On a metrization of the hyperspace of a metric space

by

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Abstract. Let 2^X denote the collection of all compact, non-empty subset of a metric space X. If $A, B \in 2^X$, then setting $\varrho_C(A, B)$ equal to the least lower bound of the set of all numbers $\varepsilon > 0$ such that there are a map $f\colon A \to B$ and a map $g\colon B \to A$ satisfying the condition $\varrho(f(x), x) < \varepsilon$ for every $x \in A$ and $\varrho(g(y), y) < \varepsilon$ for every $y \in B$, we get a well-known continuity metric ϱ_C in 2^X. Replacing, in this definition, maps by fundamental sequences, one obtains another metric ϱ_F in 2^X. It is $\varrho_C(A, B) \geqslant \varrho_F(A, B)$ and if $A, B \in \mathrm{ANR}$, then $\varrho_C(A, B) = \varrho_F(A, B)$. It is shown that if $A_0, A_1, A_n, \ldots \in 2^X$ and if $\lim_{n=\infty} \varrho_F(A_0, A_n) = 0$, then some shape properties of A_n (for $n = 1, 2, \ldots$) pass onto A_0.

1. Introduction. By a *compactum* we understand a metric, compact and non-empty space. It is well known that for every metric space X the collection 2^X of all compacta lying in X may be regarded as a metric space 2^X_H in which the distance $\varrho_H(A, B)$ of two compacta $A, B \in 2^X$ is given by the Hausdorff formula

$$\varrho_H(A, B) = \operatorname{Max}[\operatorname*{Sup}_{x \in A} \varrho(x, B), \operatorname*{Sup}_{y \in B} \varrho(y, A)]. \tag{1.1}$$

The Hausdorff metric ϱ_H plays an important role in topology, though the topological properties of compacta A, B have no influence on the distance $\varrho_H(A, B)$. So, for instance, each compactum $A \subset X$ is in 2^X_H the limit of a sequence of finite compacta.

There have been several attempts (see [2] and [4]) to introduce in 2^X other metrics, for which if compacta $A_1, A_2, \ldots$ converge to a compactum A_0, then some topological properties of all A_n pass onto the limit A_0. This is the case, in particular, with the metric ϱ_C (called the *metric of continuity*), defined as follows (compare [2], p. 169):

Let $A, B \in 2^X$ and let B^A denote the collection of all maps (= continuous functions) of A into B. If $f \in B^A$, then let us denote by $|f|$ the supremum of numbers $\varrho(x, f(x))$ for $x \in A$. Setting

$$\varrho_C(A, B) = \operatorname{Max}[\operatorname*{Inf}_{f \in B^A} |f|, \operatorname*{Inf}_{\hat{f} \in A^B} |\hat{f}|], \tag{1.2}$$

we get the metric of the continuity ϱ_C. Let 2^X_C denote the collection 2^X metrized by ϱ_C. One shows that if $A_0 \in 2^X$ is in 2^X_C the limit of a sequence $A_1, A_2, \ldots$ of compacta, then some global topological properties of the sets A_n pass onto A_0 (see Corollary (7.7) in the sequel). Observe that the same holds for the important fixed point property.

In fact, if there exists a map $g: A_0 \to A_0$ and a number $\varepsilon > 0$ such that $\varrho(x, g(x)) > \varepsilon$ for every point $x \in A$, then there is an index ν_0 such that $\varrho_C(A_{\nu_0}, A_0) < \frac{1}{2}\varepsilon$. This means that there are two maps

$$f: A_{\nu_0} \to A_0 \quad \text{and} \quad \hat{f}: A_0 \to A_{\nu_0}$$

such that $|f| < \frac{1}{2}\varepsilon$ and $|\hat{f}| < \frac{1}{2}\varepsilon$. Then the map

$$g' = \hat{f}gf: A_{\nu_0} \to A_{\nu_0}$$

satisfies, for every point $x \in A_{\nu_0}$, the condition

$$\varrho(g'(x), x) = \varrho(\hat{f}gf(x), x) \geq \varrho(gf(x), f(x)) - \varrho(\hat{f}gf(x), gf(x)) - \varrho(f(x), x)$$
$$> \varepsilon - \tfrac{1}{2}\varepsilon - \tfrac{1}{2}\varepsilon = 0 .$$

Hence A_{ν_0} does not have the fixed point property.

Another property of the metric ϱ_C is given by the following proposition:

(1.3) *Let $A_0, A_1, A_2, \ldots$ be a sequence of compacta lying in a space X such that $\lim_{n=\infty} \varrho_C(A_n, A_0) = 0$. If A_n is, for $n = 1, 2, \ldots$, homotopically dominated by a compactum B and if $A_0 \in$ ANR, then A_0 is homotopically dominated by B.*

Proof. The hypothesis that A_n is homotopically dominated by B means that there exist two maps

$$f_n: A_n \to B \quad \text{and} \quad \hat{f}_n: B \to A_n$$

such that $\hat{f}_n f_n \simeq i/A_n$. Moreover, $\lim_{n=\infty} \varrho_C(A_n, A_0) = 0$ implies that there exists a sequence of positive numbers $\varepsilon_1, \varepsilon_2, \ldots$ converging to zero and such that for every $n = 1, 2, \ldots$ there exist maps

$$\alpha_n: A_n \to A_0 \quad \text{and} \quad \hat{\alpha}_n: A_0 \to A_n$$

such that $|\alpha_n| < \varepsilon_n$, $|\hat{\alpha}_n| < \varepsilon_n$. Setting

$$g_n = f_n \hat{\alpha}_n: A_0 \to B , \quad \hat{g}_n = \alpha_n \hat{f}_n: B \to A_0 ,$$

one gets maps such that $\hat{g}_n g_n: A_0 \to A_0$ and

$$\hat{g}_n g_n = \alpha_n \hat{f}_n f_n \hat{\alpha}_n \simeq \alpha_n \hat{\alpha}_n ,$$

where $|\alpha_n \hat{\alpha}_n| \leq 2\varepsilon_n$. But it is clear that this inequality and the hypothesis $A_0 \in$ ANR imply that for ε_n sufficiently small, $\alpha_n \hat{\alpha}_n \simeq i/A_0$. Hence $\hat{g}_n g_n \simeq i/A_0$ and consequently A_0 is homotopically dominated by B.

Observe that (1.3) fails if we omit the hypothesis $A_0 \in$ ANR. In fact, if A_n denotes the diagram of the function

$$y = \sin\left(\frac{\pi}{2} + \frac{2\pi}{x}\right) \quad \text{for} \quad \frac{1}{n} \leq x \leq 1$$

and A_0 denotes the closure of the set $\bigcup_{n=1}^{\infty} A_n$, then one easily sees that $\lim_{n=\infty} \varrho_C(A_n, A_0) = 0$ and that A_n is, for $n = 1, 2, \ldots$, homotopically dominated by the set B consisting of only one point. However, A_0 is not dominated by B.

Thus the metric ϱ_C has some important qualities, in particular if we consider compacta which are ANR-sets. However, in the case of compacta with more complicated local topological properties, this metric cases to be satisfactory. This is quite natural, because the definition of ϱ_C is based on properties of maps of one compactum into another. For compacta with complicated topological properties the collection of such maps may be very limited and it does not give a reasonable base for estimating the distance of two compacta.

In the present paper I introduce another metric in 2^X, called the fundamental metric. Its definition is a quite natural modification of the definition of the metric ϱ_C, where instead of maps we consider the fundamental sequences, which are a basic concept for the theory of shape. We assume as known the most elementary notions and results of the theory of shape. The reader can find them in [1].

2. Fundamental metric. Let A, B be two compacta lying in a metric space X and let M be an AR-space containing X. By $\varrho_{F,M}(A, B)$ we denote the infimum of the set of all positive numbers ε satisfying the following condition:

There exist two fundamental sequences

$$\underline{f} = \{f_k, A, B\}_{M,M} , \quad \underline{\hat{f}} = \{\hat{f}_k, B, A\}_{M,M}$$

such that

(2.1) *There is a neighborhood (U, V) of the pair (A, B) in (M, M) such that for almost all k: $\varrho(x, f_k(x)) < \varepsilon$ for every $x \in U$, $\varrho(y, \hat{f}_k(y)) < \varepsilon$ for every $y \in V$.*

Let us prove the following

(2.2) THEOREM. *$\varrho_{F,M}$ is a metric.*

Proof. It is clear that $\varrho_{F,M}(A, B) = \varrho_{F,M}(B, A) \geq 0$ and that $\varrho_{F,M}(A, B) = 0$ if and only if $A = B$. It remains to show that $\varrho_{F,M}$ satisfies the triangle inequality.

Let $A, B, C \in 2^X$ and let

(2.3) $$\varrho_{F,M}(A, B) < \varepsilon \quad \text{and} \quad \varrho_{F,M}(B, C) < \eta .$$

In order to prove the triangle inequality it suffices to show that then

$$\varrho_{F,M}(A, C) < \varepsilon + \eta .$$

By (2.3) there exist two fundamental sequences

$$\underline{f} = \{f_k, A, B\}_{M,M} , \quad \underline{\hat{f}} = \{\hat{f}_k, B, A\}_{M,M}$$

satisfying condition (2.1) and two fundamental sequences

$$\underline{g} = \{g_k, B, C\}_{M,M} , \quad \underline{\hat{g}} = \{\hat{g}_k, C, B\}_{M,M}$$

satisfying the following condition:

(2.4) There is a neighborhood (V', W) of (B, C) in (M, M) such that for almost all k, $\varrho(x, g_k(x))<\eta$ for every $x \in V'$, $\varrho(y, \hat{g}_k(y))<\eta$ for every $y \in W$.

Since we can replace V and V' by arbitrary smaller neighborhoods of B in M, we may fix U and assume that $V = V'$ is so small that

$$\hat{f}_k(V) \subset U \quad \text{for almost all } k.$$

When V is fixed, we can select a neighborhood $U' \subset U$ of A in M so that

$$f_k(U') \subset V \quad \text{for almost all } k,$$

and we can assume that the neighborhood W of C in M is so small that

$$\hat{g}_k(W) \subset V \quad \text{for almost all } k.$$

Now let us set

$$h_k = g_k f_k, \quad \hat{h}_k = \hat{f}_k \hat{g}_k \quad \text{for} \quad k = 1, 2, \ldots$$

Then

$$\underline{h} = \{h_k, A, C\}_{M,M} = \underline{g}\underline{f} \quad \text{and} \quad \underline{\hat{h}} = \{\hat{h}_k, C, A\}_{M,M} = \underline{\hat{f}}\underline{\hat{g}}$$

are fundamental sequences.

If $x \in U'$, then

$$\varrho(x, h_k(x)) = \varrho(x, g_k f_k(x)) \leqslant \varrho(x, f_k(x)) + \varrho(f_k(x), g_k f_k(x)) < \varepsilon + \eta$$

for almost all k, because $x \in U' \subset U$ and $f_k(x) \in V$ for almost all k.

Moreover, if $z \in W$, then $\hat{g}_k(z) \in V$ and consequently

$$\varrho(z, \hat{h}_k(z)) = \varrho(z, \hat{f}_k \hat{g}_k(z)) \leqslant \varrho(z, \hat{g}_k(z)) + \varrho(\hat{g}_k(z), \hat{f}_k \hat{g}_k(z)) < \eta + \varepsilon$$

for almost all k.

It follows that $\varrho_{F,M}(A, C) \leqslant \varepsilon + \eta$ and the proof of Theorem (2.2) is finished.

Thus we have shown that the collection 2^X of all compacta lying in X with the metric $\varrho_{F,M}$ is a metric space. We denote this space by $2^X_{F,M}$.

3. Role of the space M. Now let us show that $2^X_{F,M}$ does not depend on the choice of the space $M \in \mathrm{AR}$ containing (metrically) the space X.

(3.1) THEOREM. *If $A, B \in 2^X$ and if $A \cup B$ is metrically contained in two AR-spaces M, M', then $\varrho_{F,M'}(A, B) = \varrho_{F,M}(A, B)$.*

Proof. It suffices to show that if $\varrho_{F,M}(A, B) < \varepsilon$ and if $\varepsilon < \eta$, then $\varrho_{F,M'}(A, B) < \eta$.

It is clear that there exist two maps

$$\alpha: M \to M', \quad \hat{\alpha}: M' \to M$$

such that

(3.2) $$\alpha(x) = \hat{\alpha}(x) = x \quad \text{for every point } x \in A \cup B.$$

The inequality $\varrho_{F,M}(A, B) < \varepsilon$ implies that there exist two fundamental sequences

$$\underline{f} = \{f_k, A, B\}_{M,M}, \quad \underline{\hat{f}} = \{\hat{f}_k, B, A\}_{M,M}$$

satisfying condition (2.1).

Setting

$$g_k = \alpha f_k \hat{\alpha}: M' \to M', \quad \hat{g}_k = \alpha \hat{f}_k \hat{\alpha}: M' \to M'$$

for every $k = 1, 2, \ldots$, we get two fundamental sequences

$$\underline{g} = \{g_k, A, B\}_{M',M'}, \quad \underline{\hat{g}} = \{\hat{g}_k, B, A\}_{M',M'}.$$

It remains to show that

(3.3) There is a neighborhood (U', V') of (A, B) in (M', M') such that for almost all k: $\varrho(x, g_k(x)) < \eta$ for every $x \in U'$ and $\varrho(x, \hat{g}_k(x)) < \eta$ for every $x \in V'$.

In order to show this, let us observe that the neighborhood (U, V) of (A, B) in (M, M) satisfying (2.1) may be replaced by any smaller neighborhood of (A, B) in (M, M). By virtue of (3.2), we can select (U, V) so that

(3.4) If $x, y \in U \cup V$ and $\varrho(x, y) < \varepsilon$ then $\varrho(\alpha(x), \alpha(y)) < \varepsilon + \frac{1}{2}(\eta - \varepsilon)$.

Since $\underline{f}$ and $\underline{\hat{f}}$ are fundamental sequences, there exists a neighborhood $(U_0, V_0) \subset (U, V)$ of (A, B) in (M, M) such that

(3.5) $$f_k(U_0) \subset V \quad \text{and} \quad \hat{f}_k(V_0) \subset U \quad \text{for almost all } k.$$

Now we can select a neighborhood (U', V') of (A, B) in (M', M') so that

(3.6) $$\hat{\alpha}(U') \subset U_0, \quad \hat{\alpha}(V') \subset V_0$$

and that

(3.7) $$\varrho(\alpha\hat{\alpha}(x), x) < \tfrac{1}{2}(\eta - \varepsilon) \quad \text{for every } x \in U' \cup V'.$$

If $x \in U'$, then (3.6) and (3.5) imply that $\hat{\alpha}(x) \in U_0 \subset U$ and that $f_k \hat{\alpha}(x) \in V$ for almost all k. Thus we infer by (2.1) that

$$\varrho(f_k \hat{\alpha}(x), \hat{\alpha}(x)) < \varepsilon \quad \text{for almost all } k.$$

It follows by (3.4) that

$$\varrho(\alpha f_k \hat{\alpha}(x), \alpha\hat{\alpha}(x)) < \varepsilon + \tfrac{1}{2}(\eta - \varepsilon) \quad \text{for almost all } k.$$

Using (3.7), we infer that for every point $x \in U'$ and for almost all k:

$$\varrho(g_k(x), x) \leqslant \varrho(\alpha f_k \hat{\alpha}(x), \alpha\hat{\alpha}(x)) + \varrho(\alpha\hat{\alpha}(x), x)$$
$$< \varepsilon + \tfrac{1}{2}(\eta - \varepsilon) + \tfrac{1}{2}(\eta - \varepsilon) = \eta.$$

Similarly, using (2.1), (3.5), (3.6) and (3.7), one gets for every point $x \in V'$ and for almost all k:

$$\varrho(\hat{g}_k(x), x) \leqslant \varrho(\alpha \hat{f}_k \hat{\alpha}(x), \alpha\hat{\alpha}(x)) + \varrho(\alpha\hat{\alpha}(x), x) < \eta .$$

Thus condition (3.3) is satisfied and the proof of Theorem (3.1) is concluded.

It follows by Theorem (3.1) that the index M in the notations $\varrho_{F,M}$ and $2^X_{F,M}$ is superfluous. Thus in the sequel we shall write ϱ_F instead of $\varrho_{F,M}$ and 2^X_F instead of $2^X_{F,M}$.

4. Some relations between ϱ_H, ϱ_C and ϱ_F. Let us prove the following

(4.1) THEOREM. *If A, B are compacta lying in a space X then $\varrho_H(A, B) \leqslant \varrho_F(A, B) \leqslant \varrho_C(A, B)$.*

Proof. If $\varrho_H(A, B) > \varepsilon > 0$, then in at least one of the sets A, B (say in A) there exists a point a whose distance from the other set (hence from B) is greater than ε. If

$$\underline{f} = \{f_k, A, B\}_{M,M}, \quad \underline{\hat{f}} = \{\hat{f}_k, B, A\}_{M,M}$$

are fundamental sequences, then for almost all k the point $f_k(a)$ lies arbitrarily close to B, and hence $\varrho(a, f_k(a)) > \varepsilon$. It follows that condition (2.1) is not satisfied, and hence $\varrho_F(A, B) > \varepsilon$. Thus the inequality

$$\varrho_H(A, B) \leqslant \varrho_F(A, B)$$

is proved.

Now let us assume that $\varrho_C(A, B) < \varepsilon$. Then there exist two maps

$$f\colon A \to B, \quad \hat{f}\colon B \to A$$

such that

$$\varrho(f(x), x) < \varepsilon \quad \text{for every } x \in A \quad \text{and} \quad \varrho(\hat{f}(y), y) < \varepsilon \quad \text{for every } y \in B .$$

Then there are two maps $g, \hat{g}\colon M \to M$ such that

$$g(x) = f(x) \quad \text{for every } x \in A \quad \text{and} \quad \hat{g}(y) = \hat{f}(y) \quad \text{for every } y \in B .$$

It follows that there is a neighborhood (U, V) of (A, B) in (M, M) such that

$$\varrho(g(x), x) < \varepsilon \quad \text{for every } x \in U$$

and

$$\varrho(\hat{g}(y), y) < \varepsilon \quad \text{for every } y \in V.$$

Setting

$$g_k = g \quad \text{and} \quad \hat{g}_k = \hat{g} \quad \text{for} \quad k = 1, 2, \ldots,$$

one gets two fundamental sequences

$$\underline{g} = \{g_k, A, B\}_{M,M}, \quad \underline{\hat{g}} = \{\hat{g}_k, B, A\}_{M,M}$$

such that

$$\varrho(x, g_k(x)) < \varepsilon \quad \text{for every } x \in U \quad \text{and} \quad \varrho(y, \hat{g}_k(y)) < \varepsilon \quad \text{for every } y \in V$$

It follows that $\varrho_F(A, B) < \varepsilon$ and the proof of Theorem (4.1) is finished.

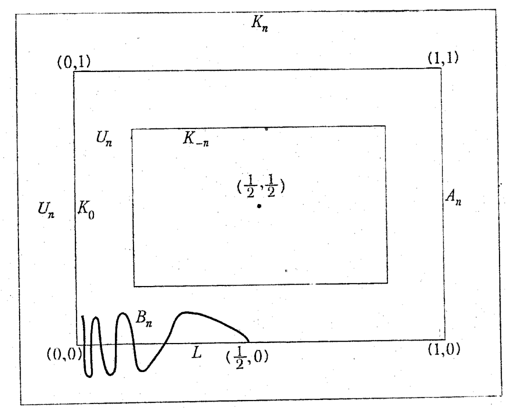
(4.2) EXAMPLE. Let A denote the segment $\langle 0, 1 \rangle$ and B — the set consisting of its two endpoints, and let $X = M = E^1$. Then $\varrho_H(A, B) = \frac{1}{2}$. Moreover, if $\underline{f} = \{f_k, A, B\}_{E^1, E^1}$ is a fundamental sequence, then for almost all k the set $f_k(A)$ lies in an arbitrarily given neighborhood of one of the points (0) and (1). It follows that for every positive number $\varepsilon < 1$, the condition $\varrho(f_k(x), x) < \varepsilon$ for every point $x \in A$ cannot be satisfied for almost all k. Hence $\varrho_F(A, B) \geqslant \varepsilon$ and consequently $\varrho_F(A, B) \geqslant 1$. Thus in this case $\varrho_H(A, B) < \varrho_F(A, B)$.

(4.3) EXAMPLE. Consider in the plane $X = E^2$ the square K_0 consisting of all points (x, y) with $0 \leqslant x, y \leqslant 1$ and denote by K_m, for every $m = \pm 1, \pm 2, \ldots$, the square consisting of all points (x, y) with $-1/3m \leqslant x, y \leqslant 1 + 1/3m$. Let L denote the segment with endpoints $(0, 0)$ and $(\frac{1}{2}, 0)$ and let B_n denote, for $n = 1, 2, \ldots$, the closure of the diagram of the function

$$y = \frac{1}{4n} \sin \frac{\pi}{x} \quad \text{where} \quad 0 < x < \tfrac{1}{2} .$$

Let A_0 denote the boundary of the square K_0 and let

$$A_n = (A_0 \setminus L) \cup B_n \quad \text{for} \quad n = 1, 2, \ldots$$



Consider now, for $n = 1, 2, \ldots$ a map

$$f_n\colon A_0 \to A_n ,$$

and let a denote the point $(\frac{1}{2}, \frac{1}{2})$. It is clear that f_n is null-homotopic in A_n, hence also in $E^2 \setminus (a)$. It follows that f_n is not homotopic in $E^2 \setminus (a)$ to the identity map $i_{A_0}\colon A_0 \to A_0$. We infer that $|f_n| \geqslant \frac{1}{2}$, because otherwise the point a would not belong to any segment with endpoints x and $f_n(x)$ and f_n would be homotopic in $E^2 \setminus (a)$ to i_{A_0}. Hence

(4.4) $$\varrho_C(A_n, A_0) \geqslant \tfrac{1}{2} \quad \text{for} \quad n = 1, 2, \ldots$$

Now let us set

$$U_n = K_n \setminus K_{-n} \quad \text{for} \quad n = 1, 2, \ldots$$

It is clear that U_n is a neighborhood of A_0 and also of A_n in E^2 for every $n = 1, 2, \ldots$ Moreover, it is clear that there exists a map

$$g_n\colon E^2 \to E^2$$

such that $g_n(p)$ is, for every point $p \in U_n$, the intersection of the ray starting from the point $(\frac{1}{2}, \frac{1}{2})$ and passing through the point p. Then

$$\varrho(g_n(p), p) \leqslant \frac{\sqrt{2}}{3n} \quad \text{for every point } p \in U_n.$$

Setting

$$g_k^{(n)} = g_n \quad \text{for every } k = 1, 2, \ldots,$$

we get, for every $n = 1, 2, \ldots$, a fundamental sequence

$$\underline{g}^{(n)} = \{g_k^{(n)}, A_n, A_0\}_{E^2, E^2},$$

satisfying the condition

(4.5) $$\varrho(g_k^{(n)}(p), p) \leqslant \frac{\sqrt{2}}{3n} \quad \text{for every point } p \in U_n \text{ and for } k = 1, 2, \ldots$$

One easily sees that A_n is a fundamental retract of the set $\overline{U}_n$. Thus, for every $n = 1, 2, \ldots$ there is a fundamental retraction

$$\underline{r}^{(n)} = \{r_k^{(n)}, \overline{U}_n, A_n\}_{E^2, E^2}$$

and it is clear that the maps $r_k^{(n)}\colon E^2 \to E^2$ can be selected so that there exists a sequence of positive numbers $\{\varepsilon_n\}$ converging to zero and such that

$$\varrho(r_k^{(n)}(p), p) \leqslant \varepsilon_n \quad \text{for every point } p \in U_n.$$

Setting

$$\hat{g}_k^{(n)} = r_k^{(n)} \quad \text{and} \quad \underline{\hat{g}}^{(n)} = \{\hat{g}_k^{(n)}, A_0, A_n\}_{E^2, E^2},$$

one gets a fundamental sequence $\underline{\hat{g}}^{(n)}\colon A_0 \to A_n$ for every $n = 1, 2, \ldots$, satisfying the condition

(4.6) $$\varrho(\hat{g}_k^{(n)}(p), p) \leqslant \varepsilon_n \quad \text{for every point } p \in U_n \text{ for } n = 1, 2, \ldots$$

It follows by (4.5) and (4.6) that

$$\varrho_F(A_0, A_n) < \frac{\sqrt{2}}{3n} + \varepsilon_n.$$

Consequently the compacta $A_1, A_2, \ldots$ constitute a sequence converging in the space 2_F^X to A_0. By virtue of (4.4) we infer that

(4.7) *The function assigning to every compactum $A \in 2_C^X$ the same compactum $A \in 2_F^X$ is not always continuous.*

Recalling Theorem (4.1), we infer that, in general, the metric ϱ_F is essentially stronger than the metric ϱ_C.

(4.8) Remark. It is clear that the compacta $A_1, A_2, \ldots$ considered in Example (4.3) have the fixed point property. However, A_0 does not have this property. Consequently the subset of the hyperspace 2_F^X consisting of all compacta with the fixed point property is, in general, not closed — contrary to the situation in the space 2_C^X.

5. Case of ANR-spaces. We have shown that for arbitrary compacta the distance $\varrho_F(A, B)$ can be less than the distance $\varrho_C(A, B)$. There is another situation as regards compact-ANR sets. Let us prove the following

(5.1) Theorem. *If A, B are compact* ANR-*spaces, then* $\varrho_F(A, B) = \varrho_C(A, B)$.

Proof. By Theorem (4.1) it suffices to show that if compacta $A, B \subset M \in \mathrm{AR}$ are ANR-sets and if $\varrho_F(A, B) < \varepsilon$, then for every $\eta > \varepsilon$ there exist maps

$$\underline{f}\colon A \to B \quad \text{and} \quad \underline{\hat{f}}\colon B \to A$$

such that $|\underline{f}|, |\underline{\hat{f}}| \leqslant \eta$.

Since $A, B \in \mathrm{ANR}$, there exist a neighborhood (U_0, V_0) of (A, B) in (M, M) and retractions:

$$r\colon U_0 \to A, \quad s\colon V_0 \to B.$$

If we replace (U_0, V_0) by a sufficiently small neighborhood of (A, B) in (M, M), then we can assume that

$$|r| < \eta - \varepsilon \quad \text{and} \quad |s| < \eta - \varepsilon.$$

Now let

$$\underline{f} = \{f_k, A, B\}_{M,M} \quad \text{and} \quad \underline{\hat{f}} = \{\hat{f}_k, B, A\}_{M,M}$$

be fundamental sequences satisfying (2.1). Then there is an index k_0 such that

$$f_{k_0}(A) \subset V_0 \quad \text{and} \quad \hat{f}_{k_0}(B) \subset U_0.$$

Setting

$$f = sf_{k_0}/A\colon A \to B, \quad \hat{f} = r\hat{f}_{k_0}/B\colon B \to A,$$

we get two maps such that

$$\varrho(x, f(x)) \leqslant \varrho(x, f_{k_0}(x)) + \varrho(f_{k_0}(x), sf_{k_0}(x)) < \varepsilon + (\eta - \varepsilon) = \eta$$

for every point $x \in A$,

$$\varrho(x, \hat{f}(x)) \leqslant \varrho(x, \hat{f}_{k_0}(x)) + \varrho(\hat{f}_{k_0}(x), r\hat{f}_{k_0}(x)) < \varepsilon + (\eta - \varepsilon) = \eta$$

for every point $x \in B$.

Hence $|f| \leqslant \eta$ and $|\hat{f}| \leqslant \eta$ and the proof of Theorem (5.1) is finished.

6. Space 2^X_F as a topological invariant of X. Now let us prove the following

(6.1) THEOREM. *If X is homeomorphic to Y, then 2^X_F is homeomorphic to 2^Y_F.*

Proof. Assume that X is a closed subset of a space $M \in \mathrm{AR}$ and Y is a closed subset of a space $N \in \mathrm{AR}$, and let $h\colon X \to Y$ be a homeomorphism. Then there are two maps

$$\alpha\colon M \to N \quad \text{and} \quad \hat{\alpha}\colon N \to M$$

such that

(6.2) $\alpha(x) = h(x)$ for every $x \in X$ and $\hat{\alpha}(y) = h^{-1}(y)$ for every $y \in Y$.

Let us show that the function assigning to every compactum $A \subset X$ the compactum $h(A) \subset Y$ is a homeomorphism of 2^X_F onto 2^Y_F. It is clear that this function is one to one. Since our hypotheses concerning h and h^{-1} are symmetric, it suffices to show that if $A_0, A_1, A_2, \ldots$ are compacta in X such that

$$\lim_{n=\infty} \varrho_F(A_n, A_0) = 0, \tag{6.3}$$

then

$$\lim_{n=\infty} \varrho_F(h(A_n), h(A_0)) = 0. \tag{6.4}$$

It follows by (6.3) that there exists a sequence $\varepsilon_1, \varepsilon_2, \ldots$ of positive numbers converging to zero and such that for every $n = 1, 2, \ldots$ there are two fundamental sequences

$$\underline{f}^{(n)} = \{f_k^{(n)}, A_n, A_0\}_{M,M} \quad \text{and} \quad \underline{\hat{f}} = \{\hat{f}_k^{(n)}, A_0, A_n\}_{M,M}$$

such that for every $n = 1, 2, \ldots$ there exists a neighborhood (U_n, U_{n0}) of (A_n, A_0) in (M, M) and an index k_n such that

$$\left.\begin{aligned} &\varrho(x, f_k^{(n)}(x)) < \varepsilon_n \quad \text{for every } x \in U_n \\ &\varrho(x, \hat{f}_k^{(n)}(x)) < \varepsilon_n \quad \text{for every } x \in U_{n0} \end{aligned}\right\} \text{for } k \geqslant k_n. \tag{6.5}$$

Since (U_n, U_{n0}) can be replaced by any smaller neighborhood of (A_n, A_0) in (M, M) and since the set

$$A = A_0 \cup \bigcup_{n=1}^{\infty} A_n$$

is compact, one easily sees by (6.2) that there exists a sequence $\eta_1, \eta_2, \ldots$ of positive numbers converging to 0 and such that

(6.6) If $x, x' \in M$, $x \in U_n$ and $\varrho(x, x') < \varepsilon_n$, then $\varrho(\alpha(x), \alpha(x')) < \eta_n$.

Setting

$$g_k^{(n)} = \alpha f_k^{(n)} \hat{\alpha}\colon N \to N, \quad \hat{g}_k^{(n)} = \alpha \hat{f}_k^{(n)} \hat{\alpha}\colon N \to N$$

for every $n = 1, 2, \ldots$ and

$$\underline{\alpha}^{(m)} = \{\alpha, A_m, h(A_m)\}_{M,N}, \quad \underline{\hat{\alpha}}^{(m)} = \{\hat{\alpha}, h(A_m), A_m\}_{N,M}$$

for $m = 0, 1, 2, \ldots$, we get two fundamental sequences

$$\underline{g}^{(n)} = \{g_k^{(n)}, h(A_n), h(A_0)\}_{N,N} = \underline{\alpha}^{(0)} \underline{f}^{(n)} \underline{\hat{\alpha}}^{(n)},$$
$$\underline{\hat{g}}^{(n)} = \{\hat{g}_k^{(n)}, h(A_0), h(A_n)\}_{N,N} = \underline{\alpha}^{(n)} \underline{f}^{(n)} \underline{\hat{\alpha}}^{(0)},$$

for $n = 1, 2, \ldots$

One readily sees that for every $n = 1, 2, \ldots$ there exists a neighborhood (V_n, V_{n0}) of $(h(A_n), h(A_0))$ in (N, N) such that

$$\hat{\alpha}(V_n) \subset U_n, \quad \hat{\alpha}(V_{n0}) \subset U_{n0} \tag{6.7}$$

and that

$$\varrho(\alpha\hat{\alpha}(y), y) < \eta_n \quad \text{for every point } y \in V_n \cup V_{n0}. \tag{6.8}$$

If $y \in V_n$, then

$$\varrho(y, g_k^{(n)}(y)) = \varrho(y, \alpha f_k^{(n)} \hat{\alpha}(y))$$
$$\leqslant \varrho(y, \alpha\hat{\alpha}(y)) + \varrho(\alpha\hat{\alpha}(y), \alpha f_k^{(n)} \hat{\alpha}(y)).$$

By (6.7), $\alpha(y) \in U_n$ and we infer by (6.5) that

$$\varrho(\hat{\alpha}(y), f_k^{(n)} \hat{\alpha}(y)) < \varepsilon_n \quad \text{for} \quad k \geqslant k_n.$$

Using (6.6), we infer that $\varrho(\alpha\hat{\alpha}(y), \alpha f_k^{(n)} \hat{\alpha}(y)) < \eta_n$ for $k \geqslant k_n$. It follows by (6.8) that

$$\varrho(y, g_k^{(n)}(y)) < 2\eta_n \quad \text{for} \quad y \in V_n \text{ and } k \geqslant k_n. \tag{6.9}$$

If $y \in V_{n0}$, then (6.7) gives $\hat{\alpha}(y) \in U_{n0}$ and we infer by (6.5) that

$$\varrho(\hat{\alpha}(y), \hat{f}_k^{(n)} \hat{\alpha}(y)) < \varepsilon_n \quad \text{for} \quad k \geqslant k_n.$$

Using (6.6), one gets $\varrho(\alpha\hat{\alpha}(y), \alpha\hat{f}_k^{(n)} \hat{\alpha}(y)) < \eta_n$ for $k \geqslant k_n$ and by virtue of (6.8) we obtain

$$\varrho(y, \hat{g}_k^{(n)}(y)) \leqslant \varrho(y, \alpha\hat{\alpha}(y)) + \varrho(\alpha\hat{\alpha}(y), \alpha f_k^{(n)} \hat{\alpha}(y)) < 2\eta_n \tag{6.10}$$

for $y \in V_{n0}$ and $k \geqslant k_n$.

It follows by (6.9) and (6.10) that

$$\varrho_F(h(A_n), h(A_0)) < 2\eta_n;$$

hence $\lim\limits_{n=\infty} \varrho_F(h(A_n), h(A_0)) = 0$ and the proof of Theorem (6.1) is finished.

7. Homology properties and the fundamental metric. We use in this paper the Vietoris homology theory. It is well known that if $\underline{f} = \{f_k, A, B\}_{M,N}$ is a fundamental sequence, then for every true cycle $\underline{\gamma} = \{\gamma_i\}$ in A there exists a sequence of indices $i_1 < i_2 < \ldots$ such that, for every sequence of indices $j_1, j_2, \ldots$ satisfying the inequality $j_k \geqslant i_k$ for $k = 1, 2, \ldots$, $\underline{\gamma}' = \{f_k(\gamma_{j_k})\}$ is a true cycle in B. The homology class of $\underline{\gamma}'$ depends only on the homology class of $\underline{\gamma}$, and it does not depend on the choice of the sequence of indices $j_1, j_2, \ldots$ In the sequel we shall say that the true cycle $\underline{\gamma}'$ (just as any true cycle homologous to $\underline{\gamma}'$ in B) is *assigned* by $\underline{f}$ to the true cycle $\underline{\gamma}$.

Let $\underline{f} = \{f_k, A, B\}_{M,N}$ and $\underline{g} = \{g_k, A, B\}_{M,N}$ be two fundamental sequences and let V be an open neighborhood of B in N. The fundamental sequences $\underline{f}$ and $\underline{g}$ are said to be *V-homotopic* (notation: $\underline{f} \underset{V}{\simeq} \underline{g}$) if the condition

(7.1) $$f_k/A \simeq g_k/A \quad \text{in } V \quad \text{for almost all } k$$

is satisfied.

Now let us prove the following

(7.2) LEMMA. *Let $\underline{f} = \{f_k, A, B\}_{M,N}$ and $\underline{g} = \{g_k, A, B\}_{M,N}$ be two V-homotopic fundamental sequences, where V is a compact neighborhood of B in N. Then for every true cycle $\underline{\gamma}$ in A the true cycle $\underline{\gamma}'$ and $\underline{\gamma}''$ in B assigned to $\underline{\gamma}$ by $\underline{f}$ and $\underline{g}$ respectively are homologous in V.*

Proof. We may assume that

$$\underline{\gamma}' = \{f_k(\gamma_{j_k})\}, \quad \underline{\gamma}'' = \{g_k(\gamma_{j_k})\},$$

where the sequence of indices $j_1 < j_2 < \dots$ can be selected so that the mesh of the cycle γ_{j_k} (i.e., the maximal diameter of simplexes of γ_{j_k}) is so small that the homotopy

$$f_k/A \simeq g_k/A \quad \text{in } V$$

implies that there exists in V a chain $\varkappa_k$ with a mesh $\leqslant 1/k$ such that

$$\partial \varkappa_k = f_k(\gamma_{j_k}) - g_k(\gamma_{j_k}).$$

It is clear that $\underline{\varkappa} = \{\varkappa_k\}$ is an infinite chain in V such that $\partial\underline{\varkappa} = \underline{\gamma}' - \underline{\gamma}''$. Hence $\underline{\gamma}' \sim \underline{\gamma}''$ in V and the proof of Lemma (7.2) is finished.

We shall limit ourselves in the sequel to the case of true cycles with rational coefficients. The maximal number of n-dimensional true cycles in A homologically independent in A is said to be the *n-th Betti number* of A. We denote it by $p_n(A)$.

(7.3) THEOREM. *Let $A_0, A_1, A_2, \dots$ be compacta lying in a space $M \in$ AR and let $\lim_{\nu=\infty} \varrho_F(A_\nu, A_0) = 0$. If $p_n(A_\nu) \leqslant m$ for every $\nu = 1, 2, \dots$, then $p_n(A_0) \leqslant m$.*

Proof. Since for $m = \infty$ the statement is obvious, we can assume that m is finite. Moreover, Theorem (5.1) allows us to limit ourselves to the case where M is the Hilbert cube Q.

Let $\varepsilon_\nu = \varrho_F(A_\nu, A_0)$. Then for every $\nu = 1, 2, \dots$ there exist two fundamental sequences

$$\underline{f}^{(\nu)} = \{f_k^{(\nu)}, A_\nu, A_0\}, \quad \underline{\hat{f}}^{(\nu)} = \{\hat{f}_k^{(\nu)}, A_0, A_\nu\}$$

and a neighborhood $(U_\nu, U_{\nu 0})$ of (A_ν, A_0) in (Q, Q) and an index k_ν such that

(7.4) $$\left.\begin{aligned} &\varrho(x, f_k^{(\nu)}(x)) < 2\varepsilon_\nu \quad \text{for} \quad x \in U_\nu, \\ &\varrho(x, \hat{f}_k^{(\nu)}(x)) < 2\varepsilon_\nu \quad \text{for} \quad x \in U_\nu \end{aligned}\right\} \quad \text{for } k \geqslant k_\nu.$$

Let U be an arbitrary compact neighborhood of A_0 in Q. By (7.4) there exists an index ν_0 such that

(7.5) U is a neighborhood of A_{ν_0} in Q,

(7.6) $\underline{f}^{(\nu_0)} \underset{U}{\simeq} \underline{i}_{A_0}$.

Consider a system $\underline{\gamma}_1, \underline{\gamma}_2, \dots, \underline{\gamma}_{m+1}$ of $m+1$ true n-dimensional cycles in A_0. We infer by (7.6) and by Lemma (7.2) that the fundamental sequence $\underline{f}^{(\nu_0)}$ assigns to those true cycles true n-dimensional cycles $\underline{\gamma}_1', \underline{\gamma}_2', \dots, \underline{\gamma}_{m+1}'$ in A_{ν_0} homological in U to the true cycles $\underline{\gamma}_1, \underline{\gamma}_2, \dots, \underline{\gamma}_{m+1}$, respectively. Since $p_n(A_{\nu_0}) \leqslant m$, there exist integers $l_1, l_2, \dots, l_{m+1}$ not all vanishing and such that

$$l_1 \cdot \underline{\gamma}_1' + l_2 \cdot \underline{\gamma}_2' + \dots + l_{m+1} \cdot \underline{\gamma}_{m+1}' \sim 0 \quad \text{in } A_{\nu_0} \subset U.$$

Since $\underline{\gamma}_i' \sim \underline{\gamma}_i$ in U for $i = 1, 2, \dots, m+1$, we infer that

$$l_1 \cdot \underline{\gamma}_1 + l_2 \cdot \underline{\gamma}_2 + \dots + l_{m+1} \cdot \underline{\gamma}_{m+1} \sim 0 \quad \text{in } U.$$

Thus we have shown that the system $\underline{\gamma}_1, \underline{\gamma}_2, \dots, \underline{\gamma}_{m+1}$ is homologically dependent in every neighborhood U of A_0 in Q. It is known ([3], p. 208) that then this system is homologically dependent in A_0; hence $p_n(A_0) \leqslant m$ and the proof of Theorem (7.3) is finished.

Theorem (7.3), combined with Theorem (4.1) give the following

(7.7) COROLLARY. *If $A_0, A_1, A_2, \dots$ are compacta lying in a space X and if $\lim_{\nu=\infty} \varrho_C(A_\nu, A_0) = 0$, then the inequality $p_n(A_\nu) \leqslant m$ for $\nu = 1, 2, \dots$ implies that $p_n(A_0) \leqslant m$.*

8. Quasi-domination and the fundamental metric. If A, B are compacta lying in spaces $M, N \in$ AR respectively and if U is a neighborhood of A in M, then A is said to be *U-dominated* by B (notation: $A \underset{U}{\leqslant} B$ in M; see [3], p. 198) if there exist two fundamental sequences

$$\underline{f} = \{f_k, A, B\}_{M,N} \quad \text{and} \quad \underline{\hat{f}} = \{\hat{f}_k, B, A\}_{N,M}$$

such that $\underline{\hat{f}}\underline{f} \underset{U}{\simeq} \underline{i}_{A,M}$. If the relation $A \underset{U}{\leqslant} B$ in M holds true for each neighborhood U of A in M, then A is said to be *quasi-dominated* by B (notation: $A \overset{q}{=} B$). It is known ([3], p. 203) that the choice of spaces $M, N \in$ AR containing A and B is immaterial for the relation $A \overset{q}{\leqslant} B$. Moreover, it is known ([3], p. 210) that the relation $A \overset{q}{\leqslant} B$ implies that $p_n(A) \leqslant p_n(B)$ for every $n = 0, 1, 2, \dots$ and that the movability of B implies the movability of A.

Let us prove the following

(8.1) THEOREM. *Let $A_0, A_1, A_2, \dots$ be compacta lying in a space $M \in$ AR and let $\lim_{\nu=\infty} \varrho_F(A_\nu, A_0) = 0$. If B is a compactum such that $A_\nu \overset{q}{\leqslant} B$ for every $\nu = 1, 2, \dots$, then $A_0 \overset{q}{\leqslant} B$.*

Proof. By Theorem (5.1) we can assume that A and B are subsets of the Hilbert space E^ω. Let $\varepsilon_\nu = \varrho_F(A_\nu, A_0)$. Hence $\lim_{\nu=\infty} \varepsilon_\nu = 0$ and for every $\nu = 1, 2, \ldots$ there exist two fundamental sequences

$$\underline{f}^{(\nu)} = \{f_k^{(\nu)}, A_\nu, A_0\}_{E^\omega, E^\omega}, \quad \underline{\hat{f}}^{(\nu)} = \{\hat{f}_k^{(\nu)} A_0, A_\nu\}_{E^\omega, E^\omega},$$

a neighborhood $(\hat{U}_\nu, \hat{U}_{\nu 0})$ of (A_ν, A_0) in (E^ω, E^ω) and an index k_ν such that

$$(8.2)\qquad \begin{aligned} &\varrho(x, f_k^{(\nu)}(x)) < 2\varepsilon_\nu \quad \text{for every } x \in \hat{U}_\nu, \\ &\varrho(x, \hat{f}_k^{(\nu)}(x)) < 2\varepsilon_\nu \quad \text{for every } x \in \hat{U}_{\nu 0} \end{aligned} \quad \text{for } k \geqslant k_\nu.$$

We can assume that $\hat{U}_\nu$ and $\hat{U}_{\nu 0}$ are closed in E^ω. Let H_ν denote the open ball in E^ω with centre $(0, 0, \ldots)$ and radius $2\varepsilon_\nu$. Setting

$$\alpha_k^{(\nu)}(x) = f_k^{(\nu)}(x) - x \quad \text{for} \quad x \in \hat{U}_\nu,$$
$$\hat{\alpha}_k^{(\nu)}(x) = \hat{f}_k^{(\nu)}(x) - x \quad \text{for} \quad x \in \hat{U}_{\nu 0},$$

we get for every $k \geqslant k_\nu$ two maps:

$$\alpha_k^{(\nu)}\colon \hat{U}_\nu \to H_\nu \quad \text{and} \quad \hat{\alpha}_k^{(\nu)}\colon \hat{U}_{\nu 0} \to H_\nu.$$

Since $H_\nu \in \mathrm{AR}$, we can extend $\alpha_k^{(\nu)}$ and $\hat{\alpha}_k^{(\nu)}$ (for $k \geqslant k_\nu$) to maps

$$\beta_k^{(\nu)}, \hat{\beta}_k^{(\nu)}\colon E^\omega \to H_\nu.$$

Setting, for $k \geqslant k_\nu$:

$$\begin{aligned} f_k'^{(\nu)}(x) &= x + \beta_k^{(\nu)}(x) \\ \hat{f}_k'^{(\nu)}(x) &= x + \hat{\beta}_k^{(\nu)}(x) \end{aligned} \quad \text{for} \quad x \in E^\omega,$$

we get maps $f'^{(\nu)}, \hat{f}_k'^{(\nu)}\colon E^\omega \to E^\omega$ which coincide on $\hat{U}_\nu$ and on $\hat{U}_{\nu 0}$ with $f_k^{(\nu)}$ and $\hat{f}_k^{(\nu)}$ respectively and which satisfy the conditions:

$$\varrho(x, f_k'^{(\nu)}(x)) < 2\varepsilon_\nu, \quad \varrho(x, \hat{f}_k'^{(\nu)}(x)) < 2\varepsilon_\nu$$

for every point $x \in E^\omega$. Setting $f_k'^{(\nu)} = f_k^{(\nu)}$ for $k < k_\nu$, we easily see that $\underline{f}'^{(\nu)} = \{f_k'^{(\nu)}, A_\nu, A_0\}_{E^\omega, E^\omega}$ is a fundamental sequence homotopic to $\underline{f}^{(\nu)}$ and $\underline{\hat{f}}'^{(\nu)} = \{\hat{f}_k'^{(\nu)}, A_0, A_\nu\}_{E^\omega, E^\omega}$ is a fundamental sequence homotopic to $\underline{\hat{f}}^{(\nu)}$.

It follows that by replacing $\underline{f}^{(\nu)}$ by $\underline{f}'^{(\nu)}$ and $\underline{\hat{f}}^{(\nu)}$ by $\underline{\hat{f}}'^{(\nu)}$ we may assume that the neighborhoods $\hat{U}_\nu$ and $\hat{U}_{\nu 0}$ in condition (8.2) coincide with the whole space E: hence

$$(8.3)\qquad |f_k^{(\nu)}| \leqslant 2\varepsilon_\nu \quad \text{and} \quad |\hat{f}_k^{(\nu)}| \leqslant 2\varepsilon_\nu \quad \text{for} \quad k \geqslant k_\nu.$$

Now consider an arbitrary open neighborhood U of A_0 in E^ω. Then, for almost all ν, U is a neighborhood of A_ν. It follows by (8.3) that there exists an open neighborhood $U' \subset U$ of A_0 in E^ω and an index ν_1 such that for $\nu \geqslant \nu_1$:

$$(8.4)\qquad f_k^{(\nu)}(U') \subset U \quad \text{for almost all } k$$

and that

(8.5) U' is a neighborhood of A_ν.

Since $A_\nu \overset{q}{\leqslant} B$, we infer by (8.5) that $A_\nu \underset{U'}{\leqslant} B$ for every $\nu \geqslant \nu_1$. Consequently for every $\nu \geqslant \nu_1$ there exist two fundamental sequences

$$\underline{g}^{(\nu)} = \{g_k^{(\nu)}, A_\nu, B\}_{E^\omega, E^\omega}, \quad \underline{\hat{g}}^{(\nu)} = \{\hat{g}_k^{(\nu)}, B, A_\nu\}_{E^\omega, E^\omega}$$

such that there exists a neighborhood $\check{U}_\nu$ of A_ν in E^ω satisfying the condition

$$(8.6)\qquad \hat{g}_k^{(\nu)} g_k^{(\nu)} / \check{U}_\nu \simeq i / \check{U}_\nu \quad \text{in } U' \quad \text{for almost all } k.$$

By (8.3) the map $f_k^{(\nu)} \hat{f}_k^{(\nu)}\colon E^\omega \to E^\omega$ satisfies the condition $|f_k^{(\nu)} \hat{f}_k^{(\nu)}| \leqslant 4\varepsilon_\nu$ for $k \geqslant k_\nu$. It follows that there exist an index $\nu_0 \geqslant \nu_1$ and a neighborhood W of A_0 in E^ω such that

$$(8.7)\qquad f_k^{(\nu_0)} \hat{f}_k^{(\nu_0)} / W \simeq i / W \quad \text{in } U \quad \text{for almost all } k.$$

Without changing ν_0, we can select the neighborhood W of A_0 so that

$$(8.8)\qquad \hat{f}_k^{(\nu_0)}(W) \subset \check{U}_{\nu_0} \quad \text{for almost all } k.$$

Now let us set

$$h_k = g_k^{(\nu_0)} \hat{f}_k^{(\nu_0)}, \quad \hat{h}_k = f_k^{(\nu_0)} \hat{g}_k^{(\nu_0)} \quad \text{for} \quad k = 1, 2, \ldots$$

Then

$$\underline{h} = \{h_k, A_0, B\}_{E^\omega, E^\omega} = \underline{g}^{(\nu_0)} \underline{\hat{f}}^{(\nu_0)}\colon A_0 \to B,$$
$$\underline{\hat{h}} = \{\hat{h}_k, B, A_0\}_{E^\omega, E^\omega} = \underline{f}^{(\nu_0)} \underline{\hat{g}}^{(\nu_0)}\colon B \to A_0$$

are fundamental sequences. By virtue of (8.8) and (8.6)

$$\hat{g}_k^{(\nu_0)} g_k^{(\nu_0)} \hat{f}_k^{(\nu_0)} / W \simeq \hat{f}_k^{(\nu_0)} / W \quad \text{in } U' \quad \text{for almost all } k.$$

It follows by (8.4) that

$$\hat{h}_k h_k / W = f_k^{(\nu_0)} \hat{g}_k^{(\nu_0)} g_k^{(\nu_0)} \hat{f}_k^{(\nu_0)} / W \simeq f_k^{(\nu_0)} \hat{f}_k^{(\nu_0)} / W \quad \text{in } U \quad \text{for almost all } k.$$

Using (8.7), we infer that $\hat{h}_k h_k / W \simeq i / W$ in U for almost all k, and hence A_0 is U-dominated by B. Since U was an arbitrary open neighborhood of A_0 in E^ω, we infer that $A_0 \overset{q}{\leqslant} B$ and the proof of Theorem (8.1) is finished.

(8.9) Problem. *Is it true that if $A_0, A_1, A_2, \ldots$ are compacta lying in a space X and satisfying the condition $\lim_{\nu=\infty} \varrho_F(A_\nu, A_0) = 0$ and if B is a compactum such that $\mathrm{Sh}(A_\nu) \leqslant \mathrm{Sh}(B)$ for $\nu = 1, 2, \ldots$, then $\mathrm{Sh}(A_0) \leqslant \mathrm{Sh}(B)$?*

9. Movability and the fundamental metric. A compactum A lying in a space $M \in \mathrm{AR}$ is said to be *movable* if for every neighborhood U of A in M there exists a neighborhood $U_0 \subset U$ of A in M with the property that for every neighborhood V of A (in M) the inclusion

$$i\colon U_0 \to U$$

is homotopic to a map with all values in V. Let us say that U_0 *realizes the movability of* A *in* U. It is well known that the movability of A does not depend on the choice of the space $M \in \mathrm{AR}$ containing A and that it is a hereditary shape invariant.

Let us prove the following

(9.1) THEOREM. *Let* $A_0, A_1, \ldots$ *be compacta lying in a space* X *and let* $\lim\limits_{\nu=\infty} \varrho_F(A_\nu, A_0) = 0$. *If* A_ν *is movable for* $\nu = 1, 2, \ldots$, *then* A_0 *is also movable.*

Proof. We may assume that X is a subset of the Hilbert cube Q and that $A_\nu \neq A_0$ for $\nu = 1, 2, \ldots$ If $\varepsilon_\nu = \varrho_F(A_\nu, A_0)$, then for every $\nu = 1, 2, \ldots$ there exist two fundamental sequences

$$\underline{f}^{(\nu)} = \{f_k^{(\nu)}, A_\nu, A_0\}, \quad \underline{\hat{f}}^{(\nu)} = \{\hat{f}_k^{(\nu)}, A_0, A_\nu\}$$

and a neighborhood $(U_\nu, U_{\nu 0})$ of (A_ν, A_0) in (Q, Q) such that:

$$\text{(9.2)} \qquad \begin{matrix} \varrho(x, f_k^{(\nu)}(x)) < 2\varepsilon_\nu & \text{for} & x \in U_\nu, \\ \varrho(x, \hat{f}_k^{(\nu)}(x)) < 2\varepsilon_\nu & \text{for} & x \in U_{\nu 0} \end{matrix} \quad \text{for almost all } k.$$

Consider a neighborhood U of A_0 in Q. One easily sees (because $\lim\limits_{\nu\to\infty} \varepsilon_\nu = 0$) that there is an index ν_0 such that

(9.3) U is a neighborhood of A_{ν_0},

(9.4) There exists a neighborhood V_0 of A_0 (in Q) such that if $x \in V_0$, $y \in Q$ and $\varrho(x, y) < 2\varepsilon_{\nu_0}$, then $tx + (1-t)y \in U$ for $0 \leqslant t \leqslant 1$.

(9.5) There exists a neighborhood W_0 of A_{ν_0} (in Q) such that if $x \in W_0$, $y \in Q$ and $\varrho(x, y) < 2\varepsilon_{\nu_0}$ then, $tx + (1-t)y \in U$ for $0 \leqslant t \leqslant 1$.

Since A_{ν_0} is movable, there is a neighborhood W_0' of A_{ν_0} in Q realizing the movability of A_{ν_0} in U. Since $\underline{\hat{f}}^{(\nu_0)}$ is a fundamental sequence, there exists a neighborhood U_0 of A_0 in Q such that $U_0 \subset V_0 \cap U_{\nu_0 0}$ and that

$$\text{(9.6)} \qquad \hat{f}_k^{(\nu_0)}(U_0) \subset W_0' \quad \text{for almost all } k.$$

Let us show that U_0 realizes the movability of A_0 in U. Consider a neighborhood V of A_0 in Q. By (9.6), (9.2) and (9.4) there is an index k_1 such that

$$\text{(9.7)} \qquad \hat{f}_k^{(\nu_0)}(U_0) \subset W_0' \quad \text{and} \quad \hat{f}_k^{(\nu_0)}/U_0 \simeq i/U_0 \quad \text{in } U \quad \text{for } k \geqslant k_1.$$

Since $\underline{f}^{(\nu_0)}$ is a fundamental sequence, we infer by (9.2) and (9.5) that there exist a neighborhood $W \subset W_0$ of A_{ν_0} in Q and an index $k_0 \geqslant k_1$ such that

$$\text{(9.8)} \qquad f_{k_0}^{(\nu_0)}(W) \subset V \quad \text{and} \quad f_{k_0}^{(\nu_0)}/W \simeq i/W \quad \text{in } U.$$

The inclusion $U_0 \subset V_0$ and (9.4) imply that

$$\hat{f}_{k_0}^{(\nu_0)}/U_0 \simeq i/U_0 \quad \text{in } U.$$

It follows by (9.7) that there is a deformation in U carrying U_0 onto the set $\hat{f}_{k_0}^{(\nu_0)}(U_0) \subset W_0'$. Since W_0' realizes the movability of A_{ν_0} in U, there is a deformation in U of the set $\hat{f}_{k_0}^{(\nu_0)}(U_0)$ onto a subset Z of W. It follows by (9.8) that Z can be carried, by a deformation in U, onto a subset of the set $f_{k_0}^{(\nu_0)}(W) \subset V$. Thus we have shown that one can obtain from U_0, by a deformation in U, a subset of an arbitrarily given neighborhood V of A_0. Hence A_0 is movable and Theorem (9.1) is proved.

(9.9) PROBLEM. *Let* α *be a hereditary shape property (that is, if* $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$ *and* $X \in \alpha$, *then* $Y \in \alpha$*). Is it true that for every sequence* $A_0, A_1, \ldots$ *of compacta lying in a compactum* X, *the two conditions*

1^0 $\lim\limits_{\nu=\infty} \varrho_F(A_\nu, A_0) = 0$,

2^0 $A_\nu \in \alpha$ *for* $\nu = 1, 2, \ldots$

imply that $A_0 \in \alpha$?

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