Approximate maxima

by

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Abstract. It is well-known that if f: $[0, 1] \to R$ it continuous, then f has an absolute maximum. An analogous, not so simple, property is proven for approximately continuous functions. Applications include a new characterization of convex functions. It is further shown that this property does not extend to approximately continuous functions of several variables.

That a real-valued continuous function defined on $[0, 1]$ has an absolute maximum is a simple elementary fact. The purpose of this paper is to prove an analogous, not so simple, property for real-valued approximately continuous functions defined on $[0, 1]$. We say that a function f has an approximate maximum at x_0 if $\{x: f(x) > f(x_0)\}$ has density zero at x_0. We show that any appıoximately continuous function has an approximate maximum. In two ways this theorem is the best possible. First, A. M. Bruckner has communicated to this author an example of a bounded approximately continuous function which has no relative extrema. Second, we provide an example to show that this result does not extend to approximately continuous functions of several variables. The applications of this property include a new characterization of convex functions.

We will require the following definitions and observations. For further development of these ideas the reader is referred to Sıks [7], O'Mılley [5] and [6], and Khintchine [4]. All sets and functions will be required to be Lebesgue measurable relative to $[0, 1]$, and m will denote Lebesgue measure.

For a fixed set E and point x_0 the upper (lower) density of E at x_0 is

$$d^-(E, x_0) = \limsup_{I \to x_0} \frac{m(E \cap I)}{m(I)},$$

$$\left(d_-(E, x_0) = \liminf_{I \to x_0} \frac{m(E \cap I)}{m(I)}\right).$$

Here the notation $I \to x_0$ (read I converges to x_0) is used to signify that we consider all possible sequences of non-degenerate intervals, containing x_0, whose measures tend to zero. It is well-known that

$$d^-(E, x) + d_-([0, 1] \sim E, x) = 1 \quad \text{for all } x,$$

and

$$d^-(E, x) = d_-(E, x) = 1 \quad \text{for almost all } x \text{ in } E.$$

When $d^-(E, x) = d_-(E, x) = \alpha$ we say that E has density α at x.

A function f is approximately continuous if and only if for every a the sets $\{x: f(x)>a\}$, $\{x: f(x)<a\}$ have density 1 at all their points. It is then clear that if f is approximately continuous and $\{x: f(x)\geqslant a\}$ has positive upper density at x, then x belongs to $\{x: f(x)\geqslant a\}$. Also, all approximately continuous functions have the Darboux (Intermediate Value) Property.

Finally, a function f has an approximate maximum at x_0 if $\{x: f(x)>f(x_0)\}$ has density zero at x_0. If f has a relative maximum at x_0 it has an approximate maximum at x_0.

THEOREM 1. *Let f: $[0, 1]\to R$ be approximately continuous. Then f has an approximate maximum at some point x_0 in $[0, 1]$.*

Proof. This statement is obvious if f is constant on any set E of positive measure. We will therefore assume that $\{x: f(x) \doteq c\}$ has measure zero for all c. In particular the image, $f(I)$, of any non-degenerate interval is a non-degenerate interval. The proof will rest on the construction of a strictly increasing sequence of numbers y_n and an associated sequence of closed intervals $[a_n, b_n]$ for which:

1) $[a_{n+1}, b_{n+1}]\subset(a_n, b_n)$;

2) $b_{n+1}-a_{n+1}\leqslant\frac{1}{2}(b_n-a_n)$;

3) $m(\{x: f(x)>y_n\}\cap[a_{n+1}, b_{n+1}])>\frac{1}{2}(b_{n+1}-a_{n+1})$; and,

4) $m(\{x: f(x)>y_{n+1}\}\cap(c, d))\leqslant\dfrac{1}{2^{n+1}}(d-c)$ for all (c, d) with $(c, d)\subset[a_n, b_n]$

and (c, d) containing either a_{n+1} or b_{n+1}.

Then 1) and 2) imply that $[a_n, b_n]$ converges to a unique point x_0. For this x_0 we have by 3) that $f(x_0)\geqslant y_n$ for all n. Finally, 4) gives that $\{x: f(x)>f(x_0)\}$ has density zero at x_0.

It will suffice to construct $y_1, [a_1, b_1], y_2$ and $[a_2, b_2]$. We will assume further that f is bounded. This will cause no loss of generality since we may substitute for f the new function $g = (f)(1+|f|)^{-1}$. This g will be approximately continuous and have the same approximate maxima as f.

We first prove a lemma.

LEMMA. *Let H be a measurable subset of an interval I and $H_i = \bigcup J$: $J = (a, b) \subset I$ and $m(H\cap J)>\dfrac{1}{2^i}m(J)$. Then for each component interval (c, d) of H_i we have that $m(H\cap(a, b))\geqslant\dfrac{1}{2^{i+1}}(b-a)$, and consequently $m(H_i)\leqslant 2^{i+1}m(H)$.*

Proof. Let (c, d) be any component interval of H_i. Let $\varepsilon>0$. We select a finite collection $J_1, J_2, \ldots, J_N$ of subintervals of (c, d) in such a way that no point is in more than two of the J's and

a) $m(H\cap J_k)>\dfrac{1}{2^i}m(J_k)$, $k = 1, \ldots, N$, and

b) $m(H_1\cup\ldots\cup J_N)>(1-\varepsilon)(d-c)$.

Then

$$\begin{aligned} m(H\cap(c, d)) &\geqslant \frac{1}{2}\Big(\sum_{k=1}^{N} m(H\cap J_k)\Big) \\ &\geqslant \frac{1}{2^{i+1}}\Big(\sum_{k=1}^{N} m(J_k)\Big) \\ &\geqslant \frac{1}{2^{i+1}}(1-\varepsilon)(d-c) \end{aligned}$$

which is enough to prove the lemma.

We now return to the proof of Theorem 1 noting that $A\subset B$ implies that $A_i\subset B_i$.

Selection of y_1 and $[a_1, b_1]$. We have that $f([0, 1]) = I_0$ is a non-degenerate interval. If there is an x_0 in $[0, 1]$ such that $f(x_0) = r_0$, where r_0 is the right end point of I_0, then f has an absolute maximum, and we are finished. Hence we must assume that r_0 is not attained. We define $H(y) = \{x: f(x)>y\}$ for y in I_0 and define $H_1(y)$, as in the lemma, as $\bigcup J$: $J = (a, b)$ and $m(\{x: f(x)>y\}\cap(a, b))>\frac{1}{2}(b-a)$. Then $H_1(y)$ is an open set with the property that in each component (c, d) of $H_1(y)$,

$$m(\{x: f(x)>y\}\cap(c, d))\geqslant\tfrac{1}{4}(d-c),$$

and also

$$m(H_1(y))\leqslant 4m(\{x: f(x)>y\}) = 4m(H(y)).$$

As a function mapping I_0 into R, $m(H_1(y)) = h(y)$ is a strictly decreasing positive continuous function with $\lim_{y\to r_0} h(y) = 0$. Let $\varepsilon = \frac{1}{2}[r_0-\max(f(0), f(1))]>0$. By the approximate continuity of f at 0 and 1 it follows that there exists a fixed $\delta>0$ such that for all $0<x<\delta<\frac{1}{2}$

$$m(\{x: f(x)>r_0-\varepsilon\}\cap[0, 1])<\tfrac{1}{4}x$$

and

$$m(\{x: f(x)>r_0-\varepsilon\}\cap[1-x, 1])<\tfrac{1}{4}x.$$

For y_1 we select a fixed real number with $r_0-\varepsilon<y_1<r_0$ and $0<m(H_1(y_1))<\delta$. This is possible because $\lim_{y\to r_0} m(H_1(y)) = 0$. We claim that no component interval of $H_1(y_1)$ can have 0 or 1 as an end point. It will suffice to show this for 0 only. Suppose that 0 is the left end point of a component of $H_1(y_1)$. This component is then of the form $(0, b)$. From the fact that $m(H_1(y_1))<\delta$ it follows that $b<\delta$.

Hence $m(\{x: f(x)>r_0-\varepsilon\} \cap (0, b))<\frac{1}{4}b$, and since $y_1>r_0-\varepsilon$ it follows that $m(\{x: f(x)>y_1\} \cap (0, b))<\frac{1}{4}b$. However, as was mentioned above, in any compact interval (c, d) of $H_1(y_1)$ we must have that $m(\{x: f(x)>y_1\} \cap (c, d))\geqslant\frac{1}{4}(d-c)$. This contradiction assures that no component interval of $H_1(y_1)$ can have 0 as a left end point. For $[a_1, b_1]$ we select the closure of any component of $H_1(y_1)$.

Selection of y_2 and $[a_2, b_2]$. Our method will be similar to the above, but among the properties 1), 2), 3), and 4) it is 3) that will present the difficulty. It will necessitate that we introduce two auxiliary sequences: u_k, a strictly increasing sequence of numbers with $u_1>y_1$, and $[c_k, d_k]$, a nested sequence of intervals contained in (a_1, b_1). Any pair, u_k and $[c_k, d_k]$, will satisfy 1), 2) and 4) relative to y_1 and $[a_1, b_1]$. From this sequence of pairs we will select y_2 and $[a_2, b_2]$. For y_1 and $[a_1, b_1]$ it is clear that

$$m(\{x: f(x)>y_1\} \cap [a_1, b_1])\geqslant\tfrac{1}{4}(b_1-a_1),$$

and

$$m(\{x: f(x)>y_1\} \cap J)\leqslant\tfrac{1}{2}m(J)$$

for all open intervals J containing either a_1 or b_1. From these two statements it follows that $y_1\geqslant\max(f(a_1), f(b_1))$ and also that $f([a_1, b_1])$ is a non-degenerate interval with right end point $s_1>y_1$. If there is an x_0 in $[a_1, b_1]$ such that $f(x_0) = s_1$ then x_0 is in (a_1, b_1), and we have found a relative maximum and are finished. We must assume, therefore, that s_1 is not attained.

We define

$$H_2^1(y) = \bigcup J: J = (a, b)\subset[a_1, b_1],$$

and

$$m(\{x: f(x)>y\} \cap J)>\tfrac{1}{4}m(J).$$

As before we can find a u_1 with $s_1>u_1>y_1$ for which $m(H_2^1(u_1))<\frac{1}{2}(b_1-a_1)$ and $a_1<c<d<b_1$ for all components (c, d) of $H_2^1(u_1)$. We select for $[c_1, d_1]$ the closure of any component of $H_2^1(u_1)$.

In general, we will define for $k\geqslant 2$

$$H_2^k(y) = \bigcup J: J = (a, b)\subset[c_{k-1}, d_{k-1}],$$

and

$$m(\{x: f(x)>y\} \cap J)>\tfrac{1}{4}m(J).$$

Then we select u_k in $f([c_{k-1}, d_{k-1}])$ with $u_k>u_{k-1}$ so that

$$m(H_2^k(u_k))<\tfrac{1}{2}(d_{k-1}-c_{k-1}),$$

and $c_{k-1}<c<d<d_{k-1}$ for all components (c, d) of $H_2^k(u_k)$. For $[c_k, d_k]$ we select the closure of any component of $H_2^k(u_k)$.

It can be easily verified that each pair u_k, $[c_k, d_k]$ satisfies 1), 2) and 4) relative to y_1 and $[a_1, b_1]$. We will also have

$$(*)\qquad m(\{x: f(x)>u_k\} \cap (c_k, d_k))\geqslant\tfrac{1}{8}(d_k-c_k).$$

The nested sequence $[c_k, d_k]$ converges to a unique point x_1. Since $\{x: f(x)>u_1\} \supset\{x: f(x)>u_k\}$, (*) implies that $\{x: f(x)>u_1\}$ has upper density at least $\frac{1}{8}$ at x_1. This in turn implies that $f(x_1)\geqslant u_1>y_1$. The set $\{x: f(x)>y_1\}$ has density 1 at all of its points. Therefore, there is a $\delta>0$ such that for all open intervals J of length less than δ and containing x_1 we have

$$m(\{x: f(x)>y_1\} \cap J)>\tfrac{1}{2}m(J).$$

We select a k such that $d_k-c_k<\delta$. We let $y_2 = u_k$ and $[a_2, b_2] = [c_k, d_k]$. Then y_2 and $[a_2, b_2]$ have properties 1), 2), 3), and 4). In all other selections we employ the same process, using $H_n^k(y)$ at stage n. This completes the proof.

The following remarks, concerning approximately continuous functions f: $[0, 1]\to R$, can be established from a perusal of the above proof.

1. There is a sequence of points $\{x_n\}$ such that f has an approximate maximum at each x_n and

$$\sup\{f(x_n): n = 1, 2, 3, \ldots\} = \sup\{f(x): 0\leqslant x\leqslant 1\}\leqslant+\infty.$$

2. Let $[a, b]\subset[0, 1]$. If f is not monotone on $[a, b]$, there is an x_0 in (a, b) at which f has an approximate maximum or minimum.

From 2 and the fact that f has the Darboux property we have:

3. Let S be the set of points where f has an approximate maximum or minimum. If S is a finite set, then f is continuous and the approximate extrema are relative extrema. See also [7].

For the rest of the paper, we will need these additional definitions. The approximate limit superior of a function g at a point x_0 is

$$\operatorname{ap-lim\,sup}_{x\to x_0} g(x) = \inf[y: \{x: g(x)>y\} \text{ has density 0 at } x_0].$$

The ap-liminf and ap-lim are defined in an obvious fashion. For a function f defined in a neighborhood of $[0, 1]$ let

$$\Delta^2 f(x, h) = f(x+h)+f(x-h)-2f(x).$$

Using $\Delta^2 f(x, h)$ we define the upper approximate Schwarz derived number of f at x as

$$\mathrm{AD}_2^- f(x) = \operatorname{ap-lim\,sup}_{h\to 0}\frac{\Delta^2 f(x, h)}{h^2}.$$

Also, f is approximately smooth at x if

$$\operatorname{ap-lim}_{h\to 0}\frac{\Delta^2 f(x, h)}{h} = 0.$$

A function f is convex if $f(\frac{1}{2}(x+y))\leqslant\frac{1}{2}(f(x)+f(y))$ for all x, y. From these definitions we have:

4. Let f be approximately smooth at x_0 and have an approximate maximum at x_0. Then

$$\operatorname*{ap-lim}_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=0\,.$$

That is, f has an approximate derivative of zero at x_0.

Theorem 1 and Remark 4 form the basis for the following results. Theorem 2 is an extension of a result by Zygmund [8]. Theorem 3 is a new characterization of convex functions analogous to one by Hardy and Rogosiński [3].

THEOREM 2. *Let f: $[0,1]\to R$ be approximately continuous and approximately smooth at every x in $[0,1]$. Let D be the set of x at which f has an approximate derivative, $f'_{\rm ap}$. Then D has the power of the continuum in each subinterval of $[0,1]$. Further, $f'_{\rm ap}$ has the Darboux property in D. That is, let x and y belong to D and $f'_{\rm ap}(x)=\alpha$, $f'_{\rm ap}(y)=\beta$ and let γ be between α and β. Then there is a z between x and y such that $f'_{\rm ap}(z)=\gamma$.*

THEOREM 3. *Let f: $[0,1]\to R$ be approximately continuous. Let $AD_2^-f(x)\geqslant 0$ except for x in a countable set E, and let f be approximately smooth at each x in E. Then f is convex.*

The proofs in [8] and [3] employ only basic methods. To obtain proofs of Theorems 2 and 3 only minor modifications, using Theorem 1, are needed. For brevity we delete the arguments.

As was mentioned in the introduction, Theorem 1 does not extend to approximately continuous functions of several variables. Here, the concept of approximate continuity for functions of several variables requires only a slight refinement of the definition of density. We now consider all balls converging to x_0 in the definition of upper and lower densities. The following is an example of an approximately continuous function f defined on the unit square,

$$\{(x,y)\colon\ 0\leqslant x\leqslant 1,\ 0\leqslant y\leqslant 1\}\,,$$

without any approximate maximum.

On the lower half unit square $=\{(x,y)\colon\ 0\leqslant x\leqslant 1,\ y\leqslant x\}$ let

$$f(x,y)=\begin{cases}\alpha(1-x) & \text{for}\quad y=x^{1+\alpha},\ 0\leqslant\alpha\leqslant 1,\ 0<x\leqslant 1\,,\\ 1-x & \text{for}\quad 0\leqslant y\leqslant x^2,\ 0<x\leqslant 1\,,\\ 0 & \text{for}\quad x=y=0\,.\end{cases}$$

On the upper half unit square let $f(x,y)=f(y,x)$. This function is continuous everywhere except the origin, and approximately continuous at the origin. Further, it is easy to verify that for every (x_0,y_0) we have

$$\{(x,y)\colon\ f(x,y)>f(x_0,y_0)\}$$

has positive upper density at (x_0,y_0).

It is worthwhile to end the paper by reinterpreting Theorem 1. In [1] and [2], a topology d, called the density topology, was introduced. The continuous functions, relative to d, are precisely the approximately continuous functions. A measurable set U is d-open if and only if U has density 1 at all its points. It is clear from the definitions that if a function f has an approximate maximum at x_0 it has a relative maximum at x_0 in the d-topology. Thus, Theorem 1 becomes:

Let f: $[0,1]\to R$ be d-continuous. There is a point x_0 in $[0,1]$ at which f has a d-relative maximum. We note that Theorem 1 could be improved and the proof simplified if $[0,1]$ were a compact set in the d-topology. However, no infinite set is compact in the d-topology.

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Accepté par la Rédaction le 10. 2. 1975