Closed curves and circle homomorphisms in groups of diffeomorphisms *

by

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Abstract. Given a smooth manifold M with diffeomorphism group Diff M, the problem of finding homomorphisms $S^1 \to \mathrm{Diff}\, M$ homotopic to given closed curves is considered. Examples are given to show this is frequently not possible. Particular attention is paid to the case $M = S^n$, for which general criteria and numerous quantitative results are obtainable. For example, certain nonlinear circle actions on spheres determine homotopy classes that might not be expected to contain homomorphisms (see 2.5).

If G is a connected finite-dimensional Lie group, then every closed curve $\gamma\colon S^1 \to G$ is homotopic to a homomorphism (a proof in the compact case is outlined in [9, Exercise 3, p. 153]; the noncompact case follows from [9, Theorem 3.1, pp. 180–181]). Examples given in this paper show this is not the case for $G = \mathrm{Diff}\, M$, the group of C^r diffeomorphisms of a smooth manifold M with the C^r topology ($1 \leqslant r \leqslant \infty$). In fact, such examples exist for spheres of suitable dimensions, although in this case the answer is *less simple than generally suspected* (as noted in the last sentence of the next paragraph). The existence of classes in π_1 (Diff M) not containing homomorphisms is strongly suggested by work of J. Palis stating that very few elements of Diff M lie on one-parameter subgroups [12].

To illustrate the widespread nature of such examples, we consider two natural but contrasting classes. The first consists of smooth manifolds with no effective smooth circle actions; in this case it merely suffices to note that $\pi_1(\mathrm{Diff}\, M^n) \neq 0$ for broad classes of manifolds by the methods and results of Antonelli, Burghelea, and Kahn [1, 2]. The other class, consisting of spheres, illustrates that the nonexistence phenomenon is basically independent of a lack of differentiable symmetry. Furthermore, in this case enough specific information is available to show that the set of classes in $\pi_1(\mathrm{Diff}\, S^n)$ containing homomorphisms often is not even a subgroup. In fact, it is possible to obtain much specific information about the representability or nonrepresentability by homomorphisms in this case, and a few such examples are discussed. One of these (Example 2.5) refutes a common misconception that no class in the image of $\pi_1(\mathrm{Diff}\, S^n \,\mathrm{rel}\, D^n_-)$ is representable by a homomorphism; on the contrary, infinitely many admit such representations.

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Acknowledgments. The problem of representing elements in $\pi_1(\mathrm{Diff}\, S^n)$ by homomorphisms was first posed to me by W. C. Hsiang; his comments on these and related matters have been extremely valuable. Proposition 2.3 was first proved to answer a question in a letter from B. Conrad.

1. Manifolds with no smooth circle actions. By a result of Montgomery and Zippin [1], pp. 208–214, if a smooth manifold M^n has no effective smooth S^1 actions, then no nonzero class in $\pi_1(\mathrm{Diff}\, M^n)$ contains a homomorphism. Antonelli, Burghelea, and Kahn have given general methods for detecting nonzero homotopy classes in such groups in [1] and [2] that apply directly to the problem at hand; their basic tool is the homomorphism $E_M\colon \pi_*(\mathrm{Diff}(D^n, S^{n-1}))\to\pi_*(\mathrm{Diff}\, M^n)$ defined and studied in [1], Ch. 2. Recall from [1] that $\mathrm{Diff}(D^n, S^{n-1})$ is the group of diffeomorphisms of D^n that are fixed near the boundary, and E has the effect of extending such diffeomorphisms to M^n via a coordinate neighborhood inclusion. Furthermore, partial information on $\pi_1(\mathrm{Diff}(D^n, S^{n-1}))$ is given by the Gromoll homomorphism

$$\lambda\colon \pi_1(\mathrm{Diff}(D^n, S^{n-1}))\to\Gamma_{n+2}$$

(see [1], § 1.1), which is onto by results of J. Cerf [8].

The following strengthens the one-dimensional case of the nontriviality criterion in [1], § 2.3; the proof of the latter result actually implies the statement given here.

LEMMA 1.1. *Suppose that $x\in\pi_1(\mathrm{Diff}(D^n, S^{n-1}))$ satisfies $E_M(x)=0$. Then $\lambda(x)$ belongs to the homotopy inertia group of $M\times S^2$.*

Remark. The homotopy inertia group $I_h(N)$ of a closed oriented manifold N^k is the group of all k-dimensional homotopy spheres Σ^k for which the canonical homotopy equivalence $N\# \Sigma\to N$ is h-cobordant to the identity (compare [6]).

Examples of pairs N, Σ with $\Sigma\notin I_h(N)$ may be given as follows:

PROPOSITION 1.2. (i) *If N^k is a π-manifold, then $I_h(N)\subseteq bP_{k+1}$.*

(ii) *Let p be an odd prime, let $k=2p^2-2p-2$, and let $\Sigma\in\Gamma_k$ have order p (such elements exist* [19], III.2). *Then $\Sigma\notin I_h(N)$ for all N^k.*

Proof. (i) Some iterated suspension of N^k has the form $S^{n+k}\vee S^n N_0$, where N_0 is a $(k-1)$-dimensional finite complex. Since the space $F/0$ that classifies normal maps is an infinite loop space, this means $[N^k, F/0]\cong\pi_k(F/0)\oplus[N_0, F/0]$, and hence the canonical homotopy equivalences in question are not even *normally* cobordant to the identity.

(ii) This result is due to H. E. Winkelnkemper [19], III.2.

Combining these observations, we immediately obtain the following examples.

THEOREM 1.3. (i) *Suppose that M^k is a π-manifold that is not a rational cohomology sphere and $\gamma\in\pi_1(\mathrm{Diff}(D^k, S^{k-1}))$ satisfies $\lambda(\gamma)\notin bP_{k+3}$. Set $N^k=T^k\# M^k$. Then $E_N(\gamma)\in\pi_1(\mathrm{Diff}(N^k))$ does not contain a homomorphism.*

(ii) *Let p be an odd prime, M^k a $(2p^2\text{-}2p\text{-}4)$-dimensional spin manifold with nonzero $\hat{A}$-genus, and $\gamma\in\pi_1(\mathrm{Diff}(D^k, S^{k-1}))$ with $\lambda(\gamma)$ of order p. Then $E_M(\gamma)$ is not representable by a circle homomorphism.*

Proof. (i) The class $E_N(\gamma)$ is nonzero by the previous two results. On the other hand, N^k has rational cup length k, so by a widely known result the orbit map for every topological S^1 action is surjective in rational cohomology (see [7] for one version). But by the Leray spectral sequence this implies $H^*(N)\cong H^*(S^1)\otimes H^*(N/S^1)$ as graded rational vector spaces, a description that is inconsistent with the construction of N. Hence N does not even admit *topological* S^1 actions. (ii) As before, $E_M(\gamma)$ is nonzero. But M has no smooth circle actions by a result of Atiyah and Hirzebruch [3].

2. Spheres. Recall the well-known splitting $\pi_*(\mathrm{Diff}\, S^n)\cong\pi_*(SO_{n+1})\oplus\oplus\pi_*(\mathrm{Diff}(D^n, S^{n-1}))$, induced by the natural inclusions of SO_{n+1} and $\mathrm{Diff}(D^n, S^{n-1})$ as subgroups of $\mathrm{Diff}\, S^n$ ([1], p. 10); we assume $n\geqslant 5$ throughout this section. Of course, all classes of the form $(\alpha, 0)$ are represented by homomorphisms; the following example shows that other such classes are also representable:

EXAMPLE 2.1. Let $\theta\in\pi_0(\mathrm{Diff}\, S^n)=\Gamma_{n+1}$ be chosen so that the element $\tau(\eta,\theta)\in\Gamma_{n+2}$ defined by the Milnor–Munkres–Novikov pairing ([1], § 1.2) is nonzero ($0\neq\eta\in\pi_1(SO_{n+1})=Z_2$; such examples exist for infinitely many n — for example, whenever $n\equiv 0, 1 \bmod 8$). Then the elements in the arc component determined by θ induce a specific automorphism $I_{\theta*}$ of $\pi_*(\mathrm{Diff}\, S^n)$ by conjugation; it follows that $\alpha\in\pi_1(\mathrm{Diff}\, S^n)$ is representable by a homomorphism if and only if $J_{\theta*}\alpha$ is. But $I_{\theta*}\eta$ has the form (η,β), and a routine verification as in [1] shows that $\lambda(\beta)=\tau(\eta,\theta)$; in particular, β is nonzero.

It is tempting to think that no class of the form $(0,\beta)$ with $\beta\neq 0$ is representable by a homomorphism, particularly since local linearity of S^1 actions at fixed points shows the nonexistence of nontrivial homomorphisms into $\mathrm{Diff}(D^n, S^{n-1})$. However, we shall give an example later to show this is false. The general problem of specifying exactly which elements of $\pi_1(\mathrm{Diff}\, S^n)$ are represented by homomorphisms seems quite subtle.

The results of this section depend upon a *smooth suspension construction* for smooth S^1 actions on S^n (compare [4], Section II). Given a homomorphism $f\colon S^1\to\mathrm{Diff}\, S^m$, call the resulting S^1-manifold (S^n, f). Form the smooth S^1-manifold

$$\Sigma^{n+2}(f)=D^2\times(S^n, f)\cup_h S^1\times(D^{n+1},\ \text{trivial})$$

where S^1 acts on itself and D^2 by complex multiplication and h is the equivariant shearing diffeomorphism $h(z, x)=(z, f(z^{-1})x)$. It is routine to check that Σ is a homotopy sphere. Define $H(f)\in\pi_{n+4}(S^{n+3})(=Z_2)$ to be the Hopf construction on the "suspended" homomorphism $S(f)\colon S^1\to\mathrm{Diff}\,\Sigma^{n+2}(f)$.

PROPOSITION 2.2. *In the above notation, let (α,β) denote the homotopy class of f. Then $\lambda(\beta)$ gives the differential structure on $\Sigma(f)$ and $H(f)=S^2(J(\alpha)+\eta)$.*

Proof. In the first place, the diffeomorphism type of $\Sigma(f)$ only depends on the homotopy class of f. Furthermore, the classes $(\alpha, 0)$ induce diffeomorphisms of $S^1\times S^n$ that extend to $S^1\times D^{n+1}$, and hence the manifold obtained by identification under (α,β) is diffeomorphic to the one obtained using $(0,\beta)$ (compare [10], 5.4). Finally, it is immediate from the definitions and [10], 2.3, p. 526 that the manifold obtained using $(0,\beta)$ is the exotic $(n+2)$-sphere with differential structure $\lambda(\beta)$.

Since the inclusion of $\mathrm{Diff}(D^n, S^{n-1})$ in the homeomorphism group of S^n is nullhomotopic by the Alexander trick, an explicit homeomorphism from $\Sigma^{n+2}(f)$ to S^{n+2} may be constructed using the topological triviality of $(0,\beta)$ and the extendibility of $(\alpha, 0)$ to $\pi_1(\mathrm{Diff}\, D^{n+1})$. Under this identification the Hopf construction on the suspended action on Σ^{n+2} corresponds to the join of α on S^n with left multiplication on S^1, and the formula for $H(f)$ is an immediate consequence of this.

In order to prove certain classes of the form $(0,\beta)$ cannot be constructed, we need a criterion showing that the above suspension construction is impossible for certain choices of Σ. Following a suggestion by B. Conrad, we say that an S^1 action on a homotopy sphere Σ^m $(m\geqslant 2)$ is *essential* or *inessential* depending on whether the Hopf construction on the adjoint map $S^1\times\Sigma^m\to\Sigma^m$ yields the essential or inessential class in $\pi_{m+2}(S^{m+1}) = Z_2$. Also recall that the *inertia group* $I(M)$ of a closed oriented manifold M^m consists of all homotopy spheres Σ^m such that $M\mathbin{\#}\Sigma$ is orientation-preservingly diffeomorphic to M.

PROPOSITION 2.3. *If* Σ^n $(n\geqslant 7)$ *admits an essential circle action, then* $I(\Sigma^n\times S^1) = 0$.

Proof. This follows from the same sort of argument used in [14], § 3, since the argument referred to uses only the fact that the action defines a nontrivial class in $\pi_1(G_{n+1})\cong\pi_{n+2}(S^{n+1})$.

It is now easy to find the desired examples.

THEOREM 2.4. *For infinitely many values of* n *(e.g.,* $n = 8k$, $8k-1$, $k\geqslant 1$*), there exist classes in* $\pi_1(\mathrm{Diff}\, S^n)$ *not representable by homomorphisms. In fact, for infinitely many values of* n *(e.g.,* $n = 8k$*) the set of classes represented by homomorphisms is not even a subgroup.*

Proof. To show the first statement, it suffices to find homotopy $(n+2)$-spheres Σ^{n+2} with $I(\Sigma^{n+2}\times S^1)\neq 0$; one reference for such examples when $n = 8k+1$ is [13]. In fact, if one takes the exotic $(8k+2)$-spheres corresponding to the elements $\mu_k\eta$ (notation as in [13]), then $I\neq 0$ for these examples too; this follows because the image of $\mu_k\eta^2$ in π_{8k+3} is the 2-torsion in the image of J, a fact which implies nontriviality by results of G. Brumfiel [5].

To see the assertion about subgroups, consider the element $I_{\mu_k *}(\eta, 0)-(\eta, 0)\in\pi_1(\mathrm{Diff}\, S^{8k})$. By Example 2.1 this is a sum of classes represented by homomorphisms and has the form $(0,\beta)$, where $\lambda(\beta) = \tau(\mu_k,\eta) = \mu_k\eta$. On the other hand, the discussion of the previous paragraph shows that such classes $(0,\beta)$ are *not* representable by homomorphisms.

EXAMPLE 2.5. Some classes of the form $(0,\beta)$ with $\lambda(\beta)\neq 0$ *are* representable by homomorphisms. For example, consider the semifree S^1 action on the exotic sphere Σ^{10} of order 3 in $\Gamma_{10} = Z_6$ constructed in [15]. W. C. Hsiang has observed that this action admits an invariant subsphere S^8 that contains all fixed points; this may be checked directly from the explicit construction for suitable choices of the equivariant diffeomorphisms in [15]. Therefore an argument due to Browder and Petrie (see [4], Prop. 2.1, p. 141) implies that Σ^{10} is a smooth suspension of some smooth semifree S^1 action f on S^8. Express the homotopy class of f as (α,β); then Proposition 2.2 implies that $\alpha = 0$ (since S^4 is the fixed point set) and $\lambda(\beta)\in\Gamma_{10}$ has order three.

An infinite family of similar examples may be constructed using the semifree circle actions in [16] and existence theorems for invariant codimension 2 subspheres from [17] (Hsiang's method also applies in this case).

EXAMPLE 2.6. It is considerably more difficult to find classes of the form (α,β) with both $\alpha\neq 0$ and $\lambda(\beta)\neq 0$ that are not representable. The only example known to the author occurs in $\pi_1(\mathrm{Diff}\, S^6)$, in which case λ maps to $\Gamma_8 = Z_2$. This is true because of the following two facts:

(i) If (α,β) is representable by a homomorphism, the suspension construction yields an S^1 action on the exotic sphere determined by $\lambda(\beta)$ that has an invariant codimension 2-subsphere and acts freely off this subsphere.

(ii) By the results of [17], the exotic 8-sphere admits no smooth S^1 actions of this type.

Finally, we mention that an elaboration of the above suspension argument yields a nonrealizability theorem for a wide class of $(8k-1)$-dimensional manifolds:

THEOREM 2.7. *Suppose that* M^{8k-1} *bounds a spin manifold and*

$$m_k\in\pi_1\big(\mathrm{Diff}(D^{8k-1}, S^{8k-2})\big)$$

satisfies $\lambda(m_k) = \mu_k\in\Gamma_{8k+1}$. *Then* $E_M(m_k)$ *is not representable by a homomorphism.*

This depends on the fact that $\mu_k\in\Gamma_{8k+1}$ does not correspond to the boundary of a spin manifold [18]. The proof uses the previous methods with M^{8k-1} replacing S^{8k-1} and a 2-connected spin manifold V^{8k} with boundary M^{8k-1} replacing D^{8k}.

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