Homogeneity, universality and saturatedness
of limit reduced powers I

by

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Abstract. We investigate homogeneity of limit ultrapowers and connections between saturated and homogeneous limit ultrapowers.

By classical results of Keisler [3] ultraproducts are known as constructions which can be used to obtain relational structures which are $\varkappa$-saturated. In [3] Keisler gave a purely combinatorical characterization of those maximal filters $\mathscr{F}$ which have the property that every product reduced by $\mathscr{F}$ is $\varkappa$-saturated. A similar characterization for $\varkappa$-universality has been given in [4]. The results of Keisler have been extended to the case of products reduced by filters which are not necessarily maximal (see [5], [7]).

It is well known that the notion of a homogeneous structure is closely related to the notion of a saturated structure. In particular a structure is $\varkappa$-saturated iff it is $\varkappa$-universal and $\varkappa$-homogeneous. This suggests that it should be possible to give a characterization of filters $\mathscr{F}$ such that the reduced power $\mathfrak{A}^I_{\mathscr{F}}$ is always $\varkappa$-homogeneous. Recently L. Pacholski proved that if $\mathfrak{A}^I_{\mathscr{F}}$ is $\varkappa$-homogeneous for every $\mathfrak{A}$, then $\mathscr{F}$ is $(\omega, \varkappa)$-regular and consequently if $\mathscr{F}$ is an ultrafilter, $\mathscr{F}$ is $\varkappa$-good. This shows that the ultrapowers which are $\varkappa$-homogeneous are to large to be considered when one looks for methods of construction of homogeneous structures. This leads us to the investigation of limit ultrapowers. But here the situation is exactly the same (see Proposition).

In this note we also show that $\varkappa$-homogeneity is not preserved under finite powers and we show that some limit ultrapowers are homogeneous, which solve some problems of B. Węglorz. The related results will be published in [8].

We use the standard notation. η is the set of all rational numbers. If $a = \langle a_0, a_1 \rangle$ then $(a)_1 = a_1$. $\varkappa$ always denote an infinite cardinal number. If f is a function then $\operatorname{do}(f)$ denotes its domain.

The necessary background can be found e.g. in [1]. A filter $\mathscr{F}$ is ω_1-complete if it is closed under countable intersections. If $\mathfrak{A}$ is a structure, $\mathscr{D}$ is a filter over a set I and $\mathscr{G}$ is a filter over I^2, then the *limit reduced power* $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ is the substructure of $\mathfrak{A}^I_{\mathscr{D}}$ with the universe $A^I_{\mathscr{D}}|\mathscr{G} = \{[f]_{\mathscr{D}}| \ \{\langle i,j \rangle| \ f(i) = f(j)\} \in \mathscr{G}\}$. $\mathfrak{A}^C$ denotes the

structure, the universe of which contains all continuous function from the Cantor set C (i.e. 2^ω with the product topology) into A (regarded as the discrete space) and $\mathfrak{A}^C \vDash R[f_1, ..., f_n]$ iff $\{i | \mathfrak{A} \vDash R[f_1(i), ..., f_n(i)]\} = C$.

We shall begin with an example, which shows that $\varkappa$-homogeneity is not preserved under finite powers. The example below is almost the same as Example 1.5 in [9].

EXAMPLE. Let $\mathfrak{A}_\varkappa$ be the following structure:

$$\mathfrak{A}_\varkappa = \left\langle \eta \cup (\eta \times \eta \times \varkappa), W, C, D, R, <; \frac{1}{n+1} \right\rangle_{n \in \omega}$$

where

W is a unary relation and $W(a)$ iff $a \in \eta$,
C is a unary relation and $C(a)$ iff $a \notin \eta$,
D is a ternary relation and $D(a, b, c)$ iff $W(a)$ and $W(b)$ and
$\quad \exists \xi \in \varkappa \; c = \langle a, b, \xi \rangle$,
R is a ternary relation and $R(a, b, c)$ iff $D(a, b, c)$ and

$$\begin{cases} f(a) < f(b) \rightarrow \exists \xi \in \varkappa \; c = \langle a, b, \xi \rangle & \text{and } \xi \text{ is even} \\ f(a) \geq f(b) \rightarrow \exists \xi \in \varkappa \; c = \langle a, b, \xi \rangle & \text{and } \xi \neq 0 \end{cases}$$

where $f: \eta \rightarrow \omega$ is a fixed 1-1 function onto ω. $<$ is the natural ordering of η. Note, that in $\mathfrak{A}_\varkappa$ we can also define an ordering of η into type ω.

LEMMA 1. $\mathfrak{A}_\varkappa$ *is $\varkappa$-homogeneous.*

Proof. Let $\alpha < \varkappa$ and assume that $(\mathfrak{A}_\varkappa, a_\xi)_{\xi < \alpha} \equiv (\mathfrak{A}_\varkappa, ga_\xi)_{\xi < \alpha}$. We are going to define an automorphism of $\mathfrak{A}_\varkappa$ which extends g. Since every element of $W^{\mathfrak{A}_\varkappa}$ is definable, we have for $x \in \mathrm{do}(g) \cap W^{\mathfrak{A}_\varkappa} g(x) = x$. Moreover, for $b_0, b_1 \in \eta$ there exist 1-1 functions h_{b_0, b_1}^R and $h_{b_0, b_1}^{\neg R}$ such that:

(i) $\quad h_{b_0, b_1}^R: \{\xi | \mathfrak{A}_\varkappa \vDash R(b_0, b_1, \xi)\} \xrightarrow{\text{onto}} \{\xi | \mathfrak{A}_\varkappa \vDash R(b_0, b_1, \xi)\}$,

$\qquad h_{b_0, b_1}^{\neg R}: \{\xi | \mathfrak{A}_\varkappa \vDash \neg R(b_0, b_1, \xi)\} \xrightarrow{\text{onto}} \{\xi | \mathfrak{A}_\varkappa \vDash \neg R(b_0, b_1, \xi)\}$,

(ii) $\quad h_{b_0, b_1}^R \supseteq g \upharpoonright \mathrm{do}(h_{b_0, b_1}^R), \; h_{b_0, b_1}^{\neg R} \supseteq g \upharpoonright \mathrm{do}(h_{b_0, b_1}^{\neg R})$.

Put

$$h(x) = \begin{cases} x & \text{if } W(x), \\ \langle x_0, x_1, h_{x_0, x_1}^R(x_2) \rangle & \text{if } \neg W(x) \text{ and } x = \langle x_0, x_1, x_2 \rangle \\ & \text{and } R(x_0, x_1, x_2), \\ \langle x_0, x_1, h_{x_0, x_1}^{\neg R}(x_2) \rangle & \text{if } \neg W(x) \text{ and } x = \langle x_0, x_1, x_2 \rangle \\ & \text{and } \neg R(x_0, x_1, x_2) \wedge D(x_0, x_1, x_2). \end{cases}$$

Then h is an automorphism of $\mathfrak{A}_\varkappa$ and $h \supseteq g$. Therefore the proof is complete.

LEMMA 2. (i) *For $n \geq 2$ $\mathfrak{A}^n$ is not ω-homogeneous.*

(ii) $\mathfrak{A}^c$ *is not ω-homogeneous.*

Proof. (i) By the Feferman–Vaught theorem ([2]) we have

$$(\mathfrak{A}^n, \langle -1, ..., -1 \rangle) \equiv (\mathfrak{A}^n, \langle 0, ..., 0 \rangle).$$

Now if $a = \langle 0, ..., 0 \rangle$, $b \in A^n$ and $a < b$, then for some k

$$\left\langle \frac{1}{k}, ..., \frac{1}{k} \right\rangle < b \quad \text{whence} \quad (\mathfrak{A}^n, \langle -1, ..., -1 \rangle, a) \not\equiv (\mathfrak{A}^n, \langle 0, ..., 0 \rangle, b).$$

Hence $\mathfrak{A}^n$ is not ω-homogeneous.

(ii) The same proof works.

L. Pacholski ([6]) characterizes those pairs $\langle \mathscr{D}, \mathscr{G} \rangle$ ($\mathscr{D}$ is an ultrafilter) for which every $\mathfrak{A}_\mathscr{D}^I | \mathscr{G}$ is $\varkappa$-saturated. We complete this result, namely we prove the following:

PROPOSITION. *Let $\mathscr{D}$ be an ultrafilter over a set I and let $\mathscr{G}$ be a filter over I^2. Suppose that for any $\mathfrak{A}$, $\mathfrak{A}_\mathscr{D}^I | \mathscr{G}$ is $\varkappa$-homogeneous. Then for any $\mathfrak{A}$, $\mathfrak{A}_\mathscr{D}^I | \mathscr{G}$ is $\varkappa$-saturated.*

Proof. Fix $\mathfrak{A}_0$ and let $\mathfrak{A}_1$ be a $\varkappa$-saturated elementary extension of $\mathfrak{A}_0$. Let $\mathfrak{B}$ be the following structure:

$$\mathfrak{B} = \langle A_0 \times \{0\} \cup A_1 \times \{1\}, \sim, R_\xi^\mathfrak{B} \rangle$$

where $\sim$ is a binary relation such that $a \sim b$ iff $(a)_1 = (b)_1$, $R_\xi^\mathfrak{B} = R_\xi^{\mathfrak{A}_0} \times \{0\} \cup \cup R_\xi^{\mathfrak{A}_1} \times \{1\}$.

Let $0 \neq X \subseteq A_{0\mathscr{D}}^I | \mathscr{G}$ and $|X| < \varkappa$. By the Łoś theorem we have:

$$(\mathfrak{B}_\mathscr{D}^I | \mathscr{G}, [fx]_\mathscr{D})_{x \in X} \equiv (\mathfrak{B}_\mathscr{D}^I | \mathscr{G}, [gx]_\mathscr{D})_{x \in X}$$

where $f(x)(i) = \langle x(i), 0 \rangle$ and $g(x)(i) = \langle x(i), 1 \rangle$.

Let p be a 1-type of the language of $\mathfrak{A}_0$ over the set X. Let fp and gp be the types obtained from p by replacing each constant $x \in X$ with fx and gx respectively.

By the hypothesis $\mathfrak{A}_{1\mathscr{D}} | \mathscr{G}$ is $\varkappa$-homogeneous. Since $\mathfrak{A}_1$ is $\varkappa$-saturated and $\mathfrak{A}_1 \prec \mathfrak{A}_{1\mathscr{D}}^I | \mathscr{G}$ it follows that $\mathfrak{A}_{1\mathscr{D}}^I | \mathscr{G}$ is also $\varkappa$-saturated. Hence p is realized in $\mathfrak{A}_{1\mathscr{D}}^I | \mathscr{G}$. From the above mentioned it follows that gp is realized in $\mathfrak{B}_\mathscr{D}^I | \mathscr{G}$ by $a = \langle a_0, 1 \rangle$. But by the hypothesis $\mathfrak{B}_\mathscr{D}^I | \mathscr{G}$ is $\varkappa$-homogeneous. Therefore fp is realized in $\mathfrak{B}_\mathscr{D}^I | \mathscr{G}$ by $b = \langle b_0, 0 \rangle$. Finally p is realized in $\mathfrak{A}_{0\mathscr{D}}^I | \mathscr{G}$ by b_0. Hence $\mathfrak{A}_{0\mathscr{D}}^I | \mathscr{G}$ is $\varkappa$-saturated.

Now we shall show that some limit ultrapowers are ω_1-homogeneous. From now on we assume that all structure are in a countable language.

THEOREM 1. *Let $\mathscr{D}$ be an ultrafilter over a set I and let $\mathscr{G}$ be an ω-complete filter over I^2. Assume that $\mathfrak{A}$ and $\mathfrak{A}_\mathscr{D}^I$ are ω_1-homogeneous. Then $\mathfrak{A}_\mathscr{D}^I | \mathscr{G}$ is also ω_1-homogeneous.*

Proof. Assume that $(\mathfrak{A}_\mathscr{D}^I | \mathscr{G}, [f_n]_\mathscr{D})_{n \in \omega} \equiv (\mathfrak{A}_\mathscr{D}^I | \mathscr{G}, [g_n]_\mathscr{D})_{n \in \omega}$ and let $f \in A^I | \mathscr{G}$. By the ω_1-completeness of $\mathscr{G}$ there are an ordinal α and sets J_ξ for $\xi < \alpha$ such that: $J_\xi \subseteq I$, $J_\xi \cap J_\eta = 0$ for $\xi < \eta < \alpha$, $\bigcup \{J_\xi | \xi < \alpha\} = I$, $\bigcup \{J_\xi^2 | \xi < \alpha\} \in \mathscr{G}$ and for $\xi < \alpha$, $n \in \omega$ f_n, g_n and f are constant on J_ξ. For $x \in J_\xi$ let $f_n(x) = \hat{f}_{n, \xi}$, $g_n(x) = \hat{g}_{n, \xi}$ and $f(x) = \hat{f}_\xi$.

Case 1. There is $\xi_0 < \alpha$ such that $J_{\xi_0} \in \mathscr{D}$. From the Łoś theorem it follows that $(\mathfrak{A}, \hat{f}_{n, \xi_0})_{n \in \omega} \equiv (\mathfrak{A}, \hat{g}_{n, \xi_0})_{n \in \omega}$. But $\mathfrak{A}$ is ω_1-homogeneous and therefore there exists $\hat{g} \in A$ such that

$$(\mathfrak{A}, \hat{f}_{n, \xi_0}, \hat{f}_{\xi_0})_{n \in \omega} \equiv (\mathfrak{A}, \hat{g}_{n, \xi_0}, \hat{g})_{n \in \omega}.$$

Again using the Łoś theorem, we obtain

$$(\mathfrak{A}_\mathscr{D}^I|\mathscr{G}, [f_n]_\mathscr{D}, [f]_\mathscr{D})_{n\in\omega} \equiv (\mathfrak{A}_\mathscr{D}^I|\mathscr{G}, [g_n]_\mathscr{D}, [g]_\mathscr{D})_{n\in\omega},$$

where $g(i) = \hat{g}$ for $i \in I$.

Case 2. For all $\xi < \alpha$ $J_\xi \notin \mathscr{D}$. Since $\mathfrak{A}_\mathscr{D}^I$ is ω_1-homogeneous, there is $[g] \in A_\mathscr{D}^I$ such that

(*) $$(\mathfrak{A}_\mathscr{D}^I, [f_n]_\mathscr{D}, [f]_\mathscr{D})_{n\in\omega} \equiv (\mathfrak{A}_\mathscr{D}^I, [g_n]_\mathscr{D}, [g]_\mathscr{D})_{n\in\omega}.$$

Let $\{\varphi_i\}_{i\in\omega}$ be an enumeration of all formulas φ such that $\mathfrak{A}_\mathscr{D}^I \vDash \varphi[[f_0]_\mathscr{D}, \dots, [f_n]_\mathscr{D}, [f]_\mathscr{D}]$. We put $A_{-1} = I$ and

$$A_n = A_{n-1} \cap \{i|\mathfrak{A} \vDash \varphi_n[g_0(i), \dots, g_{i_n}(i), g(i)]\} \cap \{i|\mathfrak{A} \vDash \varphi_n[f_0(i), \dots, f_{i_n}(i), f(i)]\},$$

$$T(n) = \{\xi|A_n \cap J_\xi \neq 0\}, \qquad F = \{\xi|\forall n \in \omega \; J_\xi \cap A_n \neq 0\}.$$

From (*) and the Łoś theorem it follows that for $n \in \omega$ $A_n \in \mathscr{D}$.

Subcase 2a. $\bigcup \{J_\xi| \; \xi \in F\} \in \mathscr{D}$. Since the proof is similar to that in Case 1 we leave it to the reader.

Subcase 2b. $\bigcup \{J_\xi| \; \xi \in F\} \notin \mathscr{D}$. By induction we define sets B_n, C_n for $n \in \omega$ and a sequence $\{b_n\}_{n\in\omega}$ such that: $B_n \subseteq \alpha$, $C_n \subsetneq C_{n+1} \subseteq I$, $b_n \in \alpha$, $C_n \notin \mathscr{D}$, $B_k \cap B_l = 0$ for $k < l \in \omega$. Put $C_{-1} = 0$ and let

$$B_n = \{\xi| \; J_\xi \cap A_n = 0 \text{ and } J_\xi \cap C_{n-1} = 0\},$$
$$b_n = \min\{k \in T(n)| \; J_k \cap C_{n-1} = 0\},$$
$$C_n = C_{n-1} \cup \bigcup \{J_\xi| \; \xi \in B_n\} \cup J_{b_n}.$$

Now we choose arbitrarily a sequence $\{a_n\}_{n\in\omega}$ such that $a_n \in J_{b_n} \cap A_n$ and a sequence $\{c_\xi^n| \; n \in \omega, \; \xi \in B_n\}$ such that $c_\xi^n \in J_\xi \cap A_{n-1}$. It is a matter of an easy calculation to find that the sets $J_{b_n} \cap A_n$ and $J_\xi \cap A_{n-1}$ are nonempty. Now we are going to define the function g' as follows:

$$g'(i) = \begin{cases} g(a_n) & \text{if} \quad i \in C_n \setminus C_{n-1} \text{ and } i \in J_{b_n}, \\ g(c_\xi^n) & \text{if} \quad i \in C_n \setminus C_{n-1} \text{ and } i \in J_\xi \text{ for } \xi \in B_n, \\ g(a_0) & \text{otherwise}. \end{cases}$$

Claim. (i) $\bigcup \{C_n| \; n \in \omega\} \in \mathscr{D}$,

(ii) $g' \in A_\mathscr{D}^I|\mathscr{G}$,

(iii) $(\mathfrak{A}_\mathscr{D}^I, [g_i]_\mathscr{G}, [g]_\mathscr{D})_{i\in\omega} \equiv (\mathfrak{A}_\mathscr{D}^I, [g_i]_\mathscr{D}, [g']_\mathscr{D})_{i\in\omega}$.

(i) follows from the fact that $\bigcup \{J_\xi| \cdot \xi \in F\} \notin \mathscr{D}$. Since $J_\xi \cap C_n \neq 0$ implies that $J_\xi \subseteq C_n$ we have (ii). To prove (iii) it is enough to see that

$$\{i|\mathfrak{A} \vDash \varphi_k[g_0(i), \dots, g_{i_k}(i), g'(i)]\} \supseteq \bigcup \{C_n| \; n \in \omega\} \setminus C_k \in \mathscr{D}.$$

This completes the proof.

THEOREM 2. *Let $\mathscr{D}$ be an ultrafilter over a set I and let $\mathscr{G}$ be a filter over I^2. Assume that $\mathfrak{A}$ is ω-homogeneous. Then $\mathfrak{A}_\mathscr{D}^I|\mathscr{G}$ is also ω-homogeneous.*

Proof. If $\mathscr{D}$ is ω_1-complete, then the proof is similar to that in Case 1 of the proof of the previous theorem. In the other case, by a theorem of Keisler [3] $\mathfrak{A}_\mathscr{D}^I|\mathscr{G}$ is ω-homogeneous and the proof of Theorem 1 works.

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