On a metrization of the hyperspace of a metric space

by

Karol Borsuk (Warszawa)

Abstract. Let $2^X$ denote the collection of all compact, non-empty subset of a metric space $X$. If $A, B \in 2^X$, then setting $\varrho_0(A, B)$ equal to the least lower bound of the set of all numbers $\varepsilon > 0$ such that there are a map $f : A \to B$ and a map $g : B \to A$ satisfying the condition $d(f(x), g(y)) < \varepsilon$ for every $x \in A$ and $g(f(x), x) < \varepsilon$ for every $y \in B$, we get a well-known continuity metric $\varrho_0$ in $2^X$. Replacing, in this definition, maps by fundamental sequences, one obtains another metric $\varrho_1$ in $2^X$. It is $\varrho_0(A, B) \geq \varrho_1(A, B)$ and if $A, B$ are ANR, then $\varrho_0(A, B) = \varrho_1(A, B)$. It is shown that if $A_1, A_2, A_3, \ldots \in 2^X$ and if $\lim_{n \to \infty} \varrho_0(A_n, A_0) = 0$, then some shape properties of $A_0$ (for $n = 1, 2, \ldots$) pass onto $A_n$.

1. Introduction. By a compactum we understand a metric, compact and non-empty space. It is well known that for every metric space $X$ the collection $2^X$ of all compacta lying in $X$ may be regarded as a metric space $2^X$ in which the distance $\varrho_0(A, B)$ of two compacta $A, B \in 2^X$ is given by the Hausdorff formula

$$\varrho_0(A, B) = \max \sup_{x \in A} \sup_{y \in B} d(x, y).$$

The Hausdorff metric $\varrho_0$ plays an important role in topology, though the topological properties of compacta $A, B$ have no influence on the distance $\varrho_0(A, B)$. So, for instance, each compactum $A \in X$ is in $2^X$ the limit of a sequence of finite compacita.

There have been several attempts (see [2] and [4]) to introduce in $2^X$ other metrics, for which if compacta $A_1, A_2, \ldots$ converge to a compactum $A_0$, then some topological properties of all $A_n$ pass onto the limit $A_0$. This is the case, in particular, with the metric $\varrho_C$ (called the metric of continuity), defined as follows (compare [2], p. 169):

Let $A, B \in 2^X$ and denote the collection of all maps (continuous functions of $A$ into $B$. If $\varphi \in B^A$, then let us denote by $|\varphi|$ the supremum of numbers $d(x, \varphi(x))$ for $x \in A$. Setting

$$\varrho_C(A, B) = \max \sup_{x \in A} \sup_{\varphi \in B^A} |\varphi(x)|,$$

we get the metric of the continuity $\varrho_C$. Let $2^X_0$ denote the collection $2^X$ metrized by $\varrho_C$. One shows that if $A_0 \in 2^X_0$ is in $2^X_0$ the limit of a sequence $A_1, A_2, \ldots$ of compacta.
On a metrication of the hyperspace of a metric space

and \( A_0 \) denotes the closure of the set \( \bigcup_{n=1}^{\infty} A_n \), then one easily sees that \( \lim_{n \to \infty} \varrho_0(A_n, A_0) = 0 \) and that \( A_n \) is, for \( n = 1, 2, \ldots \), homotopically dominated by the set \( B \) consisting of only one point. However, \( A_0 \) is not dominated by \( B \).

Thus the metric \( \varrho_0 \) has some important qualities, in particular if we consider compacta which are ANR-sets. However, in the case of compacta with more complicated local topological properties, this metric cases to be satisfactory. This is quite natural, because the definition of \( \varrho_0 \) is based on properties of maps of one compactum into another. For compacta with complicated topological properties the collection of such maps may be very limited and it does not give a reasonable basis for estimating the distance of two compacta.

In the present paper I introduce another metric in \( d^2 \), called the fundamental metric. Its definition is a quite natural modification of the definition of the metric \( \varrho_0 \), where instead of maps we consider the fundamental sequences, which are a basic concept for the theory of shape. We assume as known the most elementary notions and results of the theory of shape. The reader can find them in [1].

2. Fundamental metric. Let \( A, B \) be two compacta lying in a metric space \( X \) and let \( M \) be an AR-space containing \( X \). By \( \varrho_{F, M}(A, B) \) we denote the infimum of the set of all positive numbers \( \varepsilon \) satisfying the following condition:

There exist two fundamental sequences

\[ f = (f_n, A, B)_{M, M}, \quad \tilde{f} = (\tilde{f}_n, B, A)_{M, M} \]

such that

(2.1) There is a neighborhood \( (U, V) \) of the pair \( (A, B) \) in \( (M, M) \) such that for almost all \( k \) : \( q(x, f_k(x)) < \varepsilon \) for every \( x \in U \), \( q(y, \tilde{f}_k(y)) < \varepsilon \) for every \( y \in V \).

Let us prove the following

(2.2) Theorem \( \varrho_{F, M} \) is a metric.

Proof. It is clear that \( \varrho_{F, M}(A, B) = \varrho_{F, M}(B, A) \geq 0 \) and that \( \varrho_{F, M}(A, B) = 0 \) if and only if \( A = B \). It remains to show that \( \varrho_{F, M} \) satisfies the triangle inequality.

Let \( A, B, C \in d^2 \) and let

(2.3) \( \varrho_{F, M}(A, B) < \varepsilon \) and \( \varrho_{F, M}(B, C) < \eta \).

In order to prove the triangle inequality it suffices to show that then

\( \varrho_{F, M}(A, C) < \varepsilon + \eta \).

By (2.3) there exist two fundamental sequences

\[ f = (f_n, A, B)_{M, M}, \quad \tilde{f} = (\tilde{f}_n, B, A)_{M, M} \]

satisfying condition (2.1) and two fundamental sequences

\[ q = (q_n, B, C)_{M, M}, \quad \tilde{q} = (\tilde{q}_n, C, B)_{M, M} \]
satisfying the following condition:

(2.4) There is a neighborhood \((V', W')\) of \((B, C)\) in \((M, M)\) such that for almost all \(k\), 
\[ q(x, g_k(x)) < \eta \quad \text{for every} \quad x \in V', \quad q(y, g_k(y)) < \eta \quad \text{for every} \quad y \in W'. \]

Since we can replace \(V\) and \(V'\) by arbitrary smaller neighborhoods of \(B\) in \(M\), we may fix \(U\) and assume that \(V = V'\) is so small that

\[ f(U) \subseteq U \quad \text{for almost all} \quad k. \]

When \(V\) is fixed, we can select a neighborhood \(U' \subseteq U\) of \(A\) in \(M\) so that

\[ f(U') \subseteq V \quad \text{for almost all} \quad k, \]

and we can assume that the neighborhood \(W\) of \(C\) in \(M\) is so small that

\[ g_k(W) \subseteq V \quad \text{for almost all} \quad k. \]

Now let us set

\[ h_k = g_k, \quad h_k = f_k g_k \quad \text{for} \quad k = 1, 2, ..., \]

Then

\[ h = (h_k, C), A)_{M,M} = g_k \quad \text{and} \quad f = (f_k, C), A)_{M,M} = f_k \]

are fundamental sequences.

If \(x \in U',\) then

\[ q(x, h_k(x)) = q(x, g_k(x)) = q(x, f_k(x)) + q(f_k(x), g_k(x)) < \eta + \eta \]

for almost all \(k\), because \(x \in U' \subseteq U\) and \(f_k(x) \in V\) for almost all \(k.

Moreover, if \(x \in W',\) then \(h_k(x) \in V\) and consequently

\[ q(x, h_k(x)) = q(x, f_k g_k(x)) = q(x, f_k(x)) + q(f_k(x), g_k(x)) < \eta + \eta \]

for almost all \(k.

It follows that \(q_{f_k}(A, C) < \eta + \eta\) and the proof of Theorem (2.2) is finished.

Thus we have shown that the collection \(\mathcal{F}\) of all compacta lying in \(X\) with the metric \(q_{f_k}(A, B)\) is a metric space. We denote this space by \(\mathcal{F}_k\).

3. Role of the space \(M\). Now let us show that \(\mathcal{F}_k\) does not depend on the choice of the space \(M \in AR\) containing (metrically) the space \(X\).

(3.1) THEOREM. If \(A, B \in 2^X\) and if \(A \cup B\) is metrically contained in two AR-spaces \(M, M'\), then \(q_{f_k}(A, B) = q_{f_k}(A, B)\).

Proof. It suffices to show that if \(q_{f_k}(A, B) < \varepsilon\) and if \(\varepsilon < \eta\), then \(q_{f_k}(A, B) < \eta\).

It is clear that there exist two maps

\[ \alpha: M \rightarrow M', \quad \delta: M' \rightarrow M \]

such that

\[ \alpha(x) = \delta(x) = x \quad \text{for every point} \quad x \in A \cup B. \]

On a metrization of the hyperspace of a metric space

The inequality \(q_{f_k}(A, B) < \varepsilon\) implies that there exist two fundamental sequences

\[ f = \{f_k, A, B\}_{M,M}, \quad f = \{f_k, B, A\}_{M,M} \]

satisfying condition (2.1).

Setting

\[ g_k = \alpha f_k, \quad \delta_k = \alpha f_k, \quad M' \rightarrow M' \]

for every \(k = 1, 2, ..., \) we get two fundamental sequences

\[ g = \{g_k, A, B\}_{M,M}, \quad \delta = \{\delta_k, B, A\}_{M,M}. \]

It remains to show that

(3.3) There is a neighborhood \((U', V')\) of \((A, B)\) in \((M', M')\) such that for almost all \(k: q(x, g_k(x)) < \eta\) for every \(x \in U'\) and \(q(x, g_k(x)) < \eta\) for every \(x \in V'. \)

In order to show this, let us observe that the neighborhood \((U', V')\) of \((A, B)\) in \((M, M)\) satisfying (2.1) may be replaced by any smaller neighborhood of \((A, B)\) in \((M, M)\). By virtue of (3.2), we can select \((U, V)\) so that

\[ \text{(3.4)} \quad \text{If} \quad x, y \in U \cup V \quad \text{and} \quad q(x, y) < \varepsilon \quad \text{then} \quad q(x, g_k(x)) < \varepsilon + \eta, \]

Since \(f\) and \(f\) are fundamental sequences, there exists a neighborhood \((U_0, V_0)\) \(\subseteq (U, V)\) of \((A, B)\) in \((M, M)\) such that

\[ \text{(3.5)} \quad f(U_0) \subseteq V \quad \text{and} \quad f(V_0) \subseteq U \quad \text{for almost all} \quad k. \]

Now we can select a neighborhood \((U', V')\) of \((A, B)\) in \((M', M')\) so that

\[ \text{(3.6)} \quad g(U) \subseteq U_0, \quad g(V) \subseteq V_0 \]

and that

\[ \text{(3.7)} \quad q(\alpha f(x), x) < \varepsilon + \eta \quad \text{for every} \quad x \in U' \cup V'. \]

If \(x \in U',\) then (3.6) and (3.5) imply that \(g(x) \in U_0 \subseteq U\) and that \(f_k(x) \in V\) for almost all \(k.

Thus we infer by (2.1) that

\[ q(\alpha f_k(x), x) < \varepsilon + \eta \quad \text{for almost all} \quad k. \]

It follows by (3.4) that

\[ q(\alpha f_k(x), x) < \varepsilon + \eta \quad \text{for almost all} \quad k. \]

Using (3.7), we infer that for every point \(x \in U'\) and for almost all \(k:

\[ q(g_k(x), x) < q(\alpha f_k(x), x) + q(\alpha f_k(x), x) < \varepsilon + \eta = \eta. \]
Similarly, using (2.1), (3.5), (3.6) and (3.7), one gets for every point \( x \in V' \) and for almost all \( k \):

\[
q(\phi_k(x), x) < q(\phi_k, A(x), \phi_k(x)) + q(\phi_k(x), x) < \eta.
\]

Thus condition (3.3) is satisfied and the proof of Theorem (3.1) is concluded. It follows by Theorem (3.1) that the index \( M \) in the notations \( \Gamma_r, \overline{\Gamma}_r \) and \( 2\overline{\Gamma}_r \) is superclosely. Thus in the sequel we shall write \( \phi_k \) instead of \( \phi_k, A \) and \( 2\overline{\Gamma}_r \) instead of \( 2\overline{\Gamma}_r \).

4. Some relations between \( \phi_{\overline{\Gamma}_r} \), \( \phi_{\overline{\Gamma}_r} \) and \( \phi_{\Gamma_r} \). Let us prove the following.

(4.1) Theorem. If \( A, B \) are compacta lying in a space \( X \) then \( \phi_{\overline{\Gamma}_r}(A, B) \leq \phi_{\Gamma_r}(A, B) \leq \phi_{\overline{\Gamma}_r}(A, B) \).

Proof. If \( \phi_{\overline{\Gamma}_r}(A, B) > \varepsilon > 0 \), then in at least one of the sets \( A, B \) (say in \( A \)) there exists a point \( a \) whose distance from the other set (hence from \( B \)) is greater than \( \varepsilon \).

If \( f = (f, A, B)_{M, M} \), \( f = (f, B, A)_{M, M} \) are fundamental sequences, then for almost all \( k \) the point \( f_k(a) \) lies arbitrarily close to \( B \), and hence \( q(f_k(a), a) > \varepsilon \). It follows that condition (2.1) is not satisfied, and hence \( \phi_{\overline{\Gamma}_r}(A, B) > \varepsilon \). Thus the inequality

\[
\phi_{\overline{\Gamma}_r}(A, B) \leq \phi_{\Gamma_r}(A, B)
\]

is proved.

Now let us assume that \( \phi_{\overline{\Gamma}_r}(A, B) < \varepsilon \). Then there exist two maps

\[
f: A \rightarrow B, \quad f: B \rightarrow A
\]

such that

\[
q(f(x), x) < \varepsilon \quad \text{for every} \quad x \in A \quad \text{and} \quad q(f(y), y) < \varepsilon \quad \text{for every} \quad y \in B.
\]

Then there are two maps \( g, \beta: M \rightarrow M \) such that

\[
g(x) = f(x) \quad \text{for every} \quad x \in A \quad \text{and} \quad g(y) = f(y) \quad \text{for every} \quad y \in B.
\]

It follows that there is a neighborhood \((U, V)\) of \((A, B)\) in \((M, M)\) such that

\[
q(g(x), x) < \varepsilon \quad \text{for every} \quad x \in U
\]

and

\[
q(\beta(y), y) < \varepsilon \quad \text{for every} \quad y \in V.
\]

Setting

\[
\delta_1 = \delta \quad \text{and} \quad \delta_2 = \delta
\]

one gets two fundamental sequences

\[
\bar{f} = (\delta_1, A, B)_{M, M}, \quad \bar{g} = (\delta_2, B, A)_{M, M}
\]

such that

\[
q(x, \delta_1(x)) < \varepsilon \quad \text{for every} \quad x \in U \quad \text{and} \quad q(y, \delta_2(y)) < \varepsilon \quad \text{for every} \quad y \in V.
\]

It follows that \( \phi_{\overline{\Gamma}_r}(A, B) < \varepsilon \) and the proof of Theorem (4.1) is finished.

(4.2) Example. Let \( A \) denote the segment \((0, 1)\) and \( B \) — the set consisting of its two endpoints, and let \( X = M = E^1 \). Then \( \phi_{\overline{\Gamma}_r}(A, B) = \frac{1}{2} \). Moreover, if \( f = (f, A, B)_{M, M} \) is a fundamental sequence, then for almost all \( k \) the set \( f(A) \) lies in an arbitrarily given neighborhood of one of the points \((0)\) and \((1)\). It follows that for every positive number \( \varepsilon < 1 \), the condition \( q(f(x), x) < \varepsilon \) for every point \( x \in A \) cannot be satisfied for almost all \( k \). Hence \( \phi_{\overline{\Gamma}_r}(A, B) \geq \varepsilon \) and consequently \( \phi_{\overline{\Gamma}_r}(A, B) \geq 1 \). Thus in this case \( \phi_{\overline{\Gamma}_r}(A, B) = \phi_{\Gamma_r}(A, B) \).

(4.3) Example. Consider in the plane \( X = E^2 \) the square \( K_0 \) consisting of all points \((x, y)\) with \( 0 < x, y < 1 \) and denote by \( K_n \), for every \( m = \pm 1, \pm 2, \ldots \), the square consisting of all points \((x, y)\) with \(-1/3m < x, y < 1/3m \). Let \( L \) denote the segment with endpoints \((0, 0)\) and \((1, 0)\) and let \( B_n \) denote, for \( n = 1, 2, \ldots \), the closure of the diagram of the function

\[
y = \frac{1}{4n} \sin \frac{\pi}{x} \quad \text{where} \quad 0 < x < \frac{1}{2}.
\]

Let \( A_0 \) denote the boundary of the square \( K_0 \) and let

\[
A_n = (A_0 \setminus L) \cup B_n \quad \text{for} \quad n = 1, 2, \ldots
\]

Consider now, for \( n = 1, 2, \ldots \), a map

\[
f_n: A_0 \rightarrow A_n,
\]

and let \( a \) denote the point \((\frac{1}{2}, \frac{1}{2})\). It is clear that \( f_n \) is null-homotopic in \( A_n \), hence also in \( E^\infty(a) \). It follows that \( f_n \) is not homotopic in \( E^\infty(a) \) to the identity map \( i_{A_0}: A_0 \rightarrow A_0 \). We infer that \( |f_n| \geq \frac{1}{2} \), because otherwise the point \( a \) would not belong to any segment with endpoints \( x \) and \( f_n(x) \) and \( f_n \) would be homotopic in \( E^\infty(a) \) to \( i_{A_0} \). Hence

\[
\phi_{\Gamma_r}(A_n, A_0) \geq \frac{1}{2} \quad \text{for} \quad n = 1, 2, \ldots
\]
Now let us set
\[ U_n = K \cap K_{-n} \quad \text{for} \quad n = 1, 2, \ldots \]
It is clear that \( U_n \) is a neighborhood of \( A_n \) and also of \( A_n \) in \( E^2 \) for every \( n = 1, 2, \ldots \)
Moreover, it is clear that there exists a map
\[ g_n: E^2 \rightarrow E^2 \]
such that \( g_n(p) \) is, for every point \( p \in U_n \), the intersection of the ray starting from the point \( (1, 2) \) and passing through the point \( p \). Then
\[ \varrho(g_n(p), 0) \leq \frac{\sqrt{2}}{3n} \quad \text{for every point} \quad p \in U_n. \]
Setting
\[ g_n^0 = g_n \quad \text{for \ all \} \ k = 1, 2, \ldots, \]
we get, for every \( n = 1, 2, \ldots \), a fundamental sequence
\[ g_n^0 = (g_n^0, A_n, A_0)_{E^2}, \]
satisfying the condition
\[ \varrho(g_n^0(p), 0) \leq \frac{\sqrt{2}}{3n} \quad \text{for every point} \quad p \in U_n \quad \text{and for} \ k = 1, 2, \ldots. \]

One easily sees that \( A_n \) is a fundamental retract of the set \( U_n \). Thus, for every \( n = 1, 2, \ldots \), there is a fundamental retraction
\[ r_n^0 = (r_n^0, U_n, A_n)_{E^2}, \]
and it is clear that the maps \( r_n^0: E^2 \rightarrow E^2 \) can be selected so that there exists a sequence of positive numbers \( \{t_n\} \) converging to zero and such that
\[ \varrho (r_n^0(p), 0) \leq t_n \quad \text{for every point} \quad p \in U_n. \]
Setting
\[ g_n^1 = r_n^{00} \quad \text{and} \quad g_n^{00} = (g_n^1, A_n, A_0)_{E^2}, \]
one gets a fundamental sequence \( g_n^{00} : A_0 \rightarrow A_n \) for every \( n = 1, 2, \ldots \), satisfying the condition
\[ \varrho(g_n^{00}(p), 0) \leq t_n \quad \text{for every point} \quad p \in U_n \quad \text{for} \ n = 1, 2, \ldots. \]

It follows by (4.5) and (4.6) that
\[ \varrho(A_n, A_0) \leq \frac{\sqrt{2}}{3n} + t_n. \]

Consequently the compacta \( A_1, A_2, \ldots \) constitute a sequence converging in the space \( E^2 \) to \( A_0 \). By virtue of (4.4) we infer that
(4.7) The function assigning to every compactum \( A \in 2^E \) the same compactum \( A \in 2^E \) is not always continuous.

Recalling Theorem (4.1), we infer that, in general, the metric \( \varrho_E \) is essentially stronger than the metric \( \varrho_E \).

(4.8) Remark. It is clear that the compacta \( A_1, A_2, \ldots \) considered in Example (4.3) have the fixed point property. However, \( A_0 \) does not have this property. Consequently the subset of the hyperspace \( 2^E \) consisting of all compacta with the fixed point property is, in general, not closed — contrary to the situation in the space \( 2^E \).

5. Case of ANR-spaces. We have shown that for arbitrary compacta the distance \( \varrho(A, B) \) can be less than the distance \( \varrho(A, B) \). There is another situation as regards compact-ANR sets. Let us prove the following

(5.1) Theorem. If \( A, B \) are compact ANR-spaces, then \( \varrho(A, B) = \varrho(A, B) \).

Proof. By Theorem (4.1) it suffices to show that if compacta \( A, B \in M \in E \) are ANR-sets and if \( \varrho(A, B) \leq \varepsilon \), then for every \( \eta > \varepsilon \) there exist maps
\[ f: A \rightarrow B \quad \text{and} \quad \check{f}: B \rightarrow A \]
such that \( |f|, |\check{f}| \leq \eta \).

Since \( A, B \in ANR \), there exist a neighborhood \( (U_0, V_0) \) of \( (A, B) \) in \( (M, M) \) and retractions
\[ r: U_0 \rightarrow A, \quad s: V_0 \rightarrow B. \]

If we replace \( (U_0, V_0) \) by a sufficiently small neighborhood of \( (A, B) \) in \( (M, M) \), then we can assume that
\[ |r| < \eta - \varepsilon \quad \text{and} \quad |s| < \eta - \varepsilon. \]

Now let
\[ f = \{f_0, A, B\}_{M, M} \quad \text{and} \quad \check{f} = \{\check{f}_0, B, A\}_{M, M} \]
be fundamental sequences satisfying (2.1). Then there is an index \( k_0 \) such that
\[ f_{k_0}(A) = V_0 \quad \text{and} \quad \check{f}_{k_0}(B) = U_0. \]

Setting
\[ f = f_{k_0} A: A \rightarrow B, \quad \check{f} = \check{f}_{k_0} B: B \rightarrow A, \]
we get two maps such that
\[ \varrho(x, f(x)) \leq \varrho(x, f_0(x)) + \varrho(f_0(x), f_{k_0}(x)) < \varepsilon + (\eta - \varepsilon) = \eta \]
for every point \( x \in A \),
\[ \varrho(x, \check{f}(x)) \leq \varrho(x, \check{f}_0(x)) + \varrho(\check{f}_0(x), \check{f}_{k_0}(x)) < \varepsilon + (\eta - \varepsilon) = \eta \]
for every point \( x \in B \).

Hence \( |f| \leq \eta \) and \( |\check{f}| \leq \eta \) and the proof of Theorem (5.1) is finished.
6. Space $2^Y$ as a topological invariant of $X$. Now let us prove the following (6.1) Theorem. If $X$ is homeomorphic to $Y$, then $2^X$ is homeomorphic to $2^Y$.

Proof. Assume that $X$ is a closed subset of a space $M \in \mathcal{A}$ and $Y$ is a closed subset of the space $N \in \mathcal{A}$, and let $h: X \to Y$ be a homeomorphism. Then there are two maps

$$
\alpha: M \to N \quad \text{and} \quad \beta: N \to M
$$

such that

$$
\alpha(x) = h(x) \quad \text{for every} \ x \in X \quad \text{and} \quad \beta(y) = h^{-1}(y) \quad \text{for every} \ y \in Y.
$$

Let us show that the function assigning to every compactum $A \subseteq Y$ the compactum $h(A) \subseteq Y$ is a homeomorphism of $2^Y$ onto $2^Y$. It is clear that this function is one to one. Since our hypotheses concerning $h$ and $h^{-1}$ are symmetric, it suffices to show that if $A_0, A_1, A_2, \ldots$ are compacta in $X$ such that

$$
\lim_{n \to \infty} g_d(A_n, A_{n+1}) = 0,
$$

then

$$
\lim_{n \to \infty} g_d(h(A_n), h(A_{n+1})) = 0.
$$

It follows by (6.3) that there exists a sequence $\varepsilon_1, \varepsilon_2, \ldots$ of positive numbers converging to zero and such that for every $n = 1, 2, \ldots$ there are two fundamental sequences

$$
\mathcal{C}_n = \{f^{(n)}(A_1, A_2, \ldots)_{M,M} \in \mathcal{I} \}
$$

such that for every $n = 1, 2, \ldots$ there exists a neighborhood $U_n$ of $(A_1, A_2, \ldots)$ in $(M, M)$ and an index $k_n$ such that

$$
\alpha(x) = x_n \in U_n \quad \text{for every} \ x \in X
$$

for $n = 1, 2, \ldots$

Since $(U_n, U_{n+1})$ can be replaced by any smaller neighborhood of $(A_1, A_2, \ldots)$ in $(M, M)$ and since the set $A = A_0 \cup \bigcup_{n=1}^{\infty} A_n$ is compact, one easily sees by (6.2) that there exists a sequence $\eta_1, \eta_2, \ldots$ of positive numbers converging to zero and such that

$$
\alpha(x, x') = \beta(x, x') < \eta_n, \quad \text{for every} \ x, x' \in X
$$

Setting

$$
\delta(x, x') = \beta(x, x') \quad \text{for every} \ x, x' \in X
$$

for every $n = 1, 2, \ldots$ and

$$
\delta^{(n)} = \{\delta, h(A_n), h(A_{n+1})\}_{M,M}, \quad \delta^{(\infty)} = \{\delta, h(A_n), A_{n+1}\}_{M,M}
$$

for $m = 0, 1, 2, \ldots$, we get two fundamental sequences

$$
\delta^{(0)} = \{(\delta, h(A_0), h(A_1))_{X,X} = \delta^{(0)}(0), \delta^{(n)}(0)\},
$$

$$
\delta^{(n)} = \{\delta^{(n)}, h(A_0), h(A_1))_{X,X} = \delta^{(n)}(0), \delta^{(n)}(1)\}
$$

for $n = 1, 2, \ldots$.

One readily sees that for every $n = 1, 2, \ldots$ there exists a neighborhood $(V_n, V_{n+1})$ of $(h(A_n), h(A_{n+1}))$ in $(N, N)$ such that

$$
\delta^{(n)}(V_n) \subseteq V_n, \quad \delta^{(n)}(V_{n+1}) \subseteq V_{n+1}
$$

and that

$$
\delta^{(n)}(x, y) \leq \eta_n \quad \text{for every point} \ y \in V_n \cup V_{n+1}.
$$

If $y \in V_n$, then

$$
\delta^{(n)}(x, y) = \delta^{(n)}(x, y) \leq \eta_n + \delta^{(n)}(x, y) \leq \eta_n + \delta^{(n)}(x, y).
$$

By (6.7), $(x, y) \in U_n$ and we infer by (6.5) that

$$
\delta^{(n)}(x, y) \leq \eta_n, \quad \text{for} \ k \gg k_n.
$$

Using (6.6), we infer that $\delta^{(n)}(x, y) < \eta_n$, $k \gg k_n$. It follows by (6.8) that

$$
\delta^{(n)}(y, y) < \eta_n, \quad \text{for} \ y \in V_n \quad \text{and} \quad k \gg k_n.
$$

If $y \in V_{n+1}$, then (6.7) gives $\delta^{(n)}(y) \in V_{n+1}$ and we infer by (6.5) that

$$
\delta^{(n)}(x, y) \leq \eta_n, \quad \text{for} \ k \gg k_n.
$$

Using (6.6), one gets $\delta^{(n)}(x, y) < \eta_n$, $k \gg k_n$ and by virtue of (6.8) we obtain

$$
\delta^{(n)}(x, y) < \eta_n + \delta^{(n)}(x, y) < 2\eta_n
$$

for $y \in V_{n+1}$ and $k \gg k_n$.

It follows by (6.9) and (6.10) that

$$
\delta^{(n)}(h(A_n), h(A_{n+1})) < 2\eta_n,
$$

hence

$$
\lim_{n \to \infty} \delta^{(n)}(h(A_n), h(A_{n+1})) = 0
$$

and the proof of Theorem (6.1).
that the true cycle $\gamma'$ (just as any true cycle homologous to $\gamma'$ in $B$) is assigned by $f$ to
the true cycle $\gamma$.

Let $f = \{f_1(A,B)\}_{A,B}$ and $g = \{g_1(A,B)\}_{A,B}$ be two fundamental sequences
and let $V$ be an open neighborhood of $B$ in $N$. The fundamental sequences $f$ and
$g$ are said to be $V$-homotopic (notation: $f \cong_V g$) if the condition

$$f(A) = g(A) \quad \text{in} \quad V$$

is satisfied.

Now let us prove the following

(7.2) Lemma. Let $f = \{f_1(A,B)\}_{A,B}$ and $g = \{g_1(A,B)\}_{A,B}$ be two $V$-homotopic fundamental sequences, where $V$ is a compact neighborhood of $B$ in $N$. Then for every true cycle $\gamma$ in $A$ the true cycle $\gamma'$ and $\gamma''$ in $B$ assigned to $\gamma$ by $f$ and $g$ respectively are homologous in $V$.

Proof. We may assume that

$$\gamma' = \{f_1(A_0)\}, \quad \gamma'' = \{g_1(A_0)\},$$

where the sequence of indices $1 < j_1 < \ldots < j_k$ can be selected so that the mesh of the
cycle $\gamma_0$ (i.e., the maximal diameter of simplexes of $\gamma_0$) is so small that the homotopy

$$f(A) = g(A) \quad \text{in} \quad V$$

implies that there exists in $V$ a chain $A_k$ with a mesh $\leq 1/k$ such that

$$\partial_{k-1}(A_k) = f(A_0) - g(A_0).$$

It is clear that $\gamma = \{f_1(A_0)\}$ is an infinite chain in $V$ such that $\partial_{k-1}(A_k) = \gamma' - \gamma''$. Hence $\gamma' - \gamma''$ in $V$ and the proof of Lemma (7.2) is finished.

We shall limit ourselves in the sequel to the case of true cycles with rational coefficients. The maximal number of $n$-dimensional true cycles in $A$ homologically independent in $A$ is said to be the $n$-th Betti number of $A$. We denote it by $p_n(A)$.

(7.3) Theorem. Let $A_0, A_1, A_2, \ldots$ be compacly lying in a space $M$ and $A$ be $n$-dimensional true cycles in $A$ homologically independent in $A$. Then $p_n(A) = 0$. If $p_n(A_0) = m$ for every $n = 1, 2, \ldots$, then $p_n(A_n) = m$.

Proof. Since for $m = \infty$ the statement is obvious, we can assume that $m$ is finite. Moreover, Theorem (5.1) allows us to limit ourselves to the case where $M$ is the Hilbert cube $Q$.

Let $\hat{A}_0 = \hat{g}_0(A_0, A_0)$. Then for every $n = 1, 2, \ldots$ there exist two fundamental sequences

$$f(\hat{A}_0, A_0), \quad \hat{f}(\hat{A}_0, A_0)$$

and a neighborhood $(U, U_0)$ of $(\hat{A}_0, A_0)$ in $(Q, Q)$ and an index $k$, such that

$$\phi(x, f(\hat{A}_0, A_0)) < 2a, \quad \text{for} \quad x \in U,$$

$$\phi(x, f(\hat{A}_0, A_0)) < 2a, \quad \text{for} \quad x \in U_0, \quad \text{for} \quad k > k_0.$$

Let $U$ be an arbitrary compact neighborhood of $A_0$ in $Q$. By (7.4) there exists an index $\gamma_0$ such that

(7.5) $U$ is a neighborhood of $A_0$ in $Q$.

(7.6) $\hat{f}(\hat{A}_0, A_0) \cong_V f(\hat{A}_0, A_0)$.

Consider a system $\gamma_1, \gamma_2, \ldots, \gamma_{m+1}$ of true $n$-dimensional cycles in $A_0$. We infer by (7.6) and by Lemma (7.3) that the fundamental sequence $f(\hat{A}_0, A_0)$ assigns to those true cycles true $n$-dimensional cycles $\gamma'_1, \gamma'_2, \ldots, \gamma'_{m+1}$ in $A_0$ homological in $U$ to the true cycles $\gamma_1, \gamma_2, \ldots, \gamma_{m+1}$, respectively. Since $p_n(A_0) = m$, there exist integers $l_1, l_2, \ldots, l_{m+1}$ not all vanishing and such that

$$l_1 \gamma'_1 + l_2 \gamma'_2 + \ldots + l_{m+1} \gamma'_{m+1} = 0 \quad \text{in} \quad A_0 \cup U.$$

Since $\gamma'_i \cap U$ in $U$ for $i = 1, 2, \ldots, m+1$, we infer that

$$l_1 \gamma'_1 + l_2 \gamma'_2 + \ldots + l_{m+1} \gamma'_{m+1} = 0 \quad \text{in} \quad U.$$

Thus we have shown that the system $\gamma_1, \gamma_2, \ldots, \gamma_{m+1}$ is homologically dependent in every neighborhood $U$ of $A_0$ in $Q$. It is known ([3], p. 208) that then this system is homologically dependent in $A_0$; hence $p_n(A_0) = m$ and the proof of Theorem (7.3) is finished.

Theorem (7.3), combined with Theorem (4.1) give the following

(7.7) Corollary. If $A_0, A_1, A_2, \ldots$ are compacly lying in a space $X$ and if $\lim \phi(x, A_n) = 0$, then the inequality $p_n(A_0) = m$ for $n = 1, 2, \ldots$ implies that $p_n(A_0) = m$.

8. Quasi-domination and the fundamental metric. If $A, B$ are compacly lying in spaces $M, N \in AR$ respectively and if $U$ is a neighborhood of $A$ in $M$, then $A$ is said to be $U$-dominated by $B$ (notation: $A \leq U B$ in $M$; see [3], p. 198) if there exist two fundamental sequences

$$f = \{f_1(A, B)\}_{A, B} \quad \text{and} \quad \hat{f} = \{\hat{f}_1(A, B)\}_{A, B}$$

such that $f \leq U B$. If the relation $A \leq U B$ in $M$ holds true for each neighborhood $U$ of $A$ in $M$, then $A$ is said to be quasi-domination by $B$ (notation: $A \leq^* B$). It is known ([3], p. 205) that the choice of spaces $M, N \in AR$ containing $A$ and $B$ is immaterial for the relation $A \leq U B$. Moreover, it is known ([3], p. 210) that the relation $A \leq^* B$ implies that $p_n(A) \leq p_n(B)$ for every $n = 0, 1, 2, \ldots$ and that the movability of $B$ implies the movability of $A$.

Let us prove the following

(8.1) Theorem. Let $A_0, A_1, A_2, \ldots$ be compacly lying in a space $M \in AR$ and let $\lim \phi(x, A_0) = 0$. If $B$ is a compact such that $A_0, A_n, B$ for every $n = 1, 2, \ldots$, then $A_0 \leq^* B$. 
Proof. By Theorem (5.1) we can assume that $A$ and $B$ are subsets of the Hilbert space $E^n$. Let $e_v = \varphi(A_v, A_0)$. Hence $\lim e_v = 0$ and for every $v = 1, 2, ...$ there exist two fundamental sequences

$$f^{(v)} = \{f^{(v)}(A_v, A_0)_{E^n, E^n}\}, \quad h^{(v)} = \{h^{(v)}(A_v, A_0)_{E^n, E^n}\},$$

a neighborhood $(U_v, U_0)$ of $(A_v, A_0)$ in $(E^n, E^n)$ and an index $k_v$ such that

$$\varphi(x, f^{(v)}(x)) < 2e_v, \quad \text{for every } x \in U_v,$$

$$\varphi(x, h^{(v)}(x)) < 2e_v, \quad \text{for every } x \in U_0,$$

(8.2) and

$$\varphi(x, f^{(v)}(x)) > 2e_v, \quad \text{for every } x \in U_0,$$

$$\varphi(x, h^{(v)}(x)) > 2e_v, \quad \text{for every } x \in U_v.$$

We can assume that $U_v$ and $U_0$ are closed in $E^n$. Let $H_v$ denote the open ball in $E^n$ with centre $(0, 0, ..., 0)$ and radius $2e_v$. Setting

$$d^{(v)}(x) = f^{(v)}(x) - x \quad \text{for } x \in U_v,$$

$$d^{(v)}(x) = h^{(v)}(x) - x \quad \text{for } x \in U_0,$$

we get for every $k \geq k_v$ two maps:

$$d^{(v)}: U_v \rightarrow H_v \quad \text{and} \quad d^{(v)}: U_0 \rightarrow H_v.$$

Since $H_v \subset A_v$, we can extend $d^{(v)}$ and $d^{(v)}$ (for $k \geq k_v$) to maps

$$g^{(v)}: B^{(v)} \rightarrow E^n.$$

Setting, for $k \geq k_v$:

$$f^{(v)}(x) = x + d^{(v)}(x) \quad \text{for } x \in E^n,$$

$$g^{(v)}(x) = x + d^{(v)}(x) \quad \text{for } x \in E^n,$$

we get maps $f^{(v)}: E^n \rightarrow E^n$ which coincide on $U_v$ and on $U_0$ with $f^{(v)}$ and $h^{(v)}$ respectively and which satisfy the conditions:

$$\varphi(x, f^{(v)}(x)) < 2e_v, \quad \varphi(x, f^{(v)}(x)) < 2e_v,$$

(8.3) for every point $x \in E^n$. Setting $f^{(v)} = f^{(v)}$ for $k < k_v$, we easily see that $f^{(v)} = \{f^{(v)}(A_v, A_0)_{E^n, E^n}\}$ is a fundamental sequence homotopic to $f^{(v)}$ and $f^{(v)} = \{f^{(v)}(A_v, A_0)_{E^n, E^n}\}$ is a fundamental sequence homotopic to $f^{(v)}$.

It follows that by replacing $f^{(v)}$ by $f^{(v)}$ and $f^{(v)}$ by $f^{(v)}$ we may assume that the neighborhoods $U_v$ and $U_0$ in condition (8.2) coincide with the whole space $E^n$: hence

$$|f^{(v)}| < 2e_v \quad \text{and} \quad |f^{(v)}| < 2e_v, \quad \text{for } k \geq k_v.$$

(8.4)

Now consider an arbitrary open neighborhood $U$ of $A_v$ in $E^n$. Then, for almost all $v$, $U$ is a neighborhood of $A_v$. It follows by (8.3) that there exists an open neighborhood $U' \subset U$ of $A_0$ in $E^n$ and an index $v_1$ such that for $v > v_1$:

$$f^{(v)}(U') = U \quad \text{for almost all } k$$

and that

$$U' \text{ is a neighborhood of } A_v.$$

Since $A_v \subset B$, we infer by (8.5) that $A_v \subset B$ for every $v \geq v_1$. Consequently for every $v \geq v_1$ there exist two fundamental sequences

$$g^{(v)} = \{g^{(v)}(A_v, B)_{E^n, E^n}\}, \quad g^{(v)} = \{g^{(v)}(B, A_v)_{E^n, E^n}\}$$

such that there exists a neighborhood $U_v$ of $A_v$ in $E^n$ satisfying the condition

$$g^{(v)}(U_v) = U_v \quad \text{for almost all } k.$$

By (8.2) the map $f^{(v)}: E^n \rightarrow E^n$ satisfies the condition $|f^{(v)}(U_v)| < 4e_v$ for $k \geq k_v$. It follows that there exists an index $v_2 > v_1$ and a neighborhood $W$ of $A_0$ in $E^n$ such that

$$f^{(v_2)}(W) = U \quad \text{for almost all } k.$$

(8.7)

Without changing $v_2$, we can select the neighborhood $W$ of $A_0$ so that

$$f^{(v_2)}(W) \subset U_v \quad \text{for almost all } k.$$

(8.8)

Now let us set

$$h_k = g^{(v_2)}(x), \quad h_k = f^{(v_2)}(x) \quad \text{for } k = 1, 2, ...$$

Then

$$h_k = (h_k, A_0, B)_{E^n, E^n} = g^{(v_2)}(x) \quad \text{for } k = 1, 2, ...$$

are fundamental sequences. By virtue of (8.8) and (8.6) it follows by (8.4) that

$$h_k h_k W = f^{(v_2)}(y) h_k J_k W = f^{(v_2)}(y) J_k W \quad \text{in } U \quad \text{for almost all } k.$$

(8.9)

Using (8.7), we infer that $h_k h_k W = U W$ in $U$ for almost all $k$, hence $A_0$ is $U$-dominated by $B$. Since $U$ was an arbitrary open neighborhood of $A_0$ in $E^n$, we infer that $A_0 \subset B$ and the proof of Theorem (8.1) is finished.

9. Movability and the fundamental metric. A compactum $A$ lying in a space $M \in A$ is said to be movable if for every neighborhood $U$ of $A$ in $M$ there exists a neighborhood $U_0 \subset U$ of $A$ in $M$ with the property that for every neighborhood $V$ of $A$ (in $M$) the inclusion

$$U_0 \rightarrow U$$
is homotopic to a map with all values in \( V \). Let us say that \( U_0 \) realizes the movability of \( A \) in \( U \). It is well known that the movability of \( A \) does not depend on the choice of the space \( M \in \mathcal{R} \) containing \( A \) and that it is a hereditary shape invariant.

Let us prove the following

\( \text{(9.1) Theorem.} \) Let \( A_0, A_1, \ldots \) be compacta lying in a space \( X \) and let \( \lim_{\gamma \to \infty} (A_0, A_1, A_2, \ldots) = 0. \) If \( A_0, A_1, \ldots \) is movable for \( \gamma = 1, 2, \ldots \), then \( A_0, A_1, \ldots \) is also movable.

**Proof.** We may assume that \( X \) is a subset of the Hilbert cube \( Q \) and that \( \gamma \neq 0 \) for \( \gamma = 1, 2, \ldots \) If \( \gamma = \gamma(A_0, A_1) \), then for every \( \gamma = 1, 2, \ldots \) there exist two fundamental sequences

\[
\ell^{(1)} = (\ell_{A_0}^1, A_0, A_1, A_2, \ldots), \quad \ell^{(2)} = (\ell_{A_0}^2, A_0, A_1, A_2, \ldots)
\]

and a neighborhood \( (U_{\alpha}, U_{\beta}) \) of \( (A_0, A_1) \) in \( (Q, Q) \) such that:

\[
\rho(x, \ell_{A_0}^1(x)) < 2\epsilon_{\alpha} \quad \text{for} \quad x \in U_{\alpha}, \quad \text{for almost all } \alpha.
\]

Consider a neighborhood \( U \) of \( A_0, A_1 \) in \( Q \). One easily sees (because \( \lim_{\alpha \to \infty} \epsilon_{\alpha} = 0 \)) that there is an index \( \alpha_0 \) such that

\( (9.3) \) \( U \) is a neighborhood of \( A_0, A_1 \),

\( (9.4) \) There exists a neighborhood \( W_0 \) of \( A_0 \) in \( Q \) such that if \( x, y, \gamma \neq 0 \) and \( \rho(x, y) < 2\epsilon_{\alpha_0} \), then \( x + (1 - t)y \neq U \) for \( 0 \leq t \leq 1 \).

\( (9.5) \) There exists a neighborhood \( W_0 \) of \( A_0 \) in \( Q \) such that if \( x, y, \gamma \neq 0 \) and \( \rho(x, y) < 2\epsilon_{\alpha_0} \), then \( x + (1 - t)y \neq U \) for \( 0 \leq t \leq 1 \).

Since \( A_0, A_1 \) is movable, there is a neighborhood \( W_0 \) of \( A_0, A_1 \) in \( Q \) realizing the movability of \( A_0, A_1 \) in \( U \). Since \( \ell^{(1)} \) is a fundamental sequence, there exists a neighborhood \( U_{\alpha_0} \) of \( A_0, A_1 \) in \( Q \) such that \( U_{\alpha_0} \subset V_{\alpha_0} \cap U_{\alpha_0} \) and that

\( (9.6) \) \( \ell_{\alpha_0}(U_{\alpha_0}) = W_0 \) for almost all \( \alpha \).

Let us show that \( U_0 \) realizes the movability of \( A_0 \) in \( U \). Consider a neighborhood \( V \) of \( A_0 \) in \( Q \). By (9.6), (9.2) and (9.4) there is an index \( k_1 \) such that

\( (9.7) \) \( \ell_{\gamma}(U_{\alpha_0}) = W_0 \) and \( \ell_{\gamma}(U_{\alpha_0}) = V \) in \( U \) for \( k \geq k_1 \).

Since \( \ell_{\alpha_0} \) is a fundamental sequence, we infer from (9.2) and (9.5) that there exist a neighborhood \( W \subset W_0 \) of \( A_0 \) in \( Q \) and an index \( k_0 \geq k_1 \), such that

\( (9.8) \) \( \ell_{\alpha_0}(W) = V \) and \( \ell_{\alpha_0}(W) = U \) in \( U \).

The inclusion \( U_{\alpha_0} \subset V_{\alpha_0} \) and (9.4) imply that

\( (9.9) \) \( \ell_{\alpha_0}(U_{\alpha_0}) = U \) in \( U \).

It follows by (9.7) that there is a deformation in \( U \) carrying \( U_{\alpha_0} \) onto the set \( \ell_{\alpha_0}(U_{\alpha_0}) = W_0 \). Since \( W_0 \) realizes the movability of \( A_0, A_1 \) in \( U \), there is a deformation in \( U \) of \( \ell_{\alpha_0}(U_{\alpha_0}) \) onto a subset \( Z \) of \( W \). It follows by (9.8) that \( Z \) can be carried, by a deformation in \( U \) onto a subset of the set \( \ell_{\alpha_0}(W) \subset V \). Thus we have shown that one can obtain from \( U_0 \), by a deformation in \( U \), a subset of an arbitrarily given neighborhood \( V \) of \( A_0, A_1 \). Hence \( A_0, A_1 \) is movable and Theorem (9.1) is proved.

\( (9.9) \) **Problem.** Let \( A \in \mathcal{R} \) be a hereditary shape property (that is, if \( \text{Sh}(X) \geq \text{Sh}(Y) \) and \( X \in A \), then \( Y \in A \)). Is it true that for every sequence \( A_0, A_1, \ldots \) of compacta lying in a compactum \( X \), the two conditions

\( \lim_{\alpha \to \infty} \rho(x, A_0, A_1) = 0 \)

\( \ell_{\alpha}(A_0, A_1) \in A \)

imply that \( A_0 \in \alpha \)?

References


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