(In the terminology of [Ma], X is a $\sigma^{\#}$-*space*.) But it is known [Ma] that every $\sigma^{\#}$-space is c-semistratifiable.

5.5. Remark. Since the spaces satisfying the hypotheses of Theorem 5.3 form a hereditary class, we see that a generalized ordered space with a $\mathcal{G}$-Souslin diagonal must be *hereditarily* paracompact. Furthermore, (5.4) shows that a generalized ordered space is c-semistratifiable (and hence paracompact) if it has a quasi-G_δ diagonal (i.e., if it admits a countable collection Ψ as in the proof of (5.4)).

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Homogeneity, universality and saturatedness of limit reduced powers (II)

by

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Abstract. We give some necessary conditions on the pair $\mathcal{D}, \mathcal{G}$), where $\mathcal{D}$ is an ultrafilter on I and $\mathcal{G}$ is a filter on $I\times I$, which imply that for every structure $\mathfrak{A}$, the limit ultrapower $\mathfrak{A}^I_{\mathcal{D}}|\mathcal{G}$ is $\varkappa$-universal (or $\varkappa$-saturated).

The paper is a continuation of [5]. In § 1, we prove Embedding Theorem which says that every limit ultrapower $\mathfrak{A}^I_{\mathcal{D}}|\mathcal{G}$ contains a lot of elementary submodels which are isomorphic to ultrapowers of $\mathfrak{A}$ reduced by ultrafilters which are obtained in a natural way from $\mathcal{D}$. The idea of Embedding Theorem (in fact contained in the proof of Theorem 4 in [4]) was suggested to the author by the proof of Wierzejewski's Theorem 1 in [5].

In § 2, we apply Embedding Theorem to give some necessary combinatorial conditions on the pair $(\mathcal{D}, \mathcal{G})$ which imply that for every structure $\mathfrak{A}$, the limit ultrapower $\mathfrak{A}^I_{\mathcal{D}}|\mathcal{G}$ is $\varkappa$-universal (or $\varkappa$-saturated).

We assume that the reader is familiar with the notion and basic properties of limit reduced powers (see [1]). We also assume the familiarity with the notions of $(\varkappa, \omega)$-regular and $\varkappa$-good filters (see e.g. [2]). The only non standard notation is the following: if ϱ is an equivalence relation on I then by I/ϱ we denote the set of all ϱ-equivalence classes over I. We write $I/\varrho = \{I_j\colon j\in J\}$ to denote that I_j's are all the ϱ-equivalence classes of elements of I.

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§ 1. Embedding Theorem. Let $\mathcal{D}$ be a filter on I and ϱ an equivalence relation on I. Let $I/\varrho = \{I_j\colon j\in J\}$. The family $\mathcal{E}\subseteq\mathcal{P}(J)$ defined by: $X\in\mathcal{E}$ if and only if $\bigcup_{j\in X} I_j\in\mathcal{D}$ is called the ϱ-*image* of $\mathcal{D}$ and is denoted by $\mathcal{D}/\varrho$. It is easy to see that $\mathcal{E}$ is a filter on J. Let $X\subseteq I$, we say that X is ϱ-*composable* if there is $Y\subseteq J$ such that $X = \bigcup_{j\in Y} I_j$. Let $\mathcal{G}$ be a filter on $I\times I$, then the family of all ϱ-composable sets for $\varrho\in\mathcal{G}$ we call the family of $\mathcal{G}$-*composable sets*. This family coincides with $2^I|\mathcal{G}$.

Embedding Theorem. *Let $\mathcal{D}$ be a filter on I and let $\mathcal{G}$ be a filter on $I\times I$. Assume that $\varrho\in\mathcal{G}$ is an equivalence relation on I. Put $I/\varrho=\{I_j\colon j\in J\}$ and $\mathcal{E}=\mathcal{D}/\varrho$. Then:*

(i) *there is an isomorphism $F\colon \mathfrak{A}^J_{\mathcal{E}}\to\mathfrak{A}^I_{\mathcal{D}}|\mathcal{G}$;*

(ii) *if $f\in A^I|\mathcal{G}$ and $\mathrm{eq}(f)\supseteq\varrho$, then $[f]_{\mathcal{D}}\in\mathrm{Rng}(F)$;*

(iii) *if $\mathcal{D}$ is an ultrafilter then F is an elementary embedding of $\mathfrak{A}^J_{\mathcal{E}}$ into $\mathfrak{A}^I_{\mathcal{D}}|\mathcal{G}$.*

Proof. Let $g\in A^J$. Let us define a function F_0 from A^J into $A^I|\mathcal{G}$ by: $F_0(g)(i)=a$ if and only if $i\in I_j$ and $g(j)=a$. Then if $g_1=g_2(\mathrm{mod}\,\mathcal{E})$ then the set $X=\{j\in J\colon g_1(j)=g_2(j)\}$ is in $\mathcal{E}$. But then $\bigcup_{j\in X}I_j=\{i\in I\colon F_0(g_1)(i)=F_0(g_2)(i)\}\in\mathcal{D}$, consequently we have $F_0(g_1)=F_0(g_2)$ $(\mathrm{mod}\,\mathcal{D})$. Thus, we can define a function F from $A^J_{\mathcal{E}}$ into $A^I_{\mathcal{D}}|\mathcal{G}$ by the condition: $F([g]_{\mathcal{E}})=[F_0(g)]_{\mathcal{D}}$.

Let $\varphi=\varphi(x_1,\dots,x_n)$ be an atomic formula. Then the following statements are pairwise equivalent:

$$\mathfrak{A}^J_{\mathcal{E}}\models\varphi[[g_1]_{\mathcal{E}},\dots,[g_n]_{\mathcal{E}}],$$

$$X=\{j\in J\colon \mathfrak{A}\models\varphi[g_1(j),\dots,g_n(j)]\}\in\mathcal{E}$$

$$(*)\qquad \bigcup_{j\in X}I_j=\{i\in I\colon \mathfrak{A}\models\varphi[F_0(g_1)(i),\dots,F_0(g_n)(i)]\}\in\mathcal{D},$$

$$\mathfrak{A}^I_{\mathcal{D}}|\mathcal{G}\models\varphi[F([g_1]_{\mathcal{E}}),\dots,F([g_n]_{\mathcal{E}})].$$

So, F is an isomorphism, which proves (i).

To check (iii), it suffices to notice that if $\mathcal{D}$ is an ultrafilter, then $\mathcal{E}$ is also an ultrafilter and the statements from $(*)$ are equivalent for arbitrary formula φ.

It remains to prove (ii). Let $f\in A^I|\mathcal{G}$ satisfy $\mathrm{eq}(f)\supseteq\varrho$. Then, for each $j\in J$, the function f is constants on I_j. Consequently, we can define a function $g\in A^J$ by: $g(j)=f(i)$ for $i\in I_j$. But then we have $F_0(g)=f$, so $[f]_{\mathcal{D}}\in\mathrm{Rng}(F)$. Q.E.D.

Example 1. The assumptions of maximality of $\mathcal{D}$ in clause (iii) cannot be removed. Indeed, let $\mathfrak{A}$ be the two-elements Boolean algebra, let $\mathcal{F}$ be the Fréchet filter on ω and $\mathcal{G}$ be the filter on $\omega\times\omega$ generated by all the equivalence relations ϱ on ω such that ω/ϱ is finite. Then for any equivalence relation $\varrho\in\mathcal{G}$, the isomorphism F from (i) of Embedding Theorem is not elementary.

Corollary 1. *Let $\mathcal{D}$ be a filter on I and let $\mathcal{G}$ be a $\varkappa$-complete filter on $I\times I$. Let $[f_\xi]_{\mathcal{D}}\in A^I_{\mathcal{D}}|\mathcal{G}$, for $\xi<\lambda<\varkappa$. Then there is an equivalence relation $\varrho\in\mathcal{G}$ such that there is an isomorphism $F\colon\mathfrak{A}^J_{\mathcal{E}}\to\mathfrak{A}^I_{\mathcal{D}}|\mathcal{G}$, where $I/\varrho=\{I_j\colon j\in J\}$ and $\mathcal{E}=\mathcal{D}/\varrho$, with $[f_\xi]_{\mathcal{D}}\in\mathrm{Rng}(F)$, for all $\xi<\lambda$. Moreover, if $\mathcal{D}$ is an ultrafilter then F is an elementary embedding.*

Proof. Let $\mathrm{eq}(f_\xi)=\varrho_\xi\in\mathcal{G}$, for all $\xi<\lambda$. Since $\mathcal{G}$ is $\varkappa$-complete, we have $\varrho=\bigcap_{\xi<\lambda}\varrho_\xi\in\mathcal{G}$. Consequently, by Embedding Theorem, for $I/\varrho=\{I_j\colon j\in J\}$ and $\mathcal{E}=\mathcal{D}/\varrho$, we have an isomorphism $F\colon\mathfrak{A}^J_{\mathcal{E}}\to\mathfrak{A}^I_{\mathcal{D}}|\mathcal{G}$, which is an elementary embedding when $\mathcal{D}$ is an ultrafilter. Finally, by (ii), we have $[f_\xi]_{\mathcal{D}}\in\mathrm{Rng}(F)$, because of $\varrho\subseteq\varrho_\xi$, for all $\xi<\lambda$.

Remark. The condition of $\varkappa$-completeness of $\mathcal{G}$ in Corollary 1 is not necessary (see Example 2). To define a weaker condition which gives the thesis of Corollary 1, we need some auxiliary notions.

Definition. Let $\mathcal{D}$ be a filter on I and let $\mathcal{G}$ be a filter on $I\times I$. Let $\langle\varrho_\xi\rangle_{\xi\in\varkappa}$ be a sequence of equivalence relations from $\mathcal{G}$. Then an equivalence relation ϱ on $I\times I$ is a *$\mathcal{D}$-lower bound* of $\langle\varrho_\xi\rangle_{\xi<\varkappa}$ if and only if there is a sequence $\langle X_\xi\rangle_{\xi<\varkappa}$ of $\mathcal{G}$-composable elements of $\mathcal{D}$ such that for each $\xi<\varkappa$ we have

$$\varrho\cap(X_\xi\times X_\xi)\subseteq\varrho_\xi\cap(X_\xi\times X_\xi).$$

If for every sequence $\langle\varrho_\xi\rangle_{\xi<\varkappa}$ of elements of $\mathcal{G}$ there is a $\mathcal{D}$-lower bound of $\langle\varrho_\xi\rangle_{\xi<\varkappa}$ in $\mathcal{G}$ then we say that the pair $(\mathcal{D},\mathcal{G})$ is *$\varkappa$-closed*.

Theorem 1. *Let $\mathcal{D}$ be a filter on I and let $\mathcal{G}$ be a filter on $I\times I$ such that the pair $(\mathcal{D},\mathcal{G})$ is $\varkappa$-closed. Then if $\langle f_\xi\rangle_{\xi<\varkappa}$ is a sequence of elements of $A^I|\mathcal{G}$ then there exists an equivalence relation $\varrho\in\mathcal{G}$ such that there is an isomorphism $F\colon\mathfrak{A}^J_{\mathcal{E}}\to\mathfrak{A}^I_{\mathcal{D}}|\mathcal{G}$, where $I/\varrho=\{I_j\colon j\in J\}$ and $\mathcal{E}=\mathcal{D}/\varrho$, with $[f_\xi]_{\mathcal{D}}\in\mathrm{Rng}(F)$, for all $\xi<\varkappa$. Moreover, if $\mathcal{D}$ is an ultrafilter then F is an elementary embedding.*

Proof. Let $\varrho_\xi=\mathrm{eq}(f_\xi)$, for $\xi<\varkappa$. Consider the sequence $\langle\varrho_\xi\rangle_{\xi<\varkappa}$ of elements of $\mathcal{G}$. By our assumptions there is a $\mathcal{D}$-lower bound of $\langle\varrho_\xi\rangle_{\xi<\varkappa}$ in $\mathcal{G}$, say ϱ. Thus there is a sequence $\langle X_\xi\rangle_{\xi<\varkappa}$ of $\mathcal{G}$-composable elements of $\mathcal{D}$ such that $\varrho_\xi\cap(X_\xi\times X_\xi)\subseteq\varrho\cap(X_\xi\times X_\xi)$, for all $\xi<\varkappa$. Take $I/\varrho=\{I_j\colon j\in J\}$ and $\mathcal{E}=\mathcal{D}/\varrho$. Then by Embedding Theorem there is an isomorphism $F\colon\mathfrak{A}^J_{\mathcal{E}}\to\mathfrak{A}^I_{\mathcal{D}}|\mathcal{G}$ which is an elementary embedding when $\mathcal{D}$ is an ultrafilter. It remains to prove that $[f_\xi]_{\mathcal{D}}\in\mathrm{Rng}(F)$, for all $\xi<\varkappa$.

For every $\xi<\varkappa$, let g be a function defined in such a way that $g_\xi\restriction X_\xi=f_\xi\restriction X_\xi$ and $\varrho\subseteq\mathrm{eq}(g_\xi)$. Of course, by the construction, we have $g_\xi=f_\xi$ $(\mathrm{mod}\,\mathcal{D})$. Since $\varrho\subseteq\mathrm{eq}(g_\xi)$, by Embedding Theorem, we have $[g_\xi]_{\mathcal{D}}\in\mathrm{Rng}(F)$, for all $\xi<\varkappa$, and consequently $[f_\xi]_{\mathcal{D}}\in\mathrm{Rng}(F)$, for all $\xi<\varkappa$. Q.E.D.

Example 2. Let I be the set of all positive rationals and let $\mathcal{D}$ be a filter on I such that for each $r\in I$, the set $\{x\in I\colon r\leqslant x\}$ is in $\mathcal{D}$. For each strictly increasing sequence $\psi=\langle\psi_n\rangle_{n\in\omega}$ of rationals without any accumulation point such that $\psi_0=0$, define $\varrho_\psi\subseteq I\times I$, by $\langle i,j\rangle\in\varrho_\psi$ if and only if there is some $n\in\omega$ such that $\psi_n\leqslant i,\ j\leqslant\psi_{n+1}$. Let G be the filter on $I\times I$ generated by all ϱ_ψ's. Then G is not ω_1-complete.

On the other hand, for each sequence $\langle\varrho_n\rangle_{n\in\omega}$ of elements of $\mathcal{G}$ there is a $\mathcal{D}$-lower bound of $\langle\varrho_n\rangle_{n\in\omega}$ in $\mathcal{G}$. Thus the pair $(\mathcal{D},\mathcal{G})$ is ω-closed.

Consequently the assumptions of Theorem 1, even in the countable case are weaker than those in Corollary 1.

We have also the following converse theorem.

Theorem 2. *Let $\mathcal{D}$ be a filter on I and let $\mathcal{G}$ be a filter on $I\times I$ such that for each structure $\mathfrak{A}$ and for each sequence $\langle f_\xi\rangle_{\xi<\varkappa}$ of elements of $A^I|\mathcal{G}$ there is an equivalence relation $\varrho\in\mathcal{G}$ such that if $I/\varrho=\{I_j\colon j\in J\}$ and $\mathcal{E}=\mathcal{D}/\varrho$ then there is an isomorphism $F\colon\mathfrak{A}^J_{\mathcal{E}}\to\mathfrak{A}^I_{\mathcal{D}}|\mathcal{G}$ with $[f_\xi]_{\mathcal{D}}\in\mathrm{Rng}(F)$ for all $\xi<\varkappa$. Then the pair $(\mathcal{D},\mathcal{G})$ is $\varkappa$-closed.*

Proof. Let $\langle \varrho_\xi \rangle_{\xi<\varkappa}$ be a sequence of elements of $\mathscr{G}$. If $|A| \geqslant |I|$ then there are functions $f_\xi \in A^I|\mathscr{G}$ such that $\mathrm{eq}(f_\xi)^* = \varrho_\xi$, for all $\xi<\varkappa$. Take $\varrho \in \mathscr{G}$ such that if $I/\varrho = \{I_j\colon j \in J\}$ and $\mathscr{E} = \mathscr{D}/\varrho$ then there is an isomorphism $F\colon \mathfrak{A}^J_{\mathscr{E}} \to \mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ with $[f_\xi]_{\mathscr{D}} \in \mathrm{Rng}(F)$, for all $\xi<\varkappa$. Then there are functions $g_\xi \in A^J$ such that putting $h_\xi = F_0(g_\xi)$ we have $h_\xi = f_\xi \pmod{\mathscr{D}}$. Let $X_\xi = \{i \in I\colon h_\xi(i) = f_\xi(i)\}$. Of course X_ξ is a $\mathscr{G}$-composable element of $\mathscr{D}$. Moreover, if $\langle i,j\rangle \in \varrho$ and $\langle i,j\rangle \in X_\xi \times X_\xi$ then $\langle i,j\rangle \in \mathrm{eq}(h_\xi)$ because of $\varrho \subseteq \mathrm{eq}(h_\xi)$. Since $h_\xi \restriction X_\xi = f_\xi \restriction X_\xi$, we have $\langle i,j\rangle \in \varrho_\xi$. Whence $\varrho \cap (X_\xi \times X_\xi) \subseteq \varrho_\xi \cap (X_\xi \times X_\xi)$, for all $\xi<\varkappa$ which shows that ϱ is a $\mathscr{D}$-lower bound of $\langle \varrho_\xi \rangle_{\xi<\varkappa}$. Q.E.D.

§ 2. Applications to the universality and saturatedness. To use Embedding Theorem to the universality and saturatedness of limit ultrapowers we need the following facts:

FACT I (Keisler [2], Theorem 1.4). *An ultrafilter $\mathscr{D}$ on I is $\varkappa^+$-good if and only if for every structure $\mathfrak{A}$, the ultrapower $\mathfrak{A}^I_{\mathscr{D}}$ is $\varkappa^+$-saturated.*

FACT II (Keisler [2], Theorem 1.5). *An ultrafilter $\mathscr{D}$ on I is $(\varkappa, \omega)$-regular if and only if for every structure $\mathfrak{A}$, the ultrapower $\mathfrak{A}^I_{\mathscr{D}}$ is $\varkappa^+$-universal.*

FACT III. *The following three conditions for an ultrafilter $\mathscr{D}$ on I are equivalent:*

(a) *$\mathscr{D}$ is ω_1-good,*

(b) *$\mathscr{D}$ is (ω, ω)-regular,*

(c) *$\mathscr{D}$ is ω_1-incomplete.*

FACT IV. *If for every $\lambda<\varkappa$ and every sequence $\langle b_\xi \rangle_{\xi<\lambda}$ of elements of $\mathfrak{B}$ there is a $\varkappa$-saturated model $\mathfrak{A}$ and an elementary embedding $F\colon \mathfrak{A} \to \mathfrak{B}$, with $b_\xi \in \mathrm{Rng}(F)$, for all $\xi<\lambda$, then $\mathfrak{B}$ is $\varkappa$-saturated.*

Now these facts together with Embedding Theorem yield the following theorems:

THEOREM A. *Let $\mathscr{D}$ be an ultrafilter on I such that for some $\varrho \in \mathscr{G}$, the ϱ-image of $\mathscr{D}$ is (ω, ω)-universal. Then for every structure $\mathfrak{A}$, the limit ultrapower $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ is ω-saturated.*

Proof. Let us remark that if $\mathscr{D}/\varrho$ *is* (ω, ω)-regular then for every $\varrho_1 \in \mathscr{G}$ there is $\varrho_2 \subseteq \varrho_1$ such that $\varrho_2 \in \mathscr{G}$ and $\mathscr{D}/\varrho_2$ is also (ω, ω)-regular. In fact, we can take $\varrho_2 = \varrho \cap \varrho_1$.

Let $\langle [f_n]_{\mathscr{D}} \rangle_{n<m}$ be a finite sequence of elements of $A^I_{\mathscr{D}}|\mathscr{G}$. Let $\varrho_n = \mathrm{eq}(f_n)$ and take $\varrho^* = \varrho \cap \varrho_0 \cap \ldots \cap \varrho_{m-1}$. Let $I/\varrho^* = \{I_j\colon j \in J\}$ and $\mathscr{E} = \mathscr{D}/\varrho^*$. Then by Embedding Theorem there is an elementary embedding F of $\mathfrak{A}^J_{\mathscr{E}}$ into $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ with $[f_n]_{\mathscr{D}} \in \mathrm{Rng}(F)$, for all $n<m$. Moreover, since $\mathscr{E}$ is (ω, ω)-regular, by Facts III and I, the structure $\mathfrak{A}^J_{\mathscr{E}}$ is ω_1-saturated, whence ω-saturated. Consequently, by Fact IV, the structure $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ is ω-saturated.

EXAMPLE 3. Let $\mathfrak{N} = \langle \omega, \leqslant \rangle$ be the structure of natural numbers with the natural ordering. It is easy to construct a sequence of sets I_n and a sequence of ultrafilters $\mathscr{D}_n$ (on I_n), $n \in \omega$, such that if we define $\mathfrak{A}_0 = \mathfrak{N}$, $\mathfrak{A}_{n+1} = (\mathfrak{A}_n)^{I_n}_{\mathscr{D}_n}$ then there is a function $f_n \in A_n^{I_n}$ such that for every $a \in A_n$ we have $\{i \in I_n\colon a \leqslant f_n(i)\} \in \mathscr{D}_n$.

Put $\mathfrak{B} = \bigcup_{n\in\omega} \mathfrak{A}_n$. It is well known (see [1], Theorem 5.1) that there is a set I and an ultrafilter $\mathscr{E}$ on I such that for some filter $\mathscr{G}$ on $I \times I$, we have $\mathfrak{B} \cong \mathfrak{N}^I_{\mathscr{E}}|\mathscr{G}$. On the other hand it is easy to see that $\mathfrak{B}$ is not ω_1-saturated, for, there is a countable increasing sequence of elements of $\mathfrak{B}$ which is cofinal in $\mathfrak{B}$. Thus, in Theorem A, we cannot replace ω by ω_1.

THEOREM B. *Let $\mathscr{D}$ be an ultrafilter on I and let $\mathscr{G}$ be a filter on $I \times I$. Let $\mathfrak{A}$ be a structure such that $\mathfrak{A} \neq \mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$.*

(i) *If $\mathfrak{A}$ is ω-saturated then $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ is ω-saturated too.*

(ii) *If either $|I|$ or $|A|$ is a nonmeasurable cardinal, then $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ is ω-saturated without any assumption on $\mathfrak{A}$.*

Proof. Since $\mathfrak{A} \neq \mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$, there is a function $f \in A^I|\mathscr{G}$ which is not constant on any set from $\mathscr{D}$. Take $\varrho = \mathrm{eq}(f)$. Then for each $\varrho' \subseteq \varrho$, the filter $\mathscr{D}/\varrho'$ is non-principal.

If $\mathscr{D}/\varrho$ is not ω_1-complete then it is (ω, ω)-regular by Fact III and we can get the theses of Theorem B from Theorem A.

So, suppose that for no $\varrho' \subseteq \varrho$, $\varrho' \in \mathscr{G}$ the filter $\mathscr{D}/\varrho'$ is (ω, ω)-regular. Then both $|I|$ and $|A|$ must be measurable and we need the assumption of (i).

Let $\langle [f_n]_{\mathscr{D}} \rangle_{n<m}$ be a finite sequence of elements of $A^I_{\mathscr{D}}|\mathscr{G}$. Let $\varrho^* = \varrho \cap \varrho_0 \cap \ldots \cap \varrho_{m-1}$, where $\varrho_n = \mathrm{eq}(f_n)$, $n = 0, \ldots, m-1$. Let $I/\varrho^* = \{I_j\colon j \in J\}$. Then $\mathscr{E} = \mathscr{D}/\varrho^*$ is an ω_1-complete ultrafilter on J. By Łoś Ultraproduct Theorem for ω_1-complete ultrafilters $\mathfrak{A}^J_{\mathscr{E}}$ is ω-saturated if and only if $\mathfrak{A}$ is ω-saturated. Consequently, by Embedding Theorem we have an elementary embedding F of $\mathfrak{A}^J_{\mathscr{E}}$ into $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$, with $[f_n]_{\mathscr{D}} \in \mathrm{Rng}(F)$, for all $n<m$. Thus, by Fact IV, we see that $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ is ω-saturated. Q.E.D.

Remark. Theorem B is closely related to a theorem of Wierzejewski ([5], Theorem 2) that if $\mathfrak{A}$ is an ω-homogeneous then $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ is ω-homogeneous. But in Theorem B, for the nonmeasurable case, we have a stronger thesis without any assumption on $\mathfrak{A}$.

EXAMPLE 4. Let $\mathscr{F}$, $\mathscr{G}$ be filters from Example 1, and let $\mathfrak{A}$ be the ring of integers. Then it is easy to check that $\mathfrak{A}^{\omega}_{\mathscr{F}}|\mathscr{G}$ is not ω-saturated. Consequently, in Theorems A and B we cannot omit the assumption that $\mathscr{D}$ is maximal.

THEOREM C. *Let $\mathscr{D}$ be an ultrafilter on I and let $\mathscr{G}$ be a filter on $I \times I$. Then there exists $\varrho \in \mathscr{G}$ such that $\mathscr{D}/\varrho$ is $(\varkappa, \omega)$-regular if and only if for every structure $\mathfrak{A}$, the limit ultrapower $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ is $\varkappa^+$-universal.*

Proof. Suppose there is $\varrho \in \mathscr{G}$ such that $\mathscr{D}/\varrho$ is $(\varkappa, \omega)$-regular. Let $I/\varrho = \{I_j\colon j \in J\}$ and $\mathscr{E} = \mathscr{D}/\varrho$. Then, by Fact II, the ultrapower $\mathfrak{A}^J_{\mathscr{E}}$ is $\varkappa^+$-universal. By Embedding Theorem there is an elementary embedding $F\colon \mathfrak{A}^J_{\mathscr{E}} \to \mathfrak{A}^J_{\mathscr{D}}|\mathscr{G}$. Consequently $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ is $\varkappa^+$-universal as an elementary extension of a $\varkappa^+$-universal structure.

The converse implication follows in the same way as in Keisler's proof of Fact II (see [2]).

Remark. After we had the result that the existence of $\varrho \in \mathscr{G}$ such that $\mathscr{D}/\varrho$ is $(\varkappa, \omega)$-regular implies the $\varkappa^+$-universality of $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$, L. Pacholski has drowe our attention that the condition above is also sufficient for the $\varkappa^+$-universality of $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ and that the Keisler's proof from [2] works also in our case.

THEOREM D. *Suppose $\mathscr{D}$ is an ultrafilter on I and $\mathscr{G}$ a filter on $I\times I$ such that the pair $(\mathscr{D}, \mathscr{G})$ is $\varkappa$-closed. Suppose that for every $\varrho_1 \in \mathscr{G}$ there is $\varrho_2 \subseteq \varrho_1$, $\varrho_2 \in \mathscr{G}$ such that $\mathscr{D}/\varrho_2$ is $\varkappa$-good. Then for every structure $\mathfrak{A}$, the limit ultrapower $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ is n-saturated.*

Proof. Let $\langle [f_\xi]_{\mathscr{D}} \rangle_{\xi<\varkappa}$ be a sequence of elements of $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$. From Theorem 1, it follows that there is a relation $\varrho \in \mathscr{G}$ such that if $I/\varrho = \{I_j\colon j \in J\}$ and $\mathscr{E} = \mathscr{D}/\varrho$ then there is an elementary embedding $F\colon \mathfrak{A}^J_{\mathscr{E}} \to \mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ with $[f_\xi]_{\mathscr{D}} \in \mathrm{Rng}(F)$, for all $\xi<\varkappa$. From our hypotheses we can additionaly assume that $\mathscr{D}/\varrho$ is $\varkappa$-good. Then, by Fact I, $\mathfrak{A}^J_{\mathscr{E}}$ is $\varkappa^+$-saturated. Thus the result follows from Fact IV.

Remark. L. Pacholski has informed me that he has a combinatorial condition on a pair $(\mathscr{D}, \mathscr{G})$ which is equivalent to the statement: "*for every $\mathfrak{A}$ the limit ultrapower $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ is $\varkappa$-saturated*". For more informations see [3].

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The irreducibility of continua which are the inverse limit of a collection of Hausdorff arcs

by

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Abstract. Consider the space which is the inverse limit of a collection of generalized (non metric) arcs over a linearly ordered index set. Such a space is a hereditarily unicoherent atriodic Hausdorff continuum. It is shown that every indecomposable subcontinuum of the space is irreducible between some two points. A necessary and sufficient condition in order for a subcontinuum of the space to be indecomposable is stated. Further it is shown that the space must be a generalized arc if it is not the inverse limit over a countable subset of the index set. Thus it follows that the space must be an irreducible continuum.

Introduction. In this work a continuum is a closed connected subset of a Hausdorff space and an arc is a compact continuum which has only two non-cut points. It is known that if M is a nondegenerate compact atriodic hereditarily unicoherent continuum and every nondegenerate indecomposable subcontinuum of M is irreducible between some two points then M is irreducible between some two points. (See M. H. Proffitt [4] for a stronger result.) Suppose S is the inverse limit of a collection of Hausdorff arcs over a linearly ordered index set. Then S is a compact atriodic hereditarily unicoherent continuum. In this paper we show that every nondegenerate indecomposable subcontinuum of S is irreducible between some two points. Further we show that if S is not an arc then it must be the inverse limit of a collection of arcs over a countable index set (this result has also been independently discovered by G. R. Gordh and Sibe Mardešić.) Also a necessary and sufficient condition in order for a subcontinuum of S to be indecomposable is stated.

Following are some definitions used in this paper. For theorems concerning inverse limits the reader should consult Eilenberg and Steenrod [1], and for theorems concerning arcs the reader should consult Hocking and Young [2], and R. L. Moore [3].

DEFINITION. Suppose M is an arc and 0 and 1 are the two non-cut points of M. Then the statement that *M is ordered from* 0 *to* 1 means that if x and y are two points of M then $x<y$ (or x precedes y) if and only if $x \neq 1$ and it is true that $y = 1$ or $M - y$ is the sum of two mutually separated sets, one containing 0 and x and the