On point-countable collections and monotonic properties

by

J. Chaber (Warszawa)

Abstract. A class of spaces is introduced containing both p-spaces and semimetrizable spaces. It is shown that the intersection of this class with the class of regular spaces with a point-countable base is contained in the class of monotonically developable spaces. We generalize number of theorems which assert that a semimetrizable space or a θ-refinable p-space having a certain additional property has a development. Besides, we generalize some results which assert that countably compact spaces are compact.

The aim of this paper is to give a unified method of proving theorems which assert that certain spaces are developable. Our main purpose is to find a generalization of the following two results:

A) [15] *A semimetrizable spaces with a point-countable base is developable.*

B) [14] *A paracompact p-space with a point-countable base is metrizable.*

In order to do this we introduce the class of monotonically semistrafiable spaces and the class of monotonic β-spaces. These classes do not seem to be as important as the other classes of monotonic spaces (see [11]) but they allow us to find a joint background of various characterizations of developable spaces.

We shall use the terminology and notation from [13] and [11]. All spaces are assumed to be regular. If $\mathfrak{M}$ and $\mathfrak{N}$ are families of subsets of a certain space X, then $\delta\mathfrak{M} < \mathfrak{N}$ denotes that each element of $\mathfrak{N}$ contains an element of $\mathfrak{M}$. If $\mathfrak{H}$ is a well-ordered cover of X and $x \in X$, then $x(\mathfrak{H})$ denotes the first element of $\mathfrak{H}$ which contains x.

1. Preliminaries. Let us recall that a property $\mathfrak{P}$ of sequences of sets is said to be *monotonic* if the following condition is satisfied:

$$\{H_m\}_{m=1}^{\infty} \in \mathfrak{P} \text{ and } \delta\{W_n\}_{n=1}^{\infty} < \{H_m\}_{m=1}^{\infty} \text{ implies } \{W_n\}_{n=1}^{\infty} \in \mathfrak{P}.$$

In [11] the concept of a sieve was introduced in order to simplify the generalized base of countable order theory.

DEFINITION 1.1. A sequence $\mathfrak{G} = \{\langle \mathfrak{G}_n, A_n, \pi_n \rangle\}_{n=1}^{\infty}$ will be called a *sieve* of X if, for an arbitrary n, $\mathfrak{G}_n = \{G(\alpha)\}_{\alpha \in A_n}$ is an open covering of X and π_n: $A_{n+1} \to A_n$ is such that if $\alpha \in A_n$, then $G(\alpha) = \bigcup \{G(\alpha') \colon \pi_n(\alpha') = \alpha\}$.

A sequence $\{G(\alpha_n)\}_{n=1}^{\infty}$, where $\alpha_n \in A_n$ and $\pi_n(\alpha_{n+1}) = \alpha_n$, will be called a *thread* of $\mathfrak{G}$. If each thread of $\mathfrak{G}$ satisfies a monotonic property $(\mathfrak{m})$, then $\mathfrak{G}$ will be called an $(\mathfrak{m})$-*sieve*.

It is easy to see that the proof of Lemma 1.1 from [11] can be used in order to prove

LEMMA 1.2. *Let* $(\mathfrak{m})$ *be a monotonic property. The following conditions are equivalent for an arbitrary space* X:

(a) X *has an* $(\mathfrak{m})$-*sieve*,

(b) *if, for each* $x \in X$, $\{\mathfrak{B}_n(x)\}_{n=1}^{\infty}$ *is a (decreasing) sequence of bases of* X *at the point* x, *then, for each* $x \in X$, *there exists a (decreasing) sequence* $\{\mathfrak{W}_n(x)\}_{n=1}^{\infty}$ *of bases at* x *such that* $\mathfrak{W}_n(x) \subseteq \mathfrak{B}_n(x)$ *and* $\{\bigcup \{\mathfrak{W}_n(x)\colon x \in X\}\}_{n=1}^{\infty}$ *is an* $(\mathfrak{m})$-*sequence of bases of* X.

We shall define the class of monotonically semistratifiable spaces.

DEFINITION 1.3. A space X is said to be *monotonically semistratifiable* if, for each point $x \in X$, there exists a decreasing sequence $\{\mathfrak{B}_n(x)\}_{n=1}^{\infty}$ of bases of X at the point x such that if $B_n \in \mathfrak{B}_n(x_n)$, $B_{n+1} \subseteq B_n$, for all $n \in N$, and $x \in \bigcap_{n=1}^{\infty} B_n$ then x is a limit point of the sequence $\{x_n\}_{n=1}^{\infty}$. The family $\{\{\mathfrak{B}_n(x)\}_{n=1}^{\infty}\colon x \in X\}$ will be called a *monotonic semistratification of* X.

Note that if, in the above definition, we do not assume that the sequence $\{B_n\}_{n=1}^{\infty}$ is decreasing, then we get a characterization of semistratifiable spaces [12]. The following two easy propositions show that monotonic semistratifiability is, in fact, a monotonic equivalent of semistratifiability.

PROPOSITION 1.4. *A space* X *is monotonically semistratifiable if and only if, for each closed subset* F *of* X, *there exists a decreasing* (W)-*sequence* $\{\mathfrak{B}_n(F)\}_{n=1}^{\infty}$ *of bases of* F *in* X (¹) *such that* $F \subseteq F'$ *implies* $\mathfrak{B}_n(F) \subseteq \mathfrak{B}(F')$ *for* $n \in N$.

PROPOSITION 1.5. *Monotonically developable spaces are monotonically semistratifiable.*

An example of a semimetrizable nondevelopable space [18, Example 3.1] shows (see, for example, [11, Theorems 2.8 and 4.2(d)]) that there exist monotonically semistratifiable spaces which are not p-spaces.

In order to obtain a class of spaces which contains all semimetrizable spaces and all p-spaces, we introduce the class of monotonic β-spaces.

DEFINITION 1.6. A space X is said to be a *monotonic* β-*space* if, for each point $x \in X$, there exists a decreasing sequence $\{\mathfrak{B}_n(x)\}_{n=1}^{\infty}$ of bases of X at the point x such that if $B_n \in \mathfrak{B}_n(x_n)$, $B_{n+1} \subseteq B_n$ and $\bigcap_{n=1}^{\infty} B_n$ is non-empty, then the sequence $\{x_n\}_{n=1}^{\infty}$ has a cluster point. The family $\{\{\mathfrak{B}_n(x)\}_{n=1}^{\infty}\colon x \in X\}$ will be called a *monotonic* β-*system of* X.

(¹) Let us recall that a sequence $\{\mathfrak{B}_n(F)\}_{n=1}^{\infty}$ of bases of F in X is called a (W)-*sequence* if each decreasing sequence $\{B_n\}_{n=1}^{\infty}$ such that $B_n \in \mathfrak{B}_n(F)$ satisfies the condition $\bigcap_{n=1}^{\infty} B_n \subseteq F$.

The class of monotonic β-spaces is a monotonic equivalent of the class of β-spaces [16] (again, dropping the assumption that $B_{n+1} \subseteq B_n$, we obtain a characterization of β-spaces).

It is easy to see that the following proposition holds.

PROPOSITION 1.7. *Each monotonic* p-*space is a monotonic* β-*space.*

Let us recall that a monotonic p-space is monotonically developable if and only if it has a W_δ-diagonal. Using Lemma 1.2, we obtain

PROPOSITION 1.8. *A monotonic* β-*space is monotonically semistratifiable if and only if it has a* W_δ-*diagonal.*

2. Non-complete monotonic properties and point-countable collections. A monotonic property $(\mathfrak{m})$ will be called *non-complete* if it contains all sequences with the empty intersection.

We shall consider the following non-complete monotonic properties of sequences of subsets of a space X:

(d) $\bigcap_{n=1}^{\infty} B_n \ni x$, then $\{B_n\}_{n=1}^{\infty}$ is a base at x,

(Δ) $\bigcap_{n=1}^{\infty} B_n \ni x$, then $\bigcap_{n=1}^{\infty} B_n = \{x\}$,

(p) $\bigcap_{n=1}^{\infty} B_n \ni x$, then, for each centred family $\mathfrak{A}$, $\delta\mathfrak{A} < \{B_n\}_{n=1}^{\infty}$ implies $\bigcap \{\bar{A}\colon A \in \mathfrak{A}\} \neq \emptyset$,

(cp) $\bigcap_{n=1}^{\infty} B_n \ni x$, then, for each centred and countable family $\mathfrak{A}$, $\delta\mathfrak{A} < \{B_n\}_{n=1}^{\infty}$ implies $\bigcap \{\bar{A}\colon A \in \mathfrak{A}\} \neq \emptyset$.

DEFINITION 2.1. A sequence $\{\mathfrak{H}_j\}_{j=1}^{\infty}$ of well-ordered open covers of a space X is said to be an $(\mathfrak{m})$-*sequence of ordered covers* if, for each $x \in X$, the sequence $\{x(\mathfrak{H}_j)\}_{j=1}^{\infty}$ satisfies $(\mathfrak{m})$ (²).

It is easy to observe that the concept of a primitive base [25] is equivalent to the concept of a (d)-sequence of ordered covers. Hence, according to [25], if a space X has a (d)-sequence of ordered covers and closed subsets of X are W_δ-subsets, then X is monotonically developable (³). Since, obviously, each quasi-developable space [6] has a (d)-sequence of ordered covers, it follows that we may regard the concept of a primitive base ((Δ)-sequence of ordered covers) as a monotonic equivalent of the concept of a quasi-development (quasi-G_δ-diagonal [17]).

(²) $(\mathfrak{m})$ will always denote a monotonic property.

(³) It can be shown that if $(\mathfrak{m})$ is a non-complete monotonic property, X has an $(\mathfrak{m})$-sequence of ordered covers and closed subsets of X are W_δ-subsets, then X has an $(\mathfrak{m})$-sieve.

The following theorem shows that, in the result announced in [25], the assumption that closed subsets of X are W_δ-subsets can be replaced by the assumption that X is a monotonic β-space (compare with [22]).

THEOREM 2.2. *Let* $(\mathfrak{m})$ *be a non-complete monotonic property. A monotonic* β*-space* X *has an* $(\mathfrak{m})$*-sieve if and only if* X *has an* $(\mathfrak{m})$*-sequence of ordered covers.*

The "only if" part follows from

LEMMA 2.3. *If* $\mathfrak{G} = \{\langle \mathfrak{G}_n, A_n, \pi_n\rangle\}_{n=1}^{\infty}$ *is a sieve of* X*, then, for each* $n\in N$*, there exists a well-ordering of* $\mathfrak{G}_n$ *such that* $\{x(\mathfrak{G}_n)\}_{n=1}^{\infty}$ *is a thread of* $\mathfrak{G}$ *for* $x\in X$.

One can prove Lemma 2.3 by using a method exhibited in [21] (see also the proof of Lemma 1.1 in [11]).

Proof of Theorem 2.2. Let $\{\{\mathfrak{B}_n(x)\}_{n=1}^{\infty}: x\in X\}$ be a monotonic β-system of X and let $\{\mathfrak{H}_j\}_{n=1}^{\infty}$ be an $(\mathfrak{m})$-sequence of ordered covers of X. One can construct a sieve $\mathfrak{G} = \{\langle \mathfrak{G}_n, A_n, \pi_n\rangle\}_{n=1}^{\infty}$ of X such that $A_n\subseteq X^n$, π_n is the restriction of the projection of X^{n+1} onto X^n to the set A_{n+1} and

(i) $(x_1, \ldots, x_n)\in A_n$, then $G(x_1, \ldots, x_n)\in \mathfrak{B}_n(x_n)$,

(ii) $(x_1, \ldots, x_n)\in A_n$, then $\overline{G(x_1, \ldots, x_n)}\subseteq \bigcap\{x_n(\mathfrak{H}_j): j\leqslant n\}$.

We shall show that $\mathfrak{G}$ is an $(\mathfrak{m})$-sieve. Let $\{G(\alpha_n)\}_{n=1}^{\infty}$ be a thread of $\mathfrak{G}$ such that $\bigcap_{n=1}^{\infty} G(\alpha_n) \neq \emptyset$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements of X such that $\alpha_n = (x_1, \ldots, x_n)$ for $n\in N$. From condition (i) we infer that $\{x_n\}_{n=1}^{\infty}$ has a cluster point $y\in \bigcap_{n=1}^{\infty} \overline{G(\alpha_n)}$.

Since $(\mathfrak{m})$ is a monotonic property, it suffices to show that $\delta\{G(\alpha_n)\}_{n=1}^{\infty} < \{y(\mathfrak{H}_j)\}_{j=1}^{\infty}$.

Let $j\in N$. From the fact that $y(\mathfrak{H}_j)$ is a neighbourhood of y it follows that, for a certain $n\geqslant j$, $x_n\in y(\mathfrak{H}_j)$. On the other hand, (ii) implies that $y\in \overline{G(\alpha_n)}\subseteq x_n(\mathfrak{H}_j)$. Hence $G(\alpha_n)\subseteq x_n(\mathfrak{H}_j) = y(\mathfrak{H}_j)$.

COROLLARY 2.4 (compare with [7, Theorem 3.1]). *A space* X *is monotonically developable if and only if* X *is a monotonic* β*-space with a* (d)*-sequence of ordered covers.*

COROLLARY 2.5 (compare with [17, Theorem 3.2]). *A monotonic* β*-space has a* W_δ*-diagonal if and only if* X *has a* (Δ)*-sequence of order covers.*

COROLLARY 2.6. *A space* X *is a monotonic* p*-space if and only if* X *is a monotonic* β*-space with a* (p)*-sequence of ordered covers.*

DEFINITION 2.7. A point-countable open cover $\mathfrak{U}$ of a space X is said to be an $(\mathfrak{m})$*-cover* if, for each $x\in X$, the family $\mathfrak{U}(x) = \{U\in\mathfrak{U}: x\in U\}$ satisfies $(\mathfrak{m})$.

THEOREM 2.8. *Let* $(\mathfrak{m})$ *be a non-complete monotonic property. If a monotonic* β*-space* X *has a point-countable* $(\mathfrak{m})$*-cover, then* X *has an* $(\mathfrak{m})$*-sieve.*

Proof (4). Let $\mathfrak{U}$ be a point-countable $(\mathfrak{m})$-cover of X and, for each $x\in X$,

(4) We use an idea from [15].

let $\mathfrak{U}(x) = \{U_i(x): i\in N\}$ be a fixed enumeration of $\mathfrak{U}(x)$. Let $\{\{\mathfrak{B}_n(x)\}_{n=1}^{\infty}: x\in X\}$ be a monotonic β-system of X.

As in the proof of Theorem 2.2, we construct a sieve $\mathfrak{G} = \{\langle \mathfrak{G}_n, A_n, \pi_n\rangle\}_{n=1}^{\infty}$ of X such that $A_n\subseteq X^n$, π_n is the restriction of the projection of X^{n+1} onto X^n and

(i) $(x_1, \ldots, x_n)\in A_n$, then $G(x_1, \ldots, x_n)\in \mathfrak{B}_n(x_n)$,

(ii) $(x_1, \ldots, x_n)\in A_n$, then $G(x_1, \ldots, x_n)\subseteq \bigcap\{U_i(x_k): i, k\leqslant n$ and $x_n\in U_i(x_k)\}$.

Let $\{G(\alpha_n)\}_{n=1}^{\infty}$ be a thread of $\mathfrak{G}$ such that $\bigcap_{n=1}^{\infty} G(\alpha_n)\neq\emptyset$ and let $\{x_n\}_{n=1}^{\infty}$ be such that $\alpha_n = (x_1, \ldots, x_n)$. As in the proof of Theorem 2.2, $\{x_n\}_{n=1}^{\infty}$ has a cluster point $y\in X$.

Since $(\mathfrak{m})$ is a monotonic property, it suffices to show that $\delta\{G(\alpha_n)\}_{n=1}^{\infty}\geqslant\mathfrak{U}(y)$.

Let $U\in\mathfrak{U}(y)$. From the fact that y is a cluster point of $\{x_n\}_{n=1}^{\infty}$ it follows that, for a certain $k\in N$, $x_k\in U$. Hence there exists an $i\in N$ such that $U = U_i(x_k)$. Let $n\geqslant i+k$ be such that $x_n\in U$. From condition (ii) we infer that $G(\alpha_n)\subseteq U$.

COROLLARY 2.9 (compare with [15], [14], [8, Theorem 2.10] and [10, Theorem 2.7]). *A monotonic* β*-space with a point-countable base is monotonically developable.*

COROLLARY 2.10 (compare with [20], [16, Theorem 3.6] and [11, Theorem 4.2 (b)]). *A monotonic* β*-space with a point-countable separating open cover has a* W_δ*-diagonal. Hence a monotonic* p*-space with a point-countable separating open cover is monotonically developable.*

The following concept generalizes the concept of a point-countable $(\mathfrak{m})$-cover.

DEFINITION 2.11. A sequence $\{\mathfrak{U}_j\}_{j=1}^{\infty}$ of open covers of a space X is said to be a σ*-distributively point-countable* $(\mathfrak{m})$*-sequence* if, for each $x\in X$, the family $\mathfrak{U}(x) = \{U\in\mathfrak{U}_j: j\in N(x)$ and $x\in U\}$, where $N(x) = \{j\in N: |\{U\in\mathfrak{U}_j: x\in U\}|\leqslant\aleph_0\}$, satisfies $(\mathfrak{m})$.

Spaces with a σ-distributively point-countable (d)-sequence of covers are defined in [3] as spaces with a $\delta\theta$-base. In [4] it is shown that a semistratifiable space with a $\delta\theta$-base is developable. The following, more general, theorem holds.

THEOREM 2.12. *Let* $(\mathfrak{m})$ *be a non-complete monotonic property. If a monotonically semistratifiable space* X *has a* σ*-distributively point-countable* $(\mathfrak{m})$*-sequence of covers, then* X *has an* $(\mathfrak{m})$*-sieve.*

Proof. By virtue of Theorem 2.2, it suffices to show that X has an $(\mathfrak{m})$-sequence of ordered covers.

Let $\{\mathfrak{U}_j\}_{j=1}^{\infty}$ be a σ-distributively point-countable $(\mathfrak{m})$-sequence of covers of X and let $\{\{\mathfrak{B}_n(x)\}_{n=1}^{\infty}: x\in X\}$ be a monotonic semistratification of X.

As in the proof of Theorem 2.8, for each $x\in X$, we fix an enumeration $\mathfrak{U}(x) = \{U_i(x): i\in N\}$ of $\mathfrak{U}(x)$.

Let $X_j = \{x\in X: j\in N(x)\}$ and let $\mathfrak{G}_j = \{\langle \mathfrak{G}_{n,j}, A_{n,j}, \pi_{n,j}\rangle\}_{n=1}^{\infty}$ be a sieve of X_j in X which satisfies the same conditions as the sieve constructed in the proof of Theorem 2.8 ($A_{n,j}\subseteq (X_j)^n$).

By virtue of Lemma 2.3, for each $n \in N$, there exists a well-ordering $\leqslant_{n,j}$ of $\mathfrak{G}_{n,j}$ such that $\{x(\mathfrak{G}_{n,j})\}_{n=1}^{\infty}$ is a thread of $\mathfrak{G}_j$ for $x \in X_j$.

Hence, the reasoning used in the proof of Theorem 2.8 shows that, for each $x \in X$, the family $\{x(\mathfrak{G}_{n,j})\colon n \in N \text{ and } j \in N(x)\}$ satisfies $(\mathfrak{m})$. Therefore the family $\{\mathfrak{G}'_{n,j}\}_{n,j=1}^{\infty}$, where $\mathfrak{G}'_{n,j} = \mathfrak{G}_{n,j} \cup \{X\}$ ($\mathfrak{G}_{n,j}$ is well-ordered by $\leqslant_{n,j}$ and X is added as the last element of $\mathfrak{G}'_{n,j}$) is a countable $(\mathfrak{m})$-family of ordered covers of X.

COROLLARY 2.13. *A monotonically semistratifiable space X with a σ-distributively point-countable* (p)*-sequence of covers is monotonically developable.*

If X is a first countable space, then the assumption that X is monotonically semistratifiable can be replaced by the assumption that X is a monotonic β-space. In order to prove this result, we need the following simple lemma.

LEMMA 2.14 ([5]). *If $\mathfrak{U}$ is an open cover of a first countable space X, then $\{x \in X\colon |\{U \in \mathfrak{U}\colon x \in U\}| \leqslant \aleph_0$ is closed in X.*

THEOREM 2.15. *Let* $(\mathfrak{m})$ *be a non-complete monotonic property. If a first countable monotonic β-space X has a σ-distributively point-countable* $(\mathfrak{m})$*-sequence of covers, then X has an* (m)*-sieve.*

Proof. Let $\{\mathfrak{U}_j\}_{j=1}^{\infty}$ be a σ-distributively point-countable $(\mathfrak{m})$-sequence of covers of X. As before, we fix an enumeration $\mathfrak{U}(x) = \{U_i(x)\colon i \in N\}$. Let $\{\{\mathfrak{B}_n(x)\}_{n=1}^{\infty}\colon x \in X\}$ be a monotonic β-system of X.

From Lemma 2.14 it follows that the sets $X_j = \{x \in X\colon j \in N(x)\}$ are closed in X. Hence we can construct a sieve $G = \{\langle G_n, A_n, \pi_n\rangle\}_{n=1}^{\infty}$ of X which satisfies the same conditions as the sieve constructed in the proof of Theorem 2.8 and

(iii) $(x_1, \ldots, x_n) \in A_n$, then $\overline{G(x_1, \ldots, x_n)} \subseteq \bigcap \{X \setminus X_j\colon j \leqslant n \text{ and } x_n \notin X_j\}$.

Let $\{G(\alpha_n)\}_{n=1}^{\infty}$ be a thread of G such that $\alpha_n = (x_1, \ldots, x_n)$ and $\bigcap_{n=1}^{\infty} G(\alpha_n) \neq \emptyset$ and let $y \in \bigcap_{n=1}^{\infty} \overline{G(\alpha_n)}$ be a cluster point of the sequence $\{x_n\}_{n=1}^{\infty}$. We have to show that $\delta\{G(\alpha_n)\}_{n=1}^{\infty} < \mathfrak{U}(y)$.

Let $U \in \mathfrak{U}(y)$ and let $j \in N(y)$ be such that $U \in \mathfrak{U}_j$. From condition (iii) we infer that $x_n \in X_j$ for $n \geqslant j$. Hence, the reasoning used in the proof of Theorem 2.8 shows that there exists an $n \geqslant j$ such that $G(\alpha_n) \subseteq U$.

COROLLARY 2.16. *A monotonic β-space with a $\delta\theta$-base is monotonically developable.*

COROLLARY 2.17. *A monotonic p-space with a σ-distributively point-countable* (Δ)*-sequence of covers is monotonically developable. Hence a paracompact p-space X is metrizable if and only if X has a σ-distributively point-countable* (Δ)*-sequence of covers* ([6]).

([5]) This lemma was suggested to the author by K. Alster.

([6]) Corollaries 2.16 and 2.17 give positive answers to Questions 2 and 3 from [5].

Corollary 2.17 gives a new metrization theorem for paracompact p-spaces We shall show that a similar theorem holds for the class of M-spaces [19]. This is a consequence of the following result:

THEOREM 2.18 (cf. [1]). *A countably compact space X with a σ-distributively point-countable* (Δ)*-sequence of covers is compact.*

Proof. Let $\{\mathfrak{U}_j\}_{j=1}^{\infty}$ be a σ-distributively point-countable (Δ)-sequence of covers of X. Assume that X is not compact and let $\mathfrak{B}$ be an open cover of X such that no countable subcollection of $\mathfrak{B}$ covers X.

By virtue of Lemma 2.14, each set $X_j = \{x \in X\colon j \in N(x)\}$ is closed and, obviously, $\{X_j\}_{j=1}^{\infty}$ is a covering of X.

We shall construct sequences

(i) $0 < j_1 < j_2 < \ldots < j_m < \ldots$ of natural numbers,

(ii) $X \supseteq F_1 \supseteq F_2 \supseteq \ldots \supseteq F_m \supseteq \ldots$ of closed subsets of X, such that

(iii) j_{m+1} is the smallest natural number which is greater than j_m and is such that $F_m \cap X_{j_{m+1}}$ cannot be covered by any countable subcollection of $\mathfrak{B}$,

(iv) $F_m = F_{m-1} \cap X_{j_m} \setminus \bigcup \mathfrak{B}'$, where $\mathfrak{B}' \subseteq \mathfrak{B}$ is a countable cover of $\bigcup \{F_{m-1} \cap X_j\colon j_{m-1} < j < j_m\}$,

(v) $F_m \subseteq \bigcap_{k=1}^{m} X_{j_k} \setminus \bigcup \{X_j\colon j \leqslant j_m \text{ and } j \notin \{j_1, \ldots, j_m\}\}$,

(vi) F_m cannot be covered by any countable subcollection of $\mathfrak{B}$.

We can construct sequences (i) and (ii) by induction. We define $F_0 = X$. Using the assumption that X cannot be covered by any countable subcollection of $\mathfrak{B}$, we can define j_1 as the smallest natural number j such that $F_0 \cap X_j$ cannot be covered by any countable subcollection of $\mathfrak{B}$.

Having F_{m-1} and j_m, for $m \geqslant 1$, we use (iv) as the definition of F_m. In order to define j_{m+1} satisfying (iii) we have to show that, for a certain $j > j_m$, $F_m \cap X_j$ cannot be covered by any countable subcollection of $\mathfrak{B}$.

Assume that there exists a countable subcollection $\mathfrak{B}'$ of $\mathfrak{B}$ which covers $\bigcup \{F_m \cap X_j\colon j > j_m\}$. Let $F = F_m \setminus \bigcup \mathfrak{B}'$ and $M = \{j_1, \ldots, j_m\} \subseteq N$. From (v) and (vi) we infer that

(*) F cannot be covered by any countable subcollection of $\mathfrak{B}$ and $x \in F$ implies $N(x) = M$.

Hence F is a countably compact non-compact space and $\mathfrak{U} = \{U \cap F\colon U \in \mathfrak{U}_j \text{ and } j \in M\}$ is a point-countable separating open cover of F. This is a contradiction [1].

We have shown that condition (iii) and (iv) define sequences (i) and (ii) satisfying (v) and (vi).

Let $F = \bigcap_{m=1}^{\infty} F_m$ and $M = \{j_m\colon m \in N\}$. From condition (vi) and countable compactness of X it follows that F cannot be covered by any countable subcollection of $\mathfrak{B}$. Using (v) we infer that F and M satisfy (*). The contradiction shows that X is compact.

COROLLARY 2.19. *An M-space X is metrizable if and only if X has a σ-distributively point-countable* (Δ)*-sequence of covers.*

Condition (Δ) in Theorem 2.18 can be replaced by the condition

(p′) $\bigcap_{n=1}^{\infty} B_n \ni x$, then, for each centred family $\mathfrak{A}$, $\delta\mathfrak{A} < \{\bar{B}_n\}_{n=1}^{\infty}$ implies $\bigcap\{\bar{A}\colon A \in \mathfrak{A}\} \neq \emptyset$.

THEOREM 2.20. *A countably compact space X with a σ-distributively point-countable* (p′)*-sequence of covers is compact.*

It is easy to observe that Theorem 2.20 follows from

LEMMA 2.21 ([24, Theorem (iv)]). *If $\mathfrak{U} = \bigcup_{j=1}^{\infty} \mathfrak{U}_j$ is an open cover of a countably compact space X and, for each $x \in X$, there exists $j \in N$ such that $x \in \bigcup \mathfrak{U}_j$ and $|\{U \in \mathfrak{U}_j\colon x \in U\}| \leqslant \aleph_0$, then X can be covered by a countable subcollection of $\mathfrak{U}$.*

Lemma 2.21 is announced in [24, Theorem (iv)]. One can reduce Lemm 2.21 to a result from [2] using the construction exhibited in the proof of Theorem 2.18.

Let us finish this section with some remarks.

Remark 2.22. If F is a closed subset of a space X, then F is a W_δ-subset of X if and only if X has an $(\mathfrak{m})$-sieve, where $(\mathfrak{m})$ is the following non-complete monotonic property:

$$\{B_n\}_{n=1}^{\infty} \in (\mathfrak{m}) \text{ iff } \bigcap_{n=1}^{\infty} B_n \subseteq F \text{ or, for a certain } n,\ B_n \cap F = \emptyset.$$

Hence we can apply the results of this section in order to show that a closed subset F of X is a W_δ-subset of X.

Remark 2.23. We have used the terms monotonically semistratifiable and a monotonic β-space. This terminology is not fully justified, for we do not know whether the class of monotonically semistratifiable spaces (monotonic β-spaces) can be defined with the use of a monotonic property. Consequently, we cannot apply the results from [11] to these classes. In particular, we do not know whether a (closed) subset of a monotonically semistratifiable space (a monotonic β-space) is a monotonically semistratifiable space (a monotonic β-space). One can prove, by using a method from [24, Theorem 1], that these properties are hereditary with respect to open subsets. Furthermore, we do not know whether the class of semistratifiable spaces is equal to the class of subparacompact monotonically semistratifiable spaces.

3. Paracompactness and monotonic properties. The results proved in the previous section show that various conditions imply the existence of $(\mathfrak{m})$-sieves (for non-complete properties $(\mathfrak{m})$). It is easy to see that, in order to obtain similar results for arbitrary monotonic properties, it suffices to replace the assumption that X is a monotonic β-space by the assumption that X has a monotonic β-system $\{\{\mathfrak{B}_n(x)\}_{n=1}^{\infty}\colon x \in X\}$ such that $B_n \in \mathfrak{B}_n(x_n)$ and $B_{n+1} \subseteq B_n$, for $n \in N$, implies $\bigcap_{n=1}^{\infty} \bar{B}_n \neq \emptyset$ (7).

The purpose of this section is to discuss conditions which allow us to convert certain monotonic structures into corresponding non-monotonic structures.

For non-complete monotonic properties we have

THEOREM 3.1 (see [11, Theorem 2.8] and [23, Theorem 3.1]). *Let $(\mathfrak{m})$ be a non-complete monotonic property. If a θ-refinable space X has an $(\mathfrak{m})$-sieve, then X has a sequence $\{\mathfrak{W}_n\}_{n=1}^{\infty}$ of open covers such that $W_n \in \mathfrak{W}_n$, for $n \in N$, implies that $\{W_n\}_{n=1}^{\infty}$ has the property* $(\mathfrak{m})$.

Example 2.9 from [11] shows that the assumption that $(\mathfrak{m})$ is non-complete cannot be dropped even if X is assumed to be a metacompact Moore space. On the other hand, for paracompact spaces, we get

THEOREM 3.2 (compare with [23, Corollary 2.9 and Theorem 3.1]). *Let $(\mathfrak{m})$ be a monotonic property. If a paracompact space X has an $(\mathfrak{m})$-sieve, then X has a sequence $\{\mathfrak{W}\}_{n=1}^{\infty}$ of open covers such that $W_n \in \mathfrak{W}_n$ and $W_n \cap W_m \neq \emptyset$, for $n, m \in N$, implies that $\{W_n\}_{n=1}^{\infty}$ has the property* $(\mathfrak{m})$.

Proof. By virtue of Lemma 1.1 from [11], X has a sequence $\{\mathfrak{B}_n\}_{n=1}^{\infty}$ of bases such that $B_n \in \mathfrak{B}_n$ and $B_{n+1} \subseteq B_n$, for $n \in N$, implies that $\{B_n\}_{n=1}^{\infty}$ has property $(\mathfrak{m})$.

We can define by induction a sequence $\{\mathfrak{W}_n\}_{n=1}^{\infty}$ of open covers of X such that each element of $\mathfrak{W}_{n+1}$ intersects only a finite number of elements of $\mathfrak{W}_n$ and $\mathfrak{W}_{n+1}$ refines $\{B \in \mathfrak{B}_n\colon B \subseteq W$ for a certain $W \in \mathfrak{W}_n\}$.

Assume that $W_n \in \mathfrak{W}_n$ and $W_n \cap W_m \neq \emptyset$ for $n, m \in N$. It is easy to observe that the collections $\mathfrak{W}_n' = \{W \in \mathfrak{W}_n\colon \delta\{W_m\}_{m=1}^{\infty} < \{W\}\}$ are finite. Hence, for each $n \in N$, there exists a finite subcollection $\mathfrak{B}_n'$ of $\mathfrak{B}_n$ such that $\mathfrak{W}_{n+1}'$ refines $\mathfrak{B}_n'$ and $\mathfrak{B}_n'$ refines $\mathfrak{W}_n'$. Therefore, we can find (see, for example, [11, Lemma 1.4]) a decreasing sequence $\{B_n\}_{n=1}^{\infty}$ such that $B_n \in \mathfrak{B}_n'$ and each B_n contains elements of the sequence $\{W_n\}_{n=1}^{\infty}$. This implies that $\delta\{W_n\}_{n=1}^{\infty} < \{B_n\}_{n=1}^{\infty}$ and, consequently, $\{W_n\}_{n=1}^{\infty}$ has property $(\mathfrak{m})$.

The proof of Theorem 3.2 is similar to the proof of Theorem 3.1 (see [11, footnote 7]). This suggests that the assumption that X is paracompact can be replaced by the assumption that X satisfies the condition

(P) for each open cover $\mathfrak{U}$ of X there exist a sequence $\{\mathfrak{U}_n\}_{n=1}^{\infty}$ of open covers of X and an open cover $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$ of X such that each $\mathfrak{U}_n$ refines $\mathfrak{U}$ and each element of $\mathfrak{B}_n$ intersects only a finite number of elements of $\mathfrak{U}_n$.

Property (P) is related to paracompactness in the same way as θ-refinability is related to metacompactness. It is known [9] that there exist normal θ-refinable spaces which are not metacompact. On the other hand, we have

(7) Weakly complete semimetrizable spaces [18] and monotonically Čech complete spaces have monotonic β-systems which satisfy this condition.

THEOREM 3.3. *A space X is paracompact if and only if X satisfies condition* (P) ([8]).

Proof. Assume that X satisfies condition (P). It is easy to observe that X is a normal space. We shall prove that X is countably paracompact.

Let $\mathfrak{G} = \{G_m\}_{m=1}^{\infty}$ be a countable open cover of X. Using (P), we can find $\{\mathfrak{U}_n\}_{n=1}^{\infty}$ and $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$ which satisfy (P) with respect to $\mathfrak{G}$.

Let

$$E_{n,m} = X \setminus \bigcup \{U \in \mathfrak{U}_n \colon\ U \not\subseteq G_k \text{ for } k \leqslant m\}.$$

It is easy to check that $E_{n,m} \subseteq \bigcup_{k=1}^{m} G_k$ and, for each $V \in \mathfrak{B}_n$, there exists an $m \in N$ such that $V \subseteq E_{n,m}$. Hence $\{W_m\}_{m=1}^{\infty}$, where $W_m = G_m \setminus \bigcap \{E_{n,k} \colon n,\ k < m\}$, is a locally finite refinement of $\mathfrak{G}$.

Now, we are able to prove that X is paracompact. Let $\mathfrak{U}$ be an open cover of X and that $\{\mathfrak{U}_n\}_{n=1}^{\infty}$ and $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$ satisfy (P) with respect to $\mathfrak{U}$. Using normality and countable paracompactness of X, we can find a locally finite cover $\{U_n\}_{n=1}^{\infty}$ of X such that $\overline{U}_n \subseteq \bigcup \mathfrak{B}_n$ for $n \in N$. It is easy to check that $\mathfrak{H} = \{U \cap U_n \colon U \in \mathfrak{U}_n$ and $n \in N\}$ is a locally finite refinement of $\mathfrak{U}$.

References

[1] A. V. Arhangel'skiĭ and V. V. Proizvolov, *On the relation between point powers of systems of sets and the weight of a bicompact Hausdorff space*, Vestnik Moskov. Univ. Ser. I Math. Meh. 21 (1966), pp. 75–77 (Russian).
[2] G. Aquaro, *Point-countable open coverings in countably compact spaces*, General Topology and Its Relations to Modern Analysis and Algebra II, Academia, Prague (1966), pp. 39–41.
[3] C. E. Aull, *Quasi-developments and $\delta\theta$-bases*, preprint.
[4] — *A semistratifiable space is developable iff it has a $\delta\theta$-base*, Notices Amer. Math. Soc. 21 (1974), p. A-504.
[5] — *Some properties involving base axioms and metrizability* (*conversion of covering properties into base properties*), TOPO 72, (1974), pp. 41–45.
[6] H. R. Bennett, *On quasi-developable spaces*, Gen. Topology Appl. 1 (1971), pp. 253–262.
[7] — and E. S. Berney, *On certain generalizations of developable spaces*, Gen. Topology Appl. 4 (1974), pp. 43–50.
[8] D. K. Burke, *On p-spaces and WΔ-spaces*, Pacific. J. Math. 35 (1970), pp. 285–296.
[9] — *A note on R. H. Bing's example G*, preprint.
[10] — and R. A. Stoltenberg, *A note on p-spaces and Moore spaces*, Pacific J. Math. 30 (1969), pp. 601–607.
[11] J. Chaber, M. M. Čoban and K. Nagami, *On monotonic generalizations of Moore spaces, Čech complete spaces and p-spaces*, Fund. Math. 84 (1974), pp. 107–119.
[12] G. D. Creede, *Concerning semi-stratifiable spaces*, Pacific J. Math. 32 (1970), pp. 47–54.
[13] R. Engelking, *Outline of General Topology*, Amsterdam 1968.

([8]) It is easy to observe that Hausdorff spaces satisfying condition (P) are regular. Therefore, property (P) is equivalent to paracompactness in the class of Hausdorff spaces.

[14] V. V. Filippov, *On feathered paracompacta*, Dokl. Akad. Nauk SSSR 178 (1968), pp. 555–558 (Russian).
[15] R. W. Heath, *On spaces with point-countable bases*, Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys. 13 (1965), pp. 393–395.
[16] R. E. Hodel, *Moore spaces and WΔ-spaces*, Pacific J. Math. 38 (1971), pp. 641–652.
[17] — *Metrizability of topological spaces*, preprint.
[18] L. F. McAuley, *A relation between perfect separability, completeness and normality in semi-metric spaces*, Pacific J. Math. 6 (1956), pp. 315–326.
[19] K. Morita, *Products of normal spaces with metric spaces*, Math. Annalen 154 (1964), pp. 365–382.
[20] J. Nagata, *A note on Filippov's theorem*, Proc. Japan Acad. 45 (1969), pp. 30–33.
[21] H. H. Wicke and J. M. Worrell, Jr., *On topological completeness of first countable Hausdorff spaces I*, Fund. Math. 75 (1972), pp. 209–222.
[22] — — *Characterizations of absolute sets of interior condensation*, Notices Amer. Math. Soc. 20 (1973), p. A-401.
[23] — — *The concept of a θ-refinable embedding*, preprint.
[24] J. M. Worrell, Jr., and H. H. Wicke, *Characterizations of developable topological spaces*, Canad. J. Math. 17 (1965), pp. 820–830.
[25] — — *A central metrization theorem I*, Notices Amer. Math. Soc. 20 (1973), p. A-355.

WYDZIAŁ MATEMATYKI I MECHANIKI UNIWERSYTETU WARSZAWSKIEGO
DEPARTMENT OF MATHEMATICS AND MECHANICS, WARSAW UNIVERSITY

Accepté par la Rédaction le 20. 1. 1975