subrings of semisimple rings are also semisimple. By these remarks, Corollary 6, Theorem 10 and transfinite induction one can easily verify that $LN\text{-}A$ is the lower normal radical and $UN\text{-}A$ is the upper normal radical defined by A. ■

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