On decompositions of hereditarily smooth continua

by

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Abstract. It is proved that a continuum X is hereditarily smooth at p (for the definition see below) if and only if there is an upper semi-continuous monotone decomposition ω of X such that Y is an arcwise connected continuum which is hereditarily smooth at $\varphi_\omega(p)$ and for each subcontinuum Q of X such that $p \in Q$ we have $\varphi_\omega^{-1}\varphi_\omega(Q) = Q$, where Y is the decomposition space of ω and φ_ω is the canonical mapping. This result generalizes a well-known theorem for continua which are hereditarily unicoherent at some point [2].

§ 1. Preliminaries. In this paper we give a characterization of hereditarily smooth continua by their monotone decompositions having an arcwise connected decomposition space. This result generalizes a theorem obtained in [2] by G. R. Gordh and reduces the study of hereditarily smooth continua to the study of hereditarily smooth arcwise connected continua.

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The topological spaces under consideration will be assumed to be metric and compact. If the space under consideration is established, then ab denotes an arbitrary arc with endpoints a and b, and $I(a, b)$ denotes an arbitrary irreducible continuum between a and b.

The notion of smoothness of continua in a general form has been introduced in [4]. We say that the continuum X is *smooth at the point* $p \in X$ if, for each convergent sequence $\{x_n\}$ of points of X and for each subcontinuum K of X such that $p, x \in K$, where $x = \lim_{n\to\infty} x_n$, there exists a sequence $\{K_n\}$ of subcontinua of X such that $p, x_n \in K_n$ for each $n = 1, 2, \ldots$ and $\operatorname{Lim}_{n\to\infty} K_n = K$ (the topological limit).

We have (see [4], Theorem (3.1)(iv), p. 83)

PROPOSITION 1. *A continuum X is smooth at the point $p \in X$ if and only if for each subcontinuum N of X and for each open set V of X there exists a continuum K such that $p \in N \subset V$ implies $N \subset \operatorname{Int} K \subset K \subset V$.*

A continuum X is said to be *hereditarily smooth at* p provided each subcontinuum of X which contains p is smooth at p. Recall that a continuum X is *hereditarily unicoherent at p* if the intersection of any two subcontinua each of which contains p is connected (see [2], p. 52).

It is well known that for every irreducible continuum X there exists an upper semi-continuous decomposition of X into continua (called *layers* of X; see [3], § 48, IV, p. 199) and the decomposition of X into layers is the finest of all linear upper semi-continuous decompositions of X into continua (see [3], § 48, IV, Theorem 3, p. 200). If each layer of X has a void interior, than X is said to *be of type* λ (see [3], § 48, III, p. 197, footnote). It is well known (see [3], § 48, VII, Theorem 3, p. 216) that an irreducible continuum X is of type λ if and only if each indecomposable subcontinuum of X has a void interior.

We have (cf. [1], Proposition 1, p. 46)

PROPOSITION 2. *If a continuum X is hereditarily smooth at the point p, then any irreducible continuum $I(p, x)$ is of type λ.*

In fact, any irreducible continuum $I(p, x)$ is smooth at p by the hereditary smoothness of X at p. Therefore the continuum $I(p, x)$ is hereditarily unicoherent at p (see [4], Theorem (5.3), p. 88); thus $I(p, x)$ is smooth in the sense of Gordh (cf. [2], p. 52). It follows from Corollary 3.3 of [2], p. 55 that every indecomposable subcontinuum of $I(p, x)$ has a void interior in $I(p, x)$, i.e., the continuum $I(p, x)$ is of type λ.

§ 2. **Continua of convergence.** Recall that a subcontinuum K of X is called the *continuum of convergence* (see [3], § 47, VI, p. 245) provided K is the topological limit of the sequence of continua such that

$$K = \operatorname*{Lim}_{n\to\infty} K_n \quad \text{and} \quad K \cap K_n = \emptyset \quad \text{for each } n = 1, 2, \ldots$$

If X is compact, then we can assume that continua $K_1, K_2, \ldots$ are mutually disjoint.

We have the following generalization of Theorem 2 of [5].

THEOREM 3. *Let a continuum X be hereditarily smooth at the point p and suppose that L is a subcontinuum of X such that $p \in L$. Then, for each continuum K of convergence in X, the set $K \cap L$ is a continuum.*

Proof. Suppose, on the contrary, that the set $K \cap L$ is not connected. Thus there are closed, nonempty sets A and B such that

$$(1) \qquad K \cap L = A \cup B \quad \text{and} \quad A \cap B = \emptyset .$$

Let $I(a, b)$ be an arbitrary subcontinuum of K irreducible between sets A and B, where $a \in A$ and $b \in B$. It follows from Theorem 2 of [3], § 48, IX, p. 223 that

(2) the continuum $I(a, b)$ is irreducible between each point of the set $I(a, b) \cap A$ and each point of the set $I(a, b) \cap B$.

Consider two cases.

1′. The continuum $I(a, b)$ is indecomposable. Let C be a composant of the point a in $I(a, b)$ (for the definition of a composant see [3], § 48, VI, p. 208). It follows from Theorem 2 of [3], § 48, VI, p. 209, that $\overline{C} = I(a, b)$. Thus there is a sequence $\{b_n\}$ of points of C such that

$$(3) \qquad \lim_{n\to\infty} b_n = b .$$

Consider the continuum $R = L \cup I(a, b)$. Since X is hereditarily smooth at p and since $p \in L \subset R$, we conclude that the continuum R is smooth at p. Thus, because $p, b \in L$, it follows from (3) that there is a sequence $\{L_n\}$ of subcontinua of R such that

$$(4) \qquad p, b_n \in L_n \quad \text{for each } n = 1, 2, \ldots$$

and

$$(5) \qquad \operatorname*{Lim}_{n\to\infty} L_n = L .$$

Since $p \in L_n \cap L$ and $b_n \in L_n$ (cf. (4)), we infer that the continuum L_n contains an irreducible continuum $I(b_n, c_n)$ between b_n and L. Since no proper subcontinuum S of $I(b_n, c_n)$ such that $b_n \in S$ intersects L, i.e., $S \cap L = \emptyset$, we conclude that the composant C_n of the point b_n in the continuum $I(b_n, c_n)$ is contained in $I(a, b)$. Therefore $I(b_n, c_n) = \overline{C}_n \subset I(a, b)$ (cf. [3], § 48, VI, Theorem 2, p. 209), i.e.,

$$(6) \qquad I(b_n, c_n) \subset I(a, b) \quad \text{for each } n = 1, 2, \ldots$$

Moreover,

$$(7) \qquad A \cap I(b_n, c_n) \neq \emptyset \quad \text{for each } n = 1, 2, \ldots$$

In fact, if $I(b_n, c_n) = I(a, b)$, then obviously (7) holds, because $a \in I(a, b) \cap A \neq \emptyset$. Thus, to show (7), we can assume by (6) that the continuum $I(b_n, c_n)$ is a proper subcontinuum of $I(a, b)$. Since $b_n \in I(b_n, c_n)$ and $b_n \in C$, we conclude $I(b_n, c_n) \subset C$. If $C \cap B \neq \emptyset$, then there is a proper subcontinuum (contained in C) of $I(a, b)$ joining sets A and B, contrary to the choice of $I(a, b)$. Thereby $C \cap B = \emptyset$; thus $I(b_n, c_n) \cap L \subset C \cap L \subset C \cap A$. Since $I(b_n, c_n) \cap L \neq \emptyset$ by the choice of $I(b_n, c_n)$, we infer that condition (7) holds.

The set $D = \operatorname*{Ls}_{n\to\infty} I(b_n, c_n)$ is a continuum (cf. [3], § 47, II, Theorem 6, p. 171) and $b \in D \subset I(a, b)$ and $D \cap A \neq \emptyset$ by (3), (6) and (7). Thus by the irreducibility of $I(a, b)$ between A and B (cf. (2)) we infer $D = I(a, b)$. Since $I(b_n, c_n) \subset L_n$, we conclude by (5) that $I(a, b) = D = \operatorname*{Ls}_{n\to\infty} I(b_n, c_n) \subset L$. Hence we have $I(a, b) \subset K \cap L$, contrary to (1).

2′. The continuum $I(a, b)$ is decomposable. Then there are proper subcontinua M and N of $I(a, b)$ such that $I(a, b) = M \cup N$. It follows from (2) that either $M \cap A = \emptyset$ and $N \cap B = \emptyset$ or inversely

$$(8) \qquad M \cap B = \emptyset \quad \text{and} \quad N \cap A = \emptyset .$$

Without loss of generality we can assume (8).

Since K is a continuum of convergence in X, we conclude that there are subcontinua K_n of X such that

(9) $K = \operatorname{Lim}_{n\to\infty} K_n$ and $(K \cup K_m) \cap K_n = \emptyset$ for each $m \neq n$ and $m, n = 1, 2, \ldots$

Let $d \in M \cap N$. Therefore there is a sequence $\{d_n\}$ of points of X such that

(10) $$\lim_{n\to\infty} d_n = d \quad \text{and} \quad d_n \in K_n \quad \text{for each } n = 1, 2, \ldots$$

It follows from (1) and (8) by the normality of X that there are open sets U and V such that

(11) $$A \subset U, \quad B \subset V$$

and

(12) $$(U \cap V) \cup (U \cap N) \cup (V \cap M) = \emptyset .$$

Then the set $G = X \setminus (K \setminus (U \cup V))$ is open in X. Moreover, conditions (1) and (11) imply $p \in L \subset G$. Since X is smooth at p, there is, by Proposition 1, a continuum Q in X such that

(13) $$L \subset \operatorname{Int} Q \subset Q \subset G .$$

Since $\operatorname{Lim}_{n\to\infty} K_n \cap L = K \cap L \neq \emptyset$, we can assume by (13) that

(14) $$K_n \cap Q \neq \emptyset \quad \text{for each } n = 1, 2, \ldots$$

It follows from (10) and (13) that we can take irreducible subcontinua $I(d_n, a_n)$ of K_n between d_n and Q. Consider the set

$$P = Q \cup K \cup \bigcup_{n=1}^{\infty} I(d_n, a_n) .$$

Since $Q \cap K \neq \emptyset$ and $I(d_n, a_n) \cap Q \neq \emptyset$ for each $n = 1, 2, \ldots$, we conclude that the set P is connected. Moreover, $\operatorname{Ls}_{n\to\infty} I(d_n, a_n) \subset \operatorname{Lim}_{n\to\infty} K_n = K$ (cf. (9)); thus P is closed, i.e.,

(15) the set P is a continuum.

Furthermore,

(16) if F is a subcontinuum of P such that $d_n \in F$ and $F \cap Q \neq \emptyset$, then $I(d_n, a_n) \subset F$.

Indeed, since $d_n \in F$ and $F \cap Q \neq \emptyset$, we infer that the continuum F contains an irreducible continuum $I(d_n, a'_n)$ between d_n and Q. Therefore no proper subcontinuum S of $I(d_n, a'_n)$ such that $d_n \in S$ intersects the continuum Q, i.e., $S \cap Q = \emptyset$. Thus

$$S = S \cap P = (S \cap K) \cup \bigcup_{k=1}^{\infty} (S \cap I(d_k, a_k)) .$$

Since S is connected and sets $S \cap K$, $S \cap I(d_k, a_k)$ for $k = 1, 2, \ldots$ are mutually disjoint (cf. (9) and the definition of $I(d_k, a_k)$) and since $d_n \in S \cap I(d_n, a_n)$, we conclude that the equality $S = S \cap I(d_n, a_n)$ holds, i.e., $S \subset I(d_n, a_n)$. This implies that the composant C of the point d_n in the continuum $I(d_n, a'_n)$ is contained in $I(d_n, a_n)$. Therefore $I(d_n, a'_n) = \bar{C} \subset I(d_n, a_n)$ (cf. [3], § 48, VI, Theorem 2, p. 209). By the irreducibility of $I(d_n, a_n)$ between d_n and Q we infer $I(d_n, a'_n) = I(d_n, a_n)$. Thus $I(d_n, a_n) \subset F$ by the choice of $I(d_n, a'_n)$, i.e., (16) holds.

The set $W = \operatorname{Ls}_{n\to\infty} I(d_n, a_n)$ is a continuum (cf. [3], § 47, II, Theorem 6, p. 171). Moreover, $d \in W \subset K$ and $W \cap Q \neq \emptyset$ by (9), (10) and by the choice of $I(d_n, a_n)$. Let $e \in W \cap Q = K \cap W \cap Q$. Since $K \cap Q \subset K \cap G \subset (K \cap U) \cup (K \cap V)$, it suffices to consider two cases.

a) $e \in K \cap U$. Since P is a continuum (cf. (15)), $p \in L \subset Q \subset P$, we conclude that P is smooth at p by the hereditary smoothness of X at p. Thus, since $p, d \in Q \cup N \subset Q \cup I(a, b) \subset Q \cup K \subset P$, we infer by (10) that there are continua F_n such that

$$p, d_n \in F_n \subset P \quad \text{for each } n = 1, 2, \ldots$$

and

$$\operatorname{Lim}_{n\to\infty} F_n = Q \cup N .$$

It follows from (16) that $I(d_n, a_n) \subset F_n$; thus $W = \operatorname{Ls}_{n\to\infty} I(d_n, a_n) \subset \operatorname{Lim}_{n\to\infty} F_n = Q \cup N$, whence $W \cup N \subset Q \cup N$. Therefore, we have $W \cup N = (W \cup N) \cap (Q \cup N) \subset (K \cap Q) \cup N \subset (K \cap U) \cup ((K \cap V) \cup N)$. The set $W \cup N$ is a continuum, because $d \in W \cap N$. But the sets $K \cap U$ and $(K \cap V) \cup N$ are disjoint (cf. (12)), and $e \in (W \cup N) \cap (K \cap U)$ and $N \subset (W \cup N) \cap ((K \cap V) \cup N)$, contrary to the connectedness of $W \cup N$.

b) $e \in K \cap V$. The continuum P is smooth at p; thus, since $p, d \in Q \cup M \subset P$, we infer by (10) that there are continua F_n such that

$$p, d_n \in F_n \subset P \quad \text{for each } n = 1, 2, \ldots$$

and

$$\operatorname{Lim}_{n\to\infty} F_n = Q \cup M .$$

It follows from (16) that $I(d_n, a_n) \subset F_n$; thus $W \subset Q \cup M$. We obtain a contradiction in the same way as in case (a). The proof of Theorem 3 is complete.

COROLLARY 4. *Let a continuum X be hereditarily smooth at the point p and suppose that L is a subcontinuum of X such that $p \in L$. Then, for each layer T of an arbitrary irreducible continuum $I(p, x)$, the set $L \cap T$ is connected.*

Indeed, by the assumptions, the continuum $I(p,x)$ is smooth at p. Let a be an arbitrary point of $L\cap T$. Therefore, by Lemma 1 of [5], for each $y\in T$ there is a continuum of convergence K_y such that $\{a,y\}\subset K_y\subset T$. Thus $T=\bigcup\{K_y: y\in T\}$ and $L\cap T=\bigcup\{K_y\cap L: y\in T\}$. Sets $K_y\cap L$ for each $y\in T$ are connected by Theorem 3, and $a\in K_y\cap L$ for each $y\in T$. This implies that the set $L\cap T$ is connected (see [3], § 46, II, Corollary 3 (i), p. 132).

If we transform the proof of Theorem 1 of [5], then we obtain the proof of the following

Proposition 5. *Let a continuum X be hereditarily smooth at the point p and let Q be an arbitrary subcontinuum of X. If pq is an arc in X which is irreducible between p and Q, then the continuum Q is hereditarily smooth at q.*

We have also

Proposition 6. *Let a continuum X be hereditarily smooth at the point p, let $I(p,c)$ be an arbitrary subcontinuum of X (irreducible from p to c) and let T be a layer of the point c in $I(p,c)$. If there is an arc pc such that $pc\cap T=\{c\}$, then $T=\{c\}$.*

In fact, one can observe that the assumption of the arcwise connectedness of X in the proof of Theorem 3 of [5] is used only to conclude that there is an arc pc. We assume the existence of the arc pc. Now if we transform the proof of Theorem 3 of [5], putting $d=p$ and using Corollary 4 instead of Corollary 7 of [5] and Proposition 5 instead of Theorem 1 of [5], then we obtain the proof of Proposition 6.

§ 3. Monotone decompositions.

Now we prove the following

Theorem 7. *Let a continuum X be hereditarily smooth at the point p. If a monotone mapping f maps the continuum X onto Y, then the continuum Y is hereditarily smooth at $f(p)$.*

Proof. Suppose Q that is a subcontinuum of Y such that $f(p)\in Q$. Since the mapping f is monotone, we infer that the set $f^{-1}(Q)$ is a continuum. Moreover, $p\in f^{-1}(Q)$. By the hereditary smoothness of X at p, we conclude that the continuum $f^{-1}(Q)$ is smooth at p. Therefore the continuum $Q=ff^{-1}(Q)$ is smooth at $f(p)$ by Theorem (6.2) of [4], p. 90, i.e., Y is hereditarily smooth at $f(p)$.

We have

Theorem 8. *Let a continuum X be hereditarily smooth at the point p. If T and T' are layers of the point c in two irreducible continua $I(p,c)$ and $I'(p,c)$, respectively, then $T=T'$.*

Proof. Consider the continuum K of the form: $K=I(p,c)\cup I'(p,c)$. Define a monotone decomposition ϱ onto K as follows: if $x,y\in K$, then $x\varrho y$ if and only if either $x=y$ or x and y belong to the same layer in $I'(p,c)$. Let φ be the canonical mapping from K onto K/ϱ. Obviously $\varphi|I'(p,c)$ is monotone and $\varphi(I'(p,c))$ is an arc joining points $\varphi(p)$ and $\varphi(c)$ in K/ϱ. Moreover, it follows from Corollary 4 that the set $\varphi^{-1}(t)\cap I(p,c)$ is connected for each $t\in K/\varrho$. Hence the mapping $\varphi|I(p,c)$ is monotone, and thus the set $\varphi(I(p,c))$ is an irreducible continuum

between $\varphi(p)$ and each point of $\varphi(T)$ by Theorem 3 of [3], § 48, I, p. 192. It follows from Theorem 7 that the continuum K/ϱ is hereditarily smooth at the point $\varphi(p)$. Thus Proposition 6 implies that $\varphi(T)=\{\varphi(c)\}$, and so $T\subset\varphi^{-1}\varphi(T)=\varphi^{-1}\varphi(c)=T'$, whence $T=T'$, because the role of T and T' is symmetric. The proof of Theorem 8 is complete.

Corollary 9. *Let a continuum X be hereditarily smooth at the point p. If $c\in I(p,x)\cap I(p,y)$ and T and T' are layers of c in these continua, then $T=T'$.*

In fact, let $I(p,c)\subset I(p,x)$ and $I'(p,c)\subset I(p,y)$. If T_c is a layer of c in $I(p,c)$ and T is a layer of c in $I(p,x)$, then $T_c=T$ by Lemma 2 of [5], and similarly: if T'_c is a layer of c in $I'(p,c)$ and T' is a layer of c in $I(p,y)$, then $T'_c=T'$ by Lemma 2 of [5]. By Theorem 8 we have $T_c=T'_c$, and thus $T=T'$.

If ω is an equivalence relation on X, we denote by $\mathcal{D}_\omega$ the decomposition of X induced by ω and we denote by φ_ω the projection mapping from X onto X/ω.

Let X be an arbitrary continuum and let $p\in X$. We define a relation ϱ as follows:

$x\varrho y$ if and only if there are continua $I(p,x)$ and $I(p,y)$ such that $I(p,x)=I(p,y)$.

Proposition 10. *The relation ϱ is reflexive and symmetric.*

Lemma 11. *If a continuum X is hereditarily smooth at p, then ϱ is an equivalence relation and the equivalence classes are layers of continua $I(p,x)$.*

Proof. By Proposition 10, it suffices to show that the relation ϱ is transitive. Let $x\varrho y$ and $y\varrho z$. By the definition of ϱ there are continua $I(p,x)$, $I(p,y)$, $I'(p,y)$ and $I(p,z)$ such that $I(p,x)=I(p,y)$ and $I'(p,y)=I(p,z)$. By Theorem 8 the layers T and T' of the point y in $I(p,y)$ and $I'(p,y)$, respectively, are equal. The point x is a point of irreducibility of $I(p,x)$, and thus $x\in T=T'$. Therefore the point x is a point of irreducibility of $I'(p,y)$, i.e., $I'(p,y)=I(p,x)$; thus $I'(p,x)=I(p,z)$. Hence $x\varrho z$.

Further, if T is a layer of the point x of the continuum $I(p,x)$, then T is contained in the equivalence class $[x]$ of x with respect to the relation ϱ. It follows from the definition of ϱ and from Corollary 9 that $[x]=T$.

We have the following generalization of Theorem 5.2 of [2], p. 58.

Theorem 12. *If a continuum X is hereditarily smooth at the point p, then the decomposition $\mathcal{D}_\varrho$ is such that*

(i) *$\mathcal{D}_\varrho$ is upper semi-continuous,*
(ii) *the elements of $\mathcal{D}_\varrho$ are continua,*
(iii) *the decomposition space of $\mathcal{D}_\varrho$ is arcwise connected,*
(iv) *if $\mathcal{E}$ is a decomposition satisfying* (i), (ii) *and* (iii), *then $\mathcal{D}_\varrho\leqslant\mathcal{E}$ (i.e., $\mathcal{D}_\varrho$ refines $\mathcal{E}$),*
(v) *each element of $\mathcal{D}_\varrho$ has a void interior,*
(vi) *X/ϱ is hereditarily smooth at $\varphi_\varrho(p)$,*
(vii) *if Q is a continuum, then $p\in Q\subset X$, implies $Q=\varphi_\varrho^{-1}\varphi_\varrho(Q)$.*

Proof. (i) In order to prove that $\mathscr{D}_\varrho$ is upper semi-continuous it suffices to show that ϱ is a closed subset of $X\times X$ (see [3], § 43, I, Theorem 4, p. 58). Let $\{(x_n, y_n): n\in N\}$ be a sequence of points of ϱ which converges to (x, y). Then $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Let $I(p, x)$ be an arbitrary subcontinuum of X irreducible between p and x. By the smoothness of X at p there are continua $I(p, x_n)$ in X such that $\operatorname{Lim}_{n\to\infty} I(p, x_n) = I(p, x)$ (cf. [4], Theorem (2.4), p. 81). Since $x_n \,\varrho\, y_n$, there are continua $I'(p, x_n)$ and $I'(p, y)$ such that $I'(p, x_n) = I'(p, y_n)$. Thus y_n belongs to a layer T' of x_n in $I'(p, x_n)$. Therefore $y_n\in I(p, x_n)$. We infer $y\in I(p, x)$. Take an irreducible continuum $I(p, y)$ in $I(p, x)$. In a similar way, we obtain $x\in I(p, y)$. But $x\in I(p, y)\subset I(p, x)$ implies $I(p, y) = I(p, x)$ by the irreducibility of $I(p, x)$. This means $x\,\varrho\, y$.

(ii) and (v). The fact that the elements of $\mathscr{D}_\varrho$ are continua, indeed continua with void interiors, follows immediately from Lemma 11 (cf. Proposition 2).

(iii) Let $\varphi_\varrho(x)$ denote an arbitrary point of $X/\varrho\setminus\{\varphi_\varrho(p)\}$. Applying Lemma 11 to the arbitrary continuum $I(p, x)$, we find that $\varphi_\varrho(I(p, x))$ is an arc containing $\varphi_\varrho(p)$ and $\varphi_\varrho(x)$. Thus X/ϱ is arcwise connected.

(iv) Suppose that there is an equivalence relation ω such that the decomposition $\mathscr{D}_\omega = \{\varphi_\omega^{-1}(t): t\in X/\omega\}$ satisfies (i), (ii) and (iii). If $\mathscr{D}_\varrho$ does not refine $\mathscr{D}_\omega$, then there exist an element $D\in\mathscr{D}_\varrho$ and elements E_1 and E_2 of $\mathscr{D}_\omega$ such that $E_1\cap D\neq\emptyset$ and $E_2\cap D\neq\emptyset$.

Since X/ω is arcwise connected, we may assume that there exists an arc A in X/ω which contains the points $\varphi_\omega(p)$ and $\varphi_\omega(E_1)$ but misses $\varphi_\omega(E_2)$. Now $\varphi_\omega^{-1}(A)$ is a continuum which contains p and intersects D properly. This contradicts the definition of D (cf. Lemma 11); consequently:

(vi) It follows from (i), (ii) and Theorem 7 that X/ϱ is hereditarily smooth at $\varphi_\varrho(p)$.

(vii) Let Q be an arbitrary continuum such that $p\in Q\subset X$. It is obvious that $Q\subset\varphi_\varrho^{-1}\varphi_\varrho(Q)$. Let $x\in\varphi_\varrho^{-1}\varphi_\varrho(Q)$. Then there is a point $y\in Q$ such that $x\,\varrho\, y$. It follows from Lemma 11 that points x and y belong to the same layer of any continuum $I(p, y)$. But $p, y\in Q$, and thus Q contains such a continuum. Therefore $x\in Q$, i.e., $\varphi_\varrho^{-1}\varphi_\varrho(Q)\subset Q$. The proof of Theorem 12 is complete.

Theorem 3 of [5] and Theorem 12 (iii), (vi) imply

COROLLARY 13. *If a continuum X is hereditarily smooth at the point p, then X/ϱ is hereditarily arcwise connected.*

We have

THEOREM 14. *Let X be a continuum and $p\in X$. If there is an equivalence relation ω such that*

(i) *$\mathscr{D}_\omega$ is upper semi-continuous,*

(ii) *the elements of $\mathscr{D}_\omega$ are continua,*

(iii) *if Q is a continuum, then $p\in Q\subset X$ implies $Q = \varphi_\omega^{-1}\varphi_\omega(Q)$,*

(iv) *X/ω is hereditarily smooth at $\varphi_\omega(p)$,*

then the continuum X is hereditarily smooth at the point p.

Proof. Let K be an arbitrary subcontinuum of X and suppose $p\in K$. Let $\{x_n\}$ be a convergent sequence of points of K and let L be a subcontinuum of K such that $p, x\in L$, where $x = \lim_{n\to\infty} x_n$. It follows from (i) that sets $\varphi_\omega(K)$ and $\varphi_\omega(L)$ are continua and $\varphi_\omega(x) = \lim_{n\to\infty}\varphi_\omega(x_n)$. Moreover $\{\varphi_\omega(p), \varphi_\omega(x)\}\subset\varphi_\omega(L)\subset\varphi_\omega(K)$, and $\varphi_\omega(x_n)\in\varphi_\omega(K)$ for each $n = 1, 2, \ldots$ Condition (iv) implies that there are subcontinua L_n of $\varphi_\omega(K)$ such that $\{\varphi_\omega(p), \varphi_\omega(x_n)\}\subset L_n$ for each $n = 1, 2, \ldots$ and $\operatorname{Lim}_{n\to\infty} L_n = \varphi_\omega(L)$. The sets $\varphi_\omega^{-1}(L_n)$ are continua for each $n = 1, 2, \ldots$ by (ii) and $\{p, x_n\}\subset\varphi_\omega^{-1}(L_n)\subset\varphi_\omega^{-1}\varphi_\omega(K) = K$ by (iii). It follows from (i) that $\operatorname{Ls}_{n\to\infty}\varphi_\omega^{-1}(L_n) \subset\varphi_\omega^{-1}(\operatorname{Lim}_{n\to\infty} L_n) = \varphi_\omega^{-1}\varphi_\omega(L) = L$ by (iii). Put $R_n = \varphi^{-1}(L_n)\cup L$. Then sets R_n are continua, $p, x_n\in R_n\subset K$ and $\operatorname{Lim}_{n\to\infty} R_n = L$; thus K is smooth at p. Therefore X is hereditarily smooth at p.

Theorems 12 and 14 imply

COROLLARY 15. *Let X be a continuum and let $p\in X$. The continuum X is hereditarily smooth at p if and only if there is an equivalence relation ω on X such that*

(i) *$\mathscr{D}_\omega$ is upper semi-continuous,*

(ii) *the elements of $\mathscr{D}_\omega$ are continua,*

(iii) *if Q is a continuum, then $p\in Q\subset X$ implies $\varphi_\omega^{-1}\varphi_\omega(Q) = Q$,*

(iv) *X/ω is hereditarily smooth at p,*

(v) *X/ω is arcwise connected.*

One can pose the following

PROBLEM 16. Let X be a continuum and let $p\in X$. Does it follow that if a mapping f of X onto Y is such that $Q = f^{-1}f(Q)$ for each subcontinuum Q of X containing p, then for each subcontinuum K of Y containing $f(p)$ the set $f^{-1}(K)$ is connected?

If the answer to Problem 16 is positive, then one can observe that assumption (ii) in Theorem 14 may be omitted.

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