Remark. After we had the result that the existence of $\varrho \in \mathscr{G}$ such that $\mathscr{D}/\varrho$ is $(\varkappa, \omega)$-regular implies the $\varkappa^+$-universality of $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$, L. Pacholski has drowe our attention that the condition above is also sufficient for the $\varkappa^+$-universality of $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ and that the Keisler's proof from [2] works also in our case.

Theorem D. *Suppose $\mathscr{D}$ is an ultrafilter on I and $\mathscr{G}$ a filter on $I \times I$ such that the pair $(\mathscr{D}, \mathscr{G})$ is $\varkappa$-closed. Suppose that for every $\varrho_1 \in \mathscr{G}$ there is $\varrho_2 \subseteq \varrho_1$, $\varrho_2 \in \mathscr{G}$ such that $\mathscr{D}/\varrho_2$ is $\varkappa$-good. Then for every structure $\mathfrak{A}$, the limit ultrapower $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ is n-saturated.*

Proof. Let $\langle [f_\xi]_{\mathscr{D}} \rangle_{\xi<\varkappa}$ be a sequence of elements of $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$. From Theorem 1, it follows that there is a relation $\varrho \in \mathscr{G}$ such that if $I/\varrho = \{I_j \colon j \in J\}$ and $\mathscr{E} = \mathscr{D}/\varrho$ then there is an elementary embedding $F\colon \mathfrak{A}^J_{\mathscr{E}} \to \mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ with $[f_\xi]_{\mathscr{D}} \in \mathrm{Rng}(F)$, for all $\xi < \varkappa$. From our hypotheses we can additionaly assume that $\mathscr{D}/\varrho$ is $\varkappa$-good. Then, by Fact I, $\mathfrak{A}^J_{\mathscr{E}}$ is $\varkappa^+$-saturated. Thus the result follows from Fact IV.

Remark. L. Pacholski has informed me that he has a combinatorial condition on a pair $(\mathscr{D}, \mathscr{G})$ which is equivalent to the statement: "*for every $\mathfrak{A}$ the limit ultrapower $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ is $\varkappa$-saturated*". For more informations see [3].

References

[1] H. J. Keisler, *Limit ultrapowers*, Trans. Amer. Math. Soc. 107 (1963), pp. 382–408.

[2] — *Ultraproducts which are not saturated*, J. Symb. Logic 32 (1967), pp. 23–46.

[3] L. Pacholski, *Homogeneity, universality and saturatedness of limit reduced powers* (III) (to appear).

[4] B. Węglorz, *Limit generalized powers*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), pp. 449–451.

[5] J. Wierzejewski, *Homogeneity, universality and saturatedness of limit reduced powers* I, Fund. Math. 94, pp. 35–39.

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The irreducibility of continua which are the inverse limit of a collection of Hausdorff arcs

by

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Abstract. Consider the space which is the inverse limit of a collection of generalized (non metric) arcs over a linearly ordered index set. Such a space is a hereditarily unicoherent atriodic Hausdorff continuum. It is shown that every indecomposable subcontinuum of the space is irreducible between some two points. A necessary and sufficient condition in order for a subcontinuum of the space to be indecomposable is stated. Further it is shown that the space must be a generalized arc if it is not the inverse limit over a countable subset of the index set. Thus it follows that the space must be an irreducible continuum.

Introduction. In this work a continuum is a closed connected subset of a Hausdorff space and an arc is a compact continuum which has only two non-cut points. It is known that if M is a nondegenerate compact atriodic hereditarily unicoherent continuum and every nondegenerate indecomposable subcontinuum of M is irreducible between some two points then M is irreducible between some two points. (See M. H. Proffitt [4] for a stronger result.) Suppose S is the inverse limit of a collection of Hausdorff arcs over a linearly ordered index set. Then S is a compact atriodic hereditarily unicoherent continuum. In this paper we show that every nondegenerate indecomposable subcontinuum of S is irreducible between some two points. Further we show that if S is not an arc then it must be the inverse limit of a collection of arcs over a countable index set (this result has also been independently discovered by G. R. Gordh and Sibe Mardešić.) Also a necessary and sufficient condition in order for a subcontinuum of S to be indecomposable is stated.

Following are some definitions used in this paper. For theorems concerning inverse limits the reader should consult Eilenberg and Steenrod [1], and for theorems concerning arcs the reader should consult Hocking and Young [2], and R. L. Moore [3].

Definition. Suppose M is an arc and 0 and 1 are the two non-cut points of M. Then the statement that *M is ordered from 0 to 1* means that if x and y are two points of M then $x<y$ (or x precedes y) if and only if $x \neq 1$ and it is true that $y = 1$ or $M - y$ is the sum of two mutually separated sets, one containing 0 and x and the

other containing 1. *M is ordered* means that M is ordered from 0 to 1 or that M is ordered from 1 to 0.

DEFINITION. If M is a Hausdorff arc and P and Q are two points of M then $[P, Q]$ denoted the Hausdorff arc which is the subcontinuum of M which has non-cut points P and Q.

DEFINITION. $\{X, f\}_E$ is an *inverse system over the directed set* E means that X is a function with domain E and f is a function with domain the subset of $E\times E$ defined by the relation "<" such that:

(1) if $a\in E$, X_a is a set,

(2) if $a<b$ in E, then $f(a, b)$, denoted by f_a^b, is a function from X_b onto X_a, and

(3) if $a<b<c$ in E then $f_a^b(f_b^c) = f_a^c$.

DEFINITION. The space S is said to be the *inverse limit*, $\varprojlim\{X_a\}_{a\in E}$, of the inverse system $\{X, f\}_E$, where X_a is a topological space for each $a\in E$, if it is true that: (1) P is a point of S means that P is a function with domain E so that for each a in E P_a is a point of X_a, and if $a<b$ in E then $P_a = f_a^b(P_b)$, and (2) R is an open set of S means that there is a finite subset $\{a_i\}_{i=1}^n$ of E and a collection $\{R_{a_i}\}_{i=1}^n$ so that for each a_i, R_{a_i} is an open set in X_{a_i} and R is the set of points to which P belongs if and only if $P_{a_i}\in R_{a_i}$.

DEFINITION. Suppose S is the inverse limit of the inverse system $\{X, f\}_E$. Then if $a\in E$ the projection π_a is the function from S into the space X_a such that if $P\in S$ then $\pi_a(P) = P_a$.

NOTATION. If for some a in E, D is a subset of X_a then $\overleftarrow{D}$ means the set $\{x\colon x_a\in D\}$.

DEFINITION. The directed set E' is said to *run through the directed set* E if it is true that $E'\subseteq E$ and for each a in E there is an element d in E' so that $a<d$ in E.

The proofs of the following two theorems are straight forward and the proofs are left for the reader.

THEOREM 1. *Suppose S is the inverse limit of the inverse system $\{M, f\}_E$ and for each a in E M_a is a compact Hausdorff space, H is a closed subset of S, and $\pi_a(H) = M_a$ for each a in E. Then $H = S$.*

THEOREM 2. *If S is the inverse limit of the inverse system $\{X, f\}_E$ where X_a is a Hausdorff arc for each a in E, and S' is a subcontinuum of S; then S' is the inverse limit of the inverse system $\{\pi_a(S'), g\}$ where $g_a^b = f_a^b|_{\pi_b(S')}$ and $\pi_a(S')$ is an arc or a point.*

THEOREM 3. *If S is the inverse limit of the inverse system $\{M, f\}_E$ where M_a is an arc for each a in E, and E is a linearly directed set; then S is atriodic and hereditarily unicoherent.*

Proof. *S is atriodic.* Suppose T is a subcontinuum of S which is a triod. Then T is the sum of three proper subcontinua, H_x, H_y, and H_z, such that the common part of each two of them is the common part of all three of them and is a proper subset of each of them. Let

$$x\in H_x-(H_y+H_z),\quad y\in H_y-(H_x+H_z),\quad z\in H_z-(H_x+H_y),\quad t\in H_x\circ H_y\circ H_z.$$

There exist regions, S_x, S_y, and S_z, of M_n for some n of E which are pairwise mutually exclusive and which contain x_n, y_n, and z_n, respectively, so that:

no point of H_y or H_z is in $\overleftarrow{S}_x$,

no point of H_x or H_z is in $\overleftarrow{S}_y$,

no point of H_x or H_y is in $\overleftarrow{S}_z$.

Then one of x_n, y_n, or z_n lies between t_n and one of x_n, y_n, or z_n. Suppose $t_n<x_n<y_n$. Then $\pi_n(H_y)$ intersects t_n and y_n and hence x_n. But $\pi_n(H_y)$ does not intersect S_x. This is a contradiction. So S contains no triod.

S is hereditarily unicoherent. Suppose H is a subcontinuum of S and H_1 and H_2 are proper subcontinua of H whose sum is H and so that $H_1\cdot H_2$ is the sum of two mutually exclusive closed point sets, A and B. There is an element a in E so that $\pi_a(A)$ and $\pi_a(B)$ are mutually exclusive, because A and B are compact. There is a point x of $\pi_a(H)$ in M_a between some point of $\pi_a(A)$ and some point of $\pi_a(B)$. There is a point P of H such that for each $b>a$ in E, P_b lies between some point of $\pi_b(A)$ and some point of $\pi_b(B)$. (This follows from the fact that some well ordered subset of E runs through E.) Suppose that P belongs to H_1. There is an element $b>a$ in E and a region R of M_b containing P_b so that $\overleftarrow{R}$ does not contain any point of H_2. But $\pi_b(H_2)$ contains P_b since it is connected and contains every point of $\pi_b(A)$ and of $\pi_b(B)$. So $\pi_b(H_2)$ intersects R. So H_2 intersects $\overleftarrow{R}$, which is a contradiction. So every subcontinuum of S is unicoherent.

THEOREM 4. *Suppose S is the inverse limit of the inverse system $\{M, f\}_E$, E is a linearly directed set, M_a is an arc for each a in E, 1 is an element of E, and there are two points, r and t, of M_1 distinct from the end points so that for each $n>1$ in E $f_1^{n^{-1}}(r)$ is separated from $f_1^{n^{-1}}(t)$ by some point x_n of M_n. Then S is decomposable.*

Proof. Let M_1 be ordered so that $r<t$. Let 0_n and 1_n be the end points of M_n so that $f_1^{n^{-1}}(r)\subseteq[0_n, x_n]$ and $f_1^{n^{-1}}(t)\subseteq[x_n, 1_n]$ and $0_n<1_n$ in M_n. Let p and q be two sequences so that:

$$p_1 = t,\quad p_k = \operatorname{glb} f_1^{k^{-1}}(p_1)\quad \text{for}\quad k>1,$$

$$q_1 = r,\quad q_k = \operatorname{lub} f_1^{k^{-1}}(q_1)\quad \text{for}\quad k>1.$$

Then from the continuity of f_a^b for all a and b in E and from the properties of arcs, the following are true:

(a) $q_k<p_k$ for each $k>1$ in E.

(b) $p_i\leqslant f_i^k(p_k)$ and $q_i\geqslant f_i^k(q_k)$ for each $k>i$ in E.

(c) No point of $(q_n, 1_n]$ is mapped into $[0_1, q_1]$ and no point of $[0_n, p_n)$ is mapped into $[p_1, 1_1]$. Thus $f_1^{n^{-1}}(0_1)\subseteq[0_n, q_n]$, and $f_1^{n^{-1}}(1_1)\subseteq[p_n, 1_n]$.

(d) $f_k^{n-1}(0_k) \subseteq [0_n, q_n]$ and $f_k^{n-1}(1_k) \subseteq [p_n, 1_n]$, for all $n > k$.

(e) $p_i = f_i^k(p_k)$ and $q_i = f_i^k(q_k)$ for all $k > i > 1$.

Thus (a)-(e) prove that the sequences p and q as defined are points of the inverse limit S. Let

$$H = \{x\colon x_i \in [0_i, p_i]\}, \quad K = \{x\colon x_i \in [q_i, 1_i]\}.$$

From (a)-(e) $[0_i, p_i]$ is mapped onto $[0_k, p_k]$ for all $i > k$; and $[q_i, 1_i]$ is mapped onto $[q_k, 1_k]$ for all $i > k$. So H and K are continua since they are inverse limits on a system of arcs.

$$\pi_1(H) = [0_1, p_1], \quad \text{so} \quad H \neq S.$$

$$\pi_1(K) = [q_1, 1_1], \quad \text{so} \quad K \neq S.$$

$$\pi_k(H) + \pi_k(K) = M_k \quad \text{for all } k \text{ in } E.$$

So $H + K = S$. So S is decomposable.

COROLLARY TO THE PROOF. *If x and y are points of S and $x_1 < r < t < y_1$, where r and t are defined as in the theorem, then x is not in K and y is not in H, as defined above.*

THEOREM 5. *Suppose S is the inverse limit of the inverse system $\{M, f\}_E$, E is a linearly directed set, M_a is an arc for each a in E, and no countable set runs through E. Then S is an arc.*

LEMMA 1. *If r and t are two points of M_1, $1 \in E$, then there is an element u in E and an integer n_u so that if $v \geqslant u$ in E then the set $H_{v,1}^{(r,t)}$ to which (a, b) belongs if and only if $f_1^v(a) = r$, $f_1^v(b) = t$, and if $x \in (a, b)$ then $f_1^v(x) \in (r, t)$, contains exactly n_u elements.*

Proof. $H_{v,u}^{(r,t)}$ is finite for all $v > u$ in E because f_u^v is continuous. Suppose that the lemma is false. Then there is an increasing sequence of elements of E $\{u_i\}_{i=1}^{\infty}$ so that $H_{u_i,1}^{(r,t)}$ contains i elements for each positive integer i. There is an element w in E which follows all the elements of the sequence $\{u_i\}_{i=1}^{\infty}$. Suppose $H_{w,1}^{(r,t)}$ has only N elements. Consider the mapping $f_{u_{N+1}}^w$. Let $H_{u_{N+1},1}^{(r,t)} = \{(a_i, b_i)\}_{i=1}^{N+j}$, $j \geqslant 1$. Since $f_{u_{N+1}}^w$ is onto, there are points x_i and y_i in M_w so that $f_{u_{N+1}}^w(x_i) = a_i$, $f_{u_{N+1}}^w(y_i) = b_i$, and $f_{u_{N+1}}^w(x) \in (a_i, b_i)$ if $x \in (x_i, y_i)$. Thus $H_{w,1}^{(r,t)}$ contains at least $N+1$ elements because $\{(x_i, y_i)\}_{i=1}^{N+1} \subseteq H_{w,1}^{(r,t)}$, which is a contradiction. So the lemma is established.

COROLLARY TO THE PROOF OF LEMMA 1. *If u is defined as in the lemma, $H_{u,1}^{(r,t)}$ contains exactly n_u elements, and $w > u$; then if (a_i, b_i) belongs to $H_{u,1}^{(r,t)}$, $H_{w,u}^{(a_i,b_i)}$ contains only one element.*

LEMMA 2. *S is decomposable.*

Proof. Suppose 1 is an element of E, $M_1 = [0, 1]$, and $0 < r < t < 1$. Define u as in Lemma 1. There is an element (a_i, b_i) in $H_{u,1}^{(r,t)}$ so that neither a_i nor b_i is an end point of M_u. Then by the corollary to Lemma 1: if $v > u$ in E, then $f_u^{v-1}(a_i)$ is separated from $f_u^{v-1}(b_i)$ by some point x_v of M_v. Thus by Theorem 4, S is decomposable; and further, every nondegenerate subcontinuum of S is decomposable.

Thus S is atriodic and hereditarily unicoherent and contains no nondegenerate indecomposable continuum, so S is irreducible between some two points, A and B.

LEMMA 3. *If x and y are two points of S then S is the sum of two continua, one not containing x and the other not containing y.*

Proof. Suppose x and y are two points of S and 1 is some element i of E so that $x_i \neq y_i$. Suppose r and t are two points of M_1 which lie in (x_1, y_1) and $r < t$ in M_1. By the corollary to Lemma 1 there is an element u of E so that if $w > u$ in E and if (a_i, b_i) belongs to $H_{u,1}^{(r,t)}$ then $H_{w,u}^{(a_i,b_i)}$ contains only one element. Suppose $y_u < x_u$. By the intermediate value property there is an element (a_i, b_i) in $H_{u,1}^{(r,t)}$ so that $y_u < a_i < b_i < x_u$. Then there is a point z_w of M_w, $w > u$, which separates $f_u^{w-1}(a_i)$ from $f_u^{w-1}(b_i)$ in M_w. Therefore, by the corollary to Theorem 4, S can be decomposed into two continua, one containing x and not y and the other containing y and not x.

Proof of Theorem 5. By Lemma 2, S is irreducible between some two points, A and B. By Lemma 3 it follows that every point of S distinct from A or B is a cut-point of S. Thus S is an arc.

DEFINITION. Suppose M is the arc $[A, B]$, M' is an arc, and f is a continuous function from M' onto M. The interval $[a, c]$ of M' is said to be *folded over the interval $[r, t]$ of M with respect to f* if there is a point b of $[a, c]$ so that $f(a)$ and $f(c)$ lie in one component of $M - [r, t]$ and $f(b)$ lies in the other.

THEOREM 6. *If S is the inverse limit of the inverse system $\{M, f\}_E$, E is the set of positive integers, $M_n = [0_n, 1_n]$ is an arc for each n in E, and there is a subset E' of E running through E so that $\{M, f\}_{E'}$ satisfies one of the following conditions, then S is irreducible between some two points.*

(A) *For each n in E', M_n is first countable at each end and if $0_n < r < t < 1_n$ then there is an element, $k > n$, in E' so that some interval, $[a, c]$ of M_k is folded over $[r, t]$ with respect to f_n^k.*

(B) *For each n in E', M_n is first countable at 1_n but not at 0_n, and if $n \in E'$ and t is a point of M_n then there is an element, $k > n$, in E' and three points, a_k, b_k, and c_k, of M_k with b_k between a_k and c_k, and either: (1) $f_n^k(a_k)$ and $f_n^k(c_k)$ lie in $[t, 1_n]$ and $f_n^k(b_k) = 0_n$, or (2) $f_n^k(a_k) = f_n^k(c_k) = 0_n$ and $f_n^k(b_k)$ lies in $[t, 1_n]$.*

(C) *For each n in E', M_n is not first countable at both ends and if $n \in E'$ then there is an element, $k > n$, in E' and three points, a_k, b_k, and c_k, of M_k with b_k between a_k and c_k, and either: (1) $f_n^k(a_k) = f_n^k(c_k) = 0_n$ and $f_n^k(b_k) = 1_n$, or (2) $f_n^k(a_k) = f_n^k(c_k) = 1_n$ and $f_n^k(b_k) = 0_n$.*

The proofs of the different cases are similar. The following is a proof of case (A) which is the most technical. For convenience let E' be E.

LEMMA 1. *On the basis of the hypothesis of the theorem there is an integer $k > n$ so that some interval I of M_k is folded over $[r, t]$ with respect to f_n^k and I contains neither end points of M_k.*

Proof of Lemma 1. Choose the integer k defined in the hypothesis of the theorem and employ the continuity of f_n^k at the points of $f_n^{k-1}(1_n)$ and $f_n^{k-1}(0_n)$.

Lemma 2. *Suppose $0_n<r<t<1_n$ in M_n and $k>n$, then there are two points r_1 and t_1 in M_k so that: $f_n^k((r_1, t_1))\supseteq[r, t]$.*

Proof of Lemma 2. Employ the continuity of f_n^k at the points 0_n and 1_n to find the desired points.

Proof of Theorem 6(A). Let $\{r_i^1\}_{i=1}^{\infty}$ be a monotonic sequence of points of M_1 which converges to 0_1 and let $\{t_i^1\}_{i=1}^{\infty}$ be a monotonic sequence of points of M_1 which converges to 1_1 and let $r_1^1<t_1^1$. Let $x_1=y_1$ be a point of (r_1^1, t_1^1). There is a positive $n_1>1$ and an interval $[a_{n_1}, c_{n_1}]$ of M_{n_1} which is folded over $[r_1^1, t_1^1]$f $a_{n_1}<c_{n_1}$, and so that $[a_{n_1}, c_{n_1}]$ contains neither end points of M_{n_1} (Lemma 1). Le, b_{n_1} be a point of $[a_{n_1}, c_{n_1}]$ which corresponds to the point b in the definition ot folding. Let x_{n_1} and y_{n_1} be preimages of x_1 and y_1 respectively under $f_1^{n_1}$ in (a_{n_1}, b_{n_1}) and (b_{n_1}, c_{n_1}) respectively.

Let $\{r_i^{n_1}\}_{i=1}^{\infty}$ and $\{t_i^{n_1}\}_{i=1}^{\infty}$ be monotonic sequences so that: $\{r_i^{n_1}\}_{i=1}^{\infty}$ converges to 0_{n_1}, $\{t_i^{n_1}\}_{i=1}^{\infty}$ converges to 1_{n_1}, $f_1^{n_1}((r_i^{n_1}, t_i^{n_1}))\supseteq[r_i^1, t_i^1]$ (by Lemma 2) and it is true that

$$0_{n_1}<r_i^{n_1}<a_{n_1}, c_{n_1}, x_{n_1}, y_{n_1}<t_i^{n_1}<1_{n_1}.$$

Suppose then that k is a positive integer, $k>1$, and that n_j, $\{r_i^{n_j}\}_{i=1}^{\infty}$, $\{t_i^{n_j}\}_{i=1}^{\infty}$, a_{n_j}, b_{n_j}, c_{n_j}, x_{n_j}, and y_{n_j} are defined for all $j\leqslant k$. There is a positive integer $n_{k+1}>n_k$ so that some subinterval $[a_{n_{k+1}}, c_{n_{k+1}}]$ of $M_{n_{k+1}}$ is folded over $[r_k^{n_k}, t_k^{n_k}]$ with respect to $f_{n_k}^{n_{k+1}}$ which contains neither endpoints of $M_{n_{k+1}}$, $a_{n_{k+1}}<c_{n_{k+1}}$, and $b_{n_{k+1}}$ corresponds to the point b in the definition of folding. Let $x_{n_{k+1}}$ and $y_{n_{k+1}}$ be preimages under $f_{n_k}^{n_{k+1}}$ of x_{n_k} and y_{n_k} in $(a_{n_{k+1}}, b_{n_{k+1}})$ and $(b_{n_{k+1}}, c_{n_{k+1}})$ respectively. So $x_{n_{k+1}}<b_{n_{k+1}}<y_{n_{k+1}}$. Let $\{r_i^{n_{k+1}}\}_{i=1}^{\infty}$ and $\{t_i^{n_{k+1}}\}_{i=1}^{\infty}$ be monotonic sequences of $M_{n_{k+1}}$ which converge to $0_{n_{k+1}}$ and $1_{n_{k+1}}$ respectively so that:

$$f_{n_k}^{n_{k+1}}((r_i^{n_{k+1}}, t_i^{n_{k+1}}))\supseteq[r_i^{n_k}, t_i^{n_k}]$$

and

$$0_{n_{k+1}}<r_i^{n_{k+1}}<a_{n_{k+1}}, c_{n_{k+1}}, x_{n_{k+1}}, y_{n_{k+1}}<t_i^{n_{k+1}}<1_{n_{k+1}}$$

for each i. Note that neither x_{n_k} nor y_{n_k} is 0_{n_k} or 1_{n_k}.

Let x and y be the points of S defined by the sequences, $x=\{x_{n_i}\}_{i=1}^{\infty}$ and $y=\{y_{n_i}\}_{i=1}^{\infty}$ respectively. Suppose I is a subcontinuum of S containing x and y So $\pi_{n_k}(I)$ contains b_{n_k} for each $k>1$, because b_{n_k} lies between x_{n_k} and y_{n_k}.

Suppose k is a positive integer. Let $m>k+2$

$$b_{n_m}\in\pi_{n_m}(I), \quad b_{n_{m-1}}\in\pi_{n_{m-1}}(I),$$

$$f_{n_{m-1}}^{n_m}(b_{n_m}) \text{ belongs to one of } [t_{m-1}^{n_{m-1}}, 1_{n_{m-1}}] \text{ or } [0_{n_{m-1}}, r_{m-1}^{n_{m-1}}].$$

So $\pi_{n_{m-1}}(I)$ contains one of $t_{m-1}^{n_{m-1}}$ or $r_{m-1}^{n_{m-1}}$.

Case (1). $t_{m-1}^{n_{m-1}}$ belongs to $\pi_{n_{m-1}}(I)$ and $b_{n_{m-1}}\in\pi_{n_{m-1}}(I)$ so $c_{n_{m-1}}\in\pi_{n_{m-1}}(I)$,

$$f_{n_{m-2}}^{n_{m-1}}[b_{n_{m-1}}, c_{n_{m-1}}])\subseteq\pi_{n_{m-2}}(I) \quad\text{so}\quad [r_{m-1}^{n_{m-2}}, t_{m-1}^{n_{m-2}}]\subseteq\pi_{n_{m-2}}(I).$$

Case (2) is similar to Case (1). So $[r_{m-2}^{n_k}, t_{m-2}^{n_k}]\subseteq\pi_{n_k}(I)$ for all $m>k+2$, because

$$f_{n_k}^{n_{m-2}}((r_{m-1}^{n_{m-2}}, t_{m-1}^{n_{m-2}}))\supseteq[r_{m-2}^{n_k}, t_{m-2}^{n_k}].$$

Thus $[0_{n_k}, 1_{n_k}]\subseteq\pi_{n_k}(I)$ since $\pi_{n_k}(I)$ is closed. So $\pi_{n_k}(I)=[0_{n_k}, 1_{n_k}]$ for all k. So $I=S$, and hence no proper subcontinuum of S contains x and y. So S is irreducible from x to y.

Corollary to the proof. *S is indecomposable if it satisfies the hypothesis of Theorem 6.*

A third point z could have been chosen along with x and y so that $z_{n_m}\in(a_{n_m}, b_{n_m})$ if m is even and $z_{n_m}\in(b_{n_m}, c_{n_m})$ if m is odd. S can be shown to be irreducible between each pair of points of the set $\{x, y, z\}$.

Theorem 7. *Suppose that S is the inverse limit of the inverse system $\{M,f\}_E$, E is the set of positive integers, and for each i in E, $M_i=[0_i, 1_i]$ is a Hausdorff arc which is first countable at neither end. Suppose further that if n is an integer, $n>1$, there is a point x_n of M_n so that $f_1^n([x_n, 1_n])$ does not contain 0_1 and $f_1^n([0_n, x_n])$ does not contain 1_1. Then S is decomposable.*

Lemma. *Suppose n is an integer. Then there are two points u_n and v_n, of M_1 so that $v_n<u_n$ and neither u_n nor v_n is 1_1 or 0_1.*

Proof of the lemma. Suppose n is a positive integer, $n>1$. Let $a_n=\operatorname{lub}f_1^{n-1}(0_1)$ and let $b_n=\operatorname{glb}f_1^{n-1}(1_1)$. Then $a_n<x_n<b_n$, $f_1^n([0_n, a_n])$ does not contain 1_1, and $f_1^n([b_n, 1_n])$ does not contain 0_1. Suppose t is a point of $M_1-f_1^n([0_n, a_n])$ distinct from 1_1 and r is a point of $M_1-f_1^n([b_n, 1_n])$ distinct from 0_1, and let $r<t$.

Suppose r and t do not have the desired property. Then let $A=\operatorname{lub}f_1^{n-1}(r)$ and $B=\operatorname{glb}f_1^{n-1}(t)$. Then $B<A$, otherwise let $v_n=r$ and $u_n=t$ and the lemma is established, $a_n<A$, $B<b_n$, A does not belong to $[b_n, 1_n]$, and B does not belong to $[0_n, a_n]$. Thus $a_n<B<A<b_n$. So 0_1 is not in $f_1^n([B, 1_n])$ and 1_1 is not in $f_1^n([0_n, A])$.

Let t' be a point of $M_1-f_1^n([0_n, A])$ distinct from 1_1, and let r' be a point of $M_1-f_1^n([B, 1_n])$ distinct from 0_1. Suppose that there are two points x and y so that $f_1^n(x)=r'$ and $f_1^n(y)=t'$ and so that $y<x$. From the above, $f_1^n(x)$ precedes every point of $f_1^n([B, A])$ and $f_1^n(y)$ follows every point of $f_1^n([B, A])$. But $a_n<y<x<b_n$. If $y\in[0_1, A]$, then $f_1^n(y)\in f_1^n([0_n, A])$, which is a contradiction, since $t'=f_1^n(y)$. Thus $y>A$ and so $x>A$. Similarly if $x\in[B, 1_n]$, then $r'\in f_1^n([B, 1_1])$ which is a contradiction. So $x<B$ and then $y<B$, which is impossible. Thus t' and r' have the desired property. So the lemma is established.

Proof of the theorem. Suppose that for each positive integer n the points u_n and v_n are defined according to the lemma. Since M_1 is first countable at neither 0_1 nor 1_1 there are points r and t, each distinct from 0_1 and 1_1, so that for each positive integer n, $u_n<t$ and $v_n>r$. Every point of $f_1^{n-1}(t)$ follows every point of $f_1^{n-1}(r)$. Therefore, by Theorem 4, S is decomposable.

Corollary 1 to Theorem 7. *Suppose that S is the inverse limit of the inverse system $\{M,f\}_E$, E is the set of positive integers, and for each i in E, M_i is an arc which is first countable at neither end. Suppose further that $M_1=[0_1, 1_1]$ and there* *is a point x_n of M_n which separates M_n into two sets, one containing $f_1^{n-1}(1_1)$ and the other containing $f_1^{n-1}(0_1)$, for each $n>1$ in E. Then S is decomposable.*

Proof. Let M_n be ordered so that $M_n = [0_n, 1_n]$ and 0_n belongs to the component of $M_n - x_n$ containing $f_1^{n-1}(0_1)$ and 1_n belongs to the component containing $f_1^{n-1}(1_1)$. Then applying Theorem 7 we get the desired result.

COROLLARY 2 TO THEOREM 7. *If S is the inverse limit of a countable sequence of arcs each of which is first countable at neither end and S is indecomposable, then S satisfies the hypothesis of Theorem* 6 (C).

THEOREM 8. *Suppose S is the inverse limit of the inverse system $\{M, f\}_E$, E is the set of positive integers, $M_i = [0_i, 1_i]$ is an arc which is first countable at 1_i and is not first countable at 0_i for each i in E, and suppose also that S is indecomposable. Then there exists some subsequence E' of E so that $\{M, f\}_{E'}$ satisfies Condition* (B) *of Theorem* 6.

Proof. If the theorem is not true there is an integer $n>1$ and a point t of M_n distinct from 1_n so that if $a_k = \operatorname{glb} f_n^{k-1}(0_n)$ and $b_k = \operatorname{lub} f_n^k(0_n)$, then t follows every point of $f_n^k(a_k, b_k)$ for all $k>n$. For each integer $k>n$, let r_k be a point of M_n which precedes t defined as follows:

Case (1). $f_n^k([b_k, 1_k])$ contains 1_n and $f_n^k([0_k, a_k])$ does not. Then let $c_k = \operatorname{glb} f_n^{k-1}(1_n)$, so $a_k \leqslant b_k < c_k$. Let r be a point of $M_n - f_n^k([c_k, 1_k])$ distinct from 0_n. $f_n^k([c_k, 1_k]) \neq M_n$, since no point of $[c_k, 1_k]$ is mapped onto 0_n. If every point of $f_n^{k-1}(r)$ precedes every point of $f_n^{k-1}(t)$ let $r_k = r$. Otherwise let $p = \operatorname{glb} f_n^{k-1}(t)$, so $b_k < p$. $f_n^k([p, 1_k])$ does not contain 0_n. Then let r_k be a point distinct from 0_n which precedes every point of $f_n^k([p, 1_k])$. So that if $p \leqslant y \leqslant 1_k$, then $r_k < f_n^k(y)$. In either case $\operatorname{lub} f_n^{k-1}(r_k) < \operatorname{glb} f_n^{k-1}(t)$.

Case (2). $f_n^k([0_k, a_k])$ contains 1_n, and $f_n^k([b_k, 1_k])$ does not. Then let $c_k = \operatorname{lub} f_n^{k-1}(1_n)$, so $c_k < a_k \leqslant b_k$. Let r be a point of $M_n - f_n^k([0_k, c_k])$ distinct from 0_n. If every point of $f_n^{k-1}(r)$ follows $f_n^{k-1}(t)$, let $r_k = r$. Otherwise let $p = \operatorname{lub} f_n^{k-1}(t)$, $p < a_k$, then let r_k be a point distinct from 0_n which precedes every point of $f_n^k([0_k, p])$. In either case $\operatorname{glb} f_n^{k-1}(r_k) > \operatorname{lub} f_n^{k-1}(t)$.

Since M_n is not first countable at 0_n there is a point r of M_n distinct from 0_n which precedes every point of the set $\{r_i\}_{i=1}^{\infty}$. Then r and t satisfy the hypothesis of Theorem 4, since $f_n^{k-1}(r)$ is separated from $f_n^{k-1}(t)$ in M_k. So S is decomposable, but this contradicts the hypothesis. So the theorem is established.

THEOREM 9. *If S is the inverse limit of an inverse system of arcs $\{M, f\}_E$, E is the positive integers and S is indecomposable, then S satisfies the hypothesis of Theorem* 6.

Proof. The proof follows from Theorems 4, 7, and 8, by considering the three possible cases.

COROLLARY. *Every inverse limit of an inverse system of arcs on a linearly directed set is irreducible between some two points.*

References

[1] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton, New Jersey, 1952.

[2] J. G. Hocking and G. S. Young, *Topology*, Reading, Mass., 1961.

[3] R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Soc. Colloquium Publications 13 (1962).

[4] M. H. Proffitt, *On composants of Hausdorff continua*, Fund. Math. 71 (1971), pp. 207–213.

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