On Ciric's fixed point theorem

by

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Abstract. Some results related to a fixed point theorem of L. B. Ciric have been presented in this paper.

0. Let (X, d) be a complete metric space and let T be a mapping of X into itself such that

$$d(Tx, Ty) \leqslant \alpha d(x, y) \tag{1}$$

where $0 \leqslant \alpha < 1$ and $x, y \in X$. Then by Banach's [2] fixed point theorem T has a unique fixed point. According to Kannan's [3] fixed point theorem the following condition also implies that T has a unique fixed point:

$$d(Tx, Ty) \leqslant \alpha[d(x, Tx) + d(y, Ty)] \tag{2}$$

where $0 \leqslant \alpha < \frac{1}{2}$ and $x, y \in X$. Recently Chatterjee [4] has proved that if T_1 and T_2 be two selfmappings of a complete metric space X such that

$$d(T_1 x, T_2 y) \leqslant \alpha[d(x, T_2 y) + d(y, T_1 x)] \tag{3}$$

for all x, y in X and for some α with $0 < \alpha < \frac{1}{2}$, then T_1 and T_2 have a unique common fixed point. If we take $T_1 = T_2 = T$ in the result of Chatterjee, then as a Corollary we get the following:

If T be a selfmapping of a complete metric space X such that

$$d(Tx, Ty) \leqslant \alpha[d(x, Ty) + d(y, Tx)] \tag{4}$$

where $x, y \in X$, $0 < \alpha < \frac{1}{2}$, then T has a unique fixed point.

These results we unified in [1] where Ciric proved:

THEOREM 1. *If T be a selfmapping of a complete metric space X such that*

$$d(Tx, Ty) \leqslant \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)] \tag{5}$$

for all x, y in X and for some $\alpha, \beta, \gamma \in R_+$ with $\alpha + 2\beta + 2\gamma < 1$, then T has a unique fixed point.

We [11] have recently established the following:

Let T_1 and T_2 be two selfmappings of a complete metric space (X, d) such that

(6) $$d(T_1^p x, T_2^q y) \leqslant \alpha d(x, y) + \beta d(x, T_1^p x) + \gamma d(y, T_2^q y) + \delta[d(x, T_2^q y) + d(y, T_1^p x)]$$

for all $x, y \in X$ where $\alpha, \beta, \gamma, \delta \in R_+$ with $\alpha+\beta+\gamma+2\delta<1$ and p, q are positive integers, then T_1 and T_2 have a unique common fixed point.

If we take $T_1 = T_2 = T$ and $p = q = 1$ in (6) then as a Corollary we get the following theorem:

THEOREM 2. *Let T be a selfmapping of a metric space X (complete) such that*

(7) $$d(Tx, Ty) \leqslant \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$ and for some $\alpha, \beta, \gamma, \delta \in R_+$ with $\alpha+\beta+\gamma+2\delta<1$, then T has a unique fixed point.

We note that the theorem of Ciric follows from Theorem 2 by taking $\beta = \gamma$ in (7).

The aim of this paper is to generalize Theorem 1 in different directions. A few theorems on sequence of mappings have also been presented in this paper. Throughout this paper X will denote a complete metric space and d the metric on X.

1. Rakotch [5] proved the following result.

THEOREM 3. *Let T be a selfmapping of X such that*

(8) $$d(Tx, Ty) \leqslant \alpha\big(d(x, y)\big) d(x, y) \quad \textit{for each } x \neq y \in X,$$

where α: $(0, \infty) \to [0, 1)$ is monotonically decreasing function, then T has a unique fixed point.

We now generalize both Theorem 1 and Theorem 3.

THEOREM 4. *Let α, β, γ be monotonically decreasing functions from $(0, \infty)$ into $[0, 1)$ with $\alpha(t)+2\beta(t)+2\gamma(t)<1$, $t \in (0, \infty)$. Let T be a selfmapping of X such that the diagram of T is closed and that*

(9) $$\begin{aligned} d(Tx, Ty) \leqslant \alpha\big(d(x, y)\big) d(x, y) &+ \\ +\beta\big(d(x, y)\big)[d(x, Tx) + d(y, Ty)] &+ \\ +\gamma\big(d(x, y)\big)[d(x, Ty) + d(y, Tx)]& \end{aligned}$$

for each $x \neq y \in X$, then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary and let us consider $\{T^n x_0\}$. Suppose $T^{n-1}x_0 \neq T^n x_0$. Then for $n>1$ we have

$$\begin{aligned} d(T^n x_0, T^{n+1}x_0) \leqslant \alpha\big(d(T^{n-1}x_0, T^n x_0)\big) d(T^{n-1}x_0, T^n x_0) &+ \\ +\beta\big(d(T^{n-1}x_0, T^n x_0)\big)[d(T^{n-1}x_0, T^n x_0) + d(T^n x_0, T^{n+1}x_0)] &+ \\ +\gamma\big(d(T^{n-1}x_0, T^n x_0)\big)[d(T^{n-1}x_0, T^{n+1}x_0) + d(T^n x_0, T^n x_0)]& \end{aligned}$$

or

$$d(T^n x_0, T^{n+1}x_0) \leqslant \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right) d(T^{n-1}x_0, T^n x_0) < d(T^{n-1}x_0, T^n x_0).$$

Hence $\{d(T^n x_0, T^{n+1}x_0)\}$ decreases. Let

$$\lim_{n\to\infty} d(T^n x_0, T^{n+1}x_0) = S$$

and suppose $S>0$. Take

$$\frac{\alpha(S)+\beta(S)+\gamma(S)}{1-\beta(S)-\gamma(S)} = a.$$

Then $d(T^n x_0, T^{n+1}x_0) \geqslant S$ implies

$$\frac{\alpha\big(d(T^n x_0, T^{n+1}x_0)\big)+\beta\big(d(T^n x_0, T^{n+1}x_0)\big)+\gamma\big(d(T^n x_0, T^{n+1}x_0)\big)}{1-\beta\big(d(T^n x_0, T^{n+1}x_0)\big)-\gamma\big(d(T^n x_0, T^{n+1}x_0)\big)} \leqslant \frac{\alpha(S)+\beta(S)+\gamma(S)}{1-\beta(S)-\gamma(S)} = a \forall n.$$

Hence $d(T^n x_0, T^{n+1}x_0) \leqslant a d(T^{n-1}x_0, T^n x_0) \leqslant \ldots \leqslant a^n d(x_0, Tx_0)$ and $a^n d(x_0, Tx_0) \to 0$ as $n \to \infty$, since $a<1$. Now we intend to show that $\{T^n x_0\}$ is Cauchy. Suppose $T^{n-1}x_0 \neq T^{m-1}x_0$, then

$$\begin{aligned} d(T^n x_0, T^m x_0) \leqslant \alpha\big(d(T^{n-1}x_0, T^{m-1}x_0)\big) d(T^{n-1}x_0, T^{m-1}x_0) &+ \\ +\beta\big(d(T^{n-1}x_0, T^{m-1}x_0)\big)[d(T^{n-1}x_0, T^n x_0) + d(T^{m-1}x_0, T^m x_0)] &+ \\ +\gamma\big(d(T^{n-1}x_0, T^{m-1}x_0)\big)[d(T^{n-1}x_0, T^m x_0) + d(T^{m-1}x_0, T^n x_0)]&, \end{aligned}$$

i.e.,

$$d(T^n x_0, T^m x_0) \leqslant \left(\frac{\alpha+\beta+\gamma}{1-\alpha-2\gamma}\right) d(T^{n-1}x_0, T^n x_0) + \left(\frac{\alpha+\beta+\gamma}{1-\alpha-2\gamma}\right) d(T^{m-1}x_0, T^m x_0).$$

Let $\varepsilon>0$ be given. If $\alpha(\varepsilon)+\beta(\varepsilon)+\gamma(\varepsilon) \neq 0$, then we can find an N such that

$$d(T^{n-1}x_0, T^n x_0) < \frac{1}{2}\min\left\{\frac{\big(1-\alpha(\varepsilon)-2\gamma(\varepsilon)\big)\varepsilon}{\alpha(\varepsilon)+\beta(\varepsilon)+\gamma(\varepsilon)}, 2\varepsilon\right\}$$

and

$$d(T^{m-1}x_0, T^m x_0) < \frac{1}{2}\min\left\{\frac{\big(1-\alpha(\varepsilon)-2\gamma(\varepsilon)\big)\varepsilon}{\alpha(\varepsilon)+\beta(\varepsilon)+\gamma(\varepsilon)}, 2\varepsilon\right\}$$

for all $n, m \geqslant N$. If $\alpha(\varepsilon)+\beta(\varepsilon)+\gamma(\varepsilon) = 0$ for example, we require that for all $n, m \geqslant N$,

$$d(T^{n-1}x_0, T^n x_0) < \varepsilon \quad \text{and} \quad d(T^{m-1}x_0, T^m x_0) < \varepsilon.$$

Let us take any $n, m \geqslant N$. We wish to show that $d(T^n x_0, T^m x_0) < \varepsilon$. Assume $T^{n-1}x_0 \neq T^{m-1}x_0$. If $d(T^{n-1}x_0, T^{m-1}x_0) \geqslant \varepsilon$ then since α, β, γ are monotonically decreasing functions we have

$$d(T^n x_0, T^m x_0) \leqslant \frac{\alpha(\varepsilon)+\beta(\varepsilon)+\gamma(\varepsilon)}{1-\alpha(\varepsilon)-2\gamma(\varepsilon)} d(T^{n-1}x_0, T^n x_0) + \frac{\alpha(\varepsilon)+\beta(\varepsilon)+\gamma(\varepsilon)}{1-\alpha(\varepsilon)-2\gamma(\varepsilon)} d(T^{m-1}x_0, T^m x_0) < \varepsilon .$$

On the otherhand $d(T^{n-1}x_0, T^{m-1}x_0) < \varepsilon$ implies

$$d(T^n x_0, T^m x_0) \leqslant \alpha d(T^{n-1}x_0, T^{m-1}x_0) + \beta[d(T^{m-1}x_0, T^m x_0) + d(T^{n-1}x_0, T^n x_0)] + \gamma[d(T^{n-1}x_0, T^m x_0) + d(T^{m-1}x_0, T^n x_0)] < (\alpha+2\beta+2\gamma)\varepsilon < \varepsilon .$$

Thus $\{T^n x_0\}$ is Cauchy. Since X is a complete metric space, $\{T^n x_0\}$ converges to a point ξ in X.

Now since the diagram of T is closed, we have $\lim T^{n+1}x_0 = T\xi$. Thus ξ is a fixed point of T.

We need prove now that ξ is a unique fixed point of T. If possible let ξ, η, $\xi \neq \eta$ be two fixed point of T. Then

$$d(\xi, \eta) = d(T\xi, T\eta) \leqslant [\alpha(d(\xi, \eta)) + 2\gamma(d(\xi, \eta))] d(\xi, \eta),$$

which gives a contradiction. Hence ξ is a unique fixed point of T. This completes the proof.

We apply Theorem 4 to the following proposition which is a generalization of a result due to Nadler [9].

THEOREM 5. *Let T_n: $X \to X$ be a function with at least one fixed point a_n for each $n = 1, 2, \ldots$ and let T_0: $X \to X$ satisfy the hypotheses of Theorem 4 with the same α, β, γ. If the sequence T_n converges uniformly to T_0 then the sequence a_n converges to a_0, the unique fixed point of T_0.*

Proof. Assume $a_n \neq a_0$, then

$$\begin{aligned} d(a_n, a_0) &\leqslant d(T_n a_n, T_0 a_n) + d(T_0 a_n, T_0 a_0) \\ &\leqslant d(T_n a_n, T_0 a_n) + \alpha(d(a_n, a_0)) d(a_n, a_0) + \\ &\quad + \beta(d(a_n, a_0))[d(a_n, T_0 a_n) + d(a_0, T_0 a_0)] + \\ &\quad + \gamma(d(a_n, a_0))[d(a_n, T_0 a_0) + d(a_0, T_0 a_n)] . \end{aligned}$$

Hence we get

$$d(a_n, a_0) \leqslant \left[\frac{1+\beta(d(a_n, a_0)) + \gamma(d(a_n, a_0))}{1-\alpha(d(a_n, a_0)) - 2\gamma(d(a_n, a_0))}\right] d(T_n a_n, T_0 a_n). \tag{10}$$

Let $\varepsilon_0 > 0$ be arbitrary and choose $\varepsilon_1 > 0$ such that

$$\varepsilon_1 < \left[\frac{1-\alpha(\varepsilon_0) - 2\gamma(\varepsilon_0)}{1+\beta(\varepsilon_0)+\gamma(\varepsilon_0)}\right] \varepsilon_0 .$$

Since $\{T_n\}_{n=1}^{\infty}$ converges uniformly to T_0, so there is a K and a positive integer N such that for all $K \geqslant N$ and for all x, $d(T_K x, T_0 x) < \varepsilon_1$.

CLAIM. *For all $i \geqslant N$, $d(a_i, a_0) < \varepsilon_0$.*

Suppose not. Then there exists a $j \geqslant N$ such that

$$d(a_j, a_0) \geqslant \varepsilon_0 . \tag{11}$$

Since α, β, γ are monotonically decreasing functions, so by (11), the relation (10) gives

$$d(a_j, a_0) \leqslant \frac{1+\beta(\varepsilon_0)+\gamma(\varepsilon_0)}{1-\alpha(\varepsilon_0)-2\gamma(\varepsilon_0)} d(T_j a_j, T_0 a_j) < \varepsilon_0$$

which contradicts (10). Therefore $a_n \to a_0$.

THEOREM 6 (cf. [6]). *Let (X, d_n) be a complete metric space for each $n = 0, 1, 2, \ldots$ and suppose $\{d_n\}_{n=1}^{\infty}$ converges uniformly to d_0. Let T_n: $(X, d_n) \to (X, d_n)$ satisfy the hypotheses of Theorem 4 with the same continuous α, β, γ for all $n = 1, 2, \ldots$ and let a_n be the fixed points of T_n for $n = 1, 2, \ldots$ If a mapping T_0: $(X, d_0) \to (X, d_0)$ is defined as the d_0 —pointwise limits of T_n, then $a_n \xrightarrow[d_0]{} a_0$, the unique fixed point of T_0.*

Proof. First we shall show that T_0 satisfies (9) with respect to d_0. Now

$$\begin{aligned} d_0(T_0 x, T_0 y) &\leqslant d_0(T_0 x, T_n x) + d_0(T_n x, T_n y) + d_0(T_n y, T_0 y) \\ &\leqslant d_0(T_0 x, T_n x) + d_n(T_n x, T_n y) + \varepsilon + d_0(T_n y, T_0 y) \\ &\quad \text{(the latter inequality is valid for } n \geqslant N) \\ &\leqslant d_0(T_0 x, T_n x) + \alpha(d_n(x, y)) d_n(x, y) + \\ &\quad + \beta(d_n(x, y))[d_n(x, T_n x) + d_n(y, T_n y)] + \\ &\quad + \nu(d_n(x, y))[d_n(x, T_n y) + d_n(y, T_n x)] + \\ &\quad + d_0(T_n y, T_0 y) + \varepsilon \\ &\leqslant d_0(T_0 x, T_n x) + \alpha(d_n(x, y))[d_0(x, y) + \varepsilon] + \\ &\quad + \beta(d_n(x, y))[d_0(x, T_n x) + \varepsilon + d_0(y, T_n y) + \varepsilon] + \\ &\quad + \gamma(d_n(x, y))[d_0(x, T_n y) + \varepsilon + d_0(y, T_n x) + \varepsilon] + \\ &\quad + d_0(T_n y, T_0 y) + \varepsilon. \end{aligned}$$

As $n \to \infty$ we get

$$\begin{aligned} d_0(T_0 x, T_0 y) &\leqslant \alpha(d_0(x, y))[d_0(x, y) + \varepsilon] \\ &\quad + \beta(d_0(x, y))[d_0(x, T_0 x) + \varepsilon + d_0(y, T_0 y) + \varepsilon] \\ &\quad + \gamma(d_0(x, y))[d_0(x, T_0 y) + \varepsilon + d_0(y, T_0 x) + \varepsilon] + \varepsilon . \end{aligned}$$

Since this is true for every $\varepsilon>0$ we get

$$d_0(T_0x, T_0y)\leqslant\alpha\big(d_0(x, y)\big)d_0(x, y)+\beta\big(d_0(x, y)\big)[d_0(x, T_0x)+d_0(y, T_0y)]+ +\gamma\big(d_0(x, y)\big)[d_0(x, T_0y)+d_0(y, T_0x)] .$$

Now

$$\begin{aligned} d_0(a_n, a_0)&\leqslant d_0(T_na_n, T_na_0)+d_0(T_na_0, T_0a_0)\\ &\leqslant d_n(T_na_n, T_na_0)+\varepsilon+d_0(T_na_0, T_0a_0)\\ &\leqslant\alpha\big(d_n(a_n, a_0)\big)d_n(a_n, a_0)+\beta\big(d_n(a_n, a_0)\big)[d_n(a_n, T_na_n)+d_n(a_0, T_na_0)]+\\ &\quad+\gamma\big(d_n(a_n, a_0)\big)[d_n(a_n, T_na_0)+d_n(a_0, T_na_n)]+\varepsilon+d_0(T_na_0, T_0a_0)\\ &\qquad\qquad(\text{if } a_n\neq a_0)\\ &\leqslant\alpha\big(d_n(a_n, a_0)\big)[d_0(a_n, a_0)+\varepsilon]+\beta\big(d_n(a_n, a_0)\big)[d_0(T_na_0, T_0a_0)+\varepsilon]+\\ &\quad+\gamma\big(d_n(a_n, a_0)\big)[d_0(a_n, a_0)+2\varepsilon+d_0(T_0a_0, T_na_0)+\varepsilon]\\ &\quad+\varepsilon+d_0(T_na_0, T_0a_0) . \end{aligned}$$

Hence

$$\begin{aligned} d_0(a_n, a_0)\leqslant\alpha\varepsilon&+\beta\big(d_n(a_n, a_0)\big)[d_0(T_na_0, T_0a_0)+\varepsilon]+\\ &+\gamma\big(d_n(a_n, a_0)\big)[d_0(T_na_0, T_0a_0)+3\varepsilon]+\\ &+\frac{\varepsilon+d_0(T_na_0, T_0a_0)}{1-\alpha\big(d_n(a_n, a_0)\big)-2\gamma\big(d_n(a_n, a_0)\big)} . \end{aligned}$$

Thus for any $\varepsilon>0$ there exists an $N(\varepsilon)$ such that $n\geqslant N(\varepsilon)$ implies that

$$d_0(a_n, a_0)\leqslant\frac{3d_0(T_na_0, T_0a_0)+6\varepsilon}{1-\alpha\big(d_0(a_n, a_0)\big)-2\gamma\big(d_0(a_n, a_0)\big)} .$$

Let $\varepsilon_1>0$ be given. Take $\varepsilon=\frac{1}{12}[1-\alpha(\varepsilon_1)-2\gamma(\varepsilon_1)]\varepsilon_1$. Let n be so large so that $n\geqslant N(\varepsilon)$ and $d_0(T_na_0, T_0a_0)<\frac{1}{6}[1-\alpha(\varepsilon_1)-2\gamma(\varepsilon_1)]\varepsilon_1$. For these n, if $d_0(a_n, a_0)\geqslant\varepsilon_1$, we get $d_0(a_n, a_0)<\varepsilon_1$ and we arrive at a contradiction. This completes the proof.

2. There exists a local form of Banach's fixed point theorem [7]. Its analogue is

THEOREM 7 (Localization of Theorem 2). *Let*

$$S(x_0, r)=\{x\in X\colon\ d(x, x_0)\leqslant r\}$$

be a sphere in X and let $T\colon X\to X$ be such that for every $x, y\in S(x_0, r)$ we have

$$d(Tx, Ty)\leqslant\alpha d(x, y)+\beta d(x, Tx)+\gamma d(y, Ty)+\delta[d(x, Ty)+d(y, Tx)]$$

for some $\alpha, \beta, \gamma, \delta\in R_+$ with $\alpha+\beta+\gamma+2\delta<1$. If

(12) $$d(x_0, Tx_0)\leqslant(1-\lambda)r \quad\text{where}\quad \lambda=(\alpha+\beta+\delta)/(1-\gamma-\delta) ,$$

then T has a unique fixed point.

Proof. By (12) $x_1=Tx_0\in S(x_0, r)$. Now

$$\begin{aligned} d(x_1, x_2)&=d(Tx_0, Tx_1)\\ &\leqslant\alpha d(x_0, x_1)+\beta d(x_0, Tx_0)+\gamma d(x_1, Tx_1)+\delta[d(x_0, Tx_1)+d(x_1, Tx_0)] , \end{aligned}$$

i.e., $d(x_1, x_2)\leqslant(1-\lambda)\lambda r$. Hence $d(x_0, x_2)\leqslant(1+\lambda)(1-\lambda)r$. Suppose

$$d(x_0, x_n)\leqslant(1+\lambda+\ldots+\lambda^{n-1})(1-\lambda)r$$

and that

$$d(x_{n-1}, x_n)\leqslant\lambda^{n-1}(1-\lambda)r , \qquad x_n=Tx_{n-1},\ n=1, 2, \ldots$$

Then

$$\begin{aligned} d(x_n, Tx_n)&=d(Tx_{n-1}, Tx_n)\\ &\leqslant\alpha d(x_{n-1}, x_n)+\beta d(x_{n-1}, Tx_{n-1})+\gamma d(x_n, Tx_n)\\ &\quad+\delta[d(x_{n-1}, Tx_n)+d(x_n, Tx_{n-1})] , \end{aligned}$$

i.e., $d(x_n, x_{n+1})\leqslant(1-\lambda)\lambda^nr$. This implies

$$d(x_0, x_{n+1})\leqslant(1+\lambda+\ldots+\lambda^n)(1-\lambda)r\leqslant r .$$

Thus the sequence $x_0, x_{n+1}=Tx_n$, $n\geqslant0$, is contained in S. Again

$$d(x_n, x_m)\leqslant(\lambda^n+\lambda^{n+1}+\ldots+\lambda^{m-1})(1-\lambda)r\leqslant\lambda^nr\to0 \quad\text{as}\quad n\to\infty .$$

Since S is also complete, $\lim x_n=\xi$ for some $\xi\in S$. But

$$\begin{aligned} d(x_{n+1}, T\xi)&=d(Tx_n, T\xi)\\ &\leqslant\alpha d(x_n, \xi)+\beta d(x_n, Tx_n)+\gamma d(\xi, T\xi)+\delta[d(x_n, T\xi)+d(\xi, Tx_n)]\\ &\leqslant\alpha d(x_n, \xi)+\beta d(x_n, x_{n+1})+\gamma d(x_{n+1}, T\xi)+\delta d(x_n, x_{n+1})\\ &\quad+\delta d(x_{n+1}, T\xi)+\delta d(\xi, x_n)+\delta d(x_{n+1}, x_n)+\gamma d(\xi, x_{n+1}) . \end{aligned}$$

Hence $x_{n+1}\to T\xi$ as $n\to\infty$, i.e., ξ is a fixed point of T. Uniqueness of ξ is obvious.

3. Ciric's fixed point theorem can be extended to multivalued mappings. Let $\mathscr{F}(X)$ denote the family of all nonempty closed and bounded subsets of a given metric space X, $H(A, B)$ the Hausdorff metric [8] for $A, B\in\mathscr{F}(X)$ and let $D(x, A)=\inf\{d(x, y)\colon\ y\in A,$ where $A\in\mathscr{F}(X)\}$.

THEOREM 8. *Let $F\colon X\to X$ be a multivalued function such that the diagram of F is closed and that*

$$H\big(F(x), F(y)\big)\leqslant\alpha d(x, y)+\beta[D\big(x, F(x)\big)+D\big(y, F(y)\big)]+ +\gamma[D\big(x, F(y)\big)+D\big(y, F(x)\big)]$$

where $\alpha>0$, $\beta>0$, $\gamma>0$ with $\alpha+2\beta+2\gamma<1$. Then F has a fixed point.

Proof. Pick any $x_0 \in X$ and choose $x_1 \in F(x_0)$. If $H(F(x_0), F(x_1)) = 0$ then $F(x_0) = F(x_1)$ and hence $x_1 \in F(x_1)$, i.e., x_1 is a fixed point of F. Therefore we may assume that $H(F(x_0), F(x_1)) > 0$. By definition, if $h > H(F(x_0), F(x_1))$, there exists $x_2 \in F(x_1)$ such that $d(x_1, x_2) < h$. Let $h = \lambda_1^{-1} H(F(x_0), F(x_1))$ where $\lambda_1 = (\alpha+2\beta+2\gamma)^{1/2}$ ($\lambda_1 < 1$ and that we may assume $\lambda_1 > 0$). Then

$$\begin{aligned} d(x_1, x_2) &\leqslant \lambda_1^{-1} H(F(x_0), F(x_1)) \\ &\leqslant \lambda_1^{-1}[\alpha d(x_0, x_1) + \beta[D(x_0, F(x_0)) + D(x_1, F(x_1))] \\ &\quad + \gamma[D(x_0, F(x_1)) + D(x_1, F(x_0))]] \\ &\leqslant \lambda_1^{-1}[\alpha d(x_0, x_1) + \beta[d(x_0, x_1) + d(x_1, x_2)] + \gamma[d(x_0, x_2)]], \end{aligned}$$

i.e.,

$$d(x_1, x_2) \leqslant q d(x_0, x_1) \quad \text{where} \quad q = \lambda_1^{-1}(\alpha+\beta+\gamma)/(1-\lambda_1^{-1}\beta - \lambda_1^{-1}\gamma) < 1 .$$

Let us suppose that $(F(x_{i-1}), F(x_i)) > 0$ for $i \geqslant 2$. By induction we get $x_{i+1} \in F(x_i)$ such that

$$d(x_i, x_{i+1}) \leqslant q d(x_{i-1}, x_i) \leqslant \ldots \leqslant q^i d(x_0, x_1) .$$

Now, if $n > m$

$$\begin{aligned} d(x_n, x_m) &\leqslant (q^m + q^{m+1} + \ldots + q^{n-1}) d(x_0, x_1) \\ &\leqslant \frac{q^m}{1-q} d(x_0, x_1) \to 0 \quad \text{as } m \to \infty . \end{aligned}$$

Thus the sequence $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence and since X is complete, $\{x_i\}_{i=1}^{\infty}$ converges to $P_0 \in X$. Since the diagram of F is closed, $\lim\{F(x_i)\}_{i=1}^{\infty} = F(P_0)$. But $x_i \in F(x_{i-1})$ for all $i = 1, 2, \ldots$ Hence $P_0 \in F(P_0)$. This completes the proof.

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