it follows that

$$\lim_{x\to 0}\frac{1}{x}(D)\int_0^x g(t)\,dt = g(0) = 0\,.$$

If $f(x)g(x)$ is not D-integrable, there is nothing to prove. If it is, then

$$\frac{1}{b_n}(D)\int_0^{b_n} f(x)g(x)\,dx \geqslant \frac{1}{b_n}\int_{a_n}^{b_n} f(x)g(x)\,dx \geqslant \frac{1}{2}\,.$$

Since $f(0)g(0) = 0$, $f(x)g(x)$ cannot be a derivative and the proof of Theorem 10 is complete.

THEOREM 11. *A function $f(x)$ belongs to A if and only if it is of distant bounded variation at each point x of* $[0, 1]$.

Proof. Necessity is shown in Theorem 10. To see that this condition is sufficient, we note that the definition of distant bounded variation entails that $f(x)$ be a bounded derivative. If there were infinitely many points at which $f(x)$ is of unbounded variation, they would have a limit point x_0 in $[0, 1]$. Then $f(x)$ would be of unbounded variation in every interval of the form $[x_0+\delta_1, x_0+\delta_2]$ (or $[x_0-\delta_2, x_0-\delta_1]$) for all $\delta_2>0$ and all sufficiently small $\delta_1>0$. Consequently, $f\notin$ BVD at x_0. Hence, there are at most finitely many such points and f is of distant bounded variation at each of them. Thus sufficiency follows from Theorem 7 and the proof of Theorem 11 is complete.

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Baire category from an abstract viewpoint *

by

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Abstract. The object of Point Set Theory is the investigation of methods of classifying point sets, their common properties, and their interrelationships. Included in the domain of this subject are Baire category, Lebesgue measure, Hausdorff measure, dimension, and sets of uniqueness for trigonometric series. In this paper we present a general framework for these investigations.

Introduction. In 1905 H. Lebesgue proved some basic theorems concerning sets which have the Baire property and, in particular, he proved the fundamental theorem that a linear set of the second category is everywhere of the second category in some interval (see [10], pp. 185–186). This theorem was undoubtedly known to R. Baire and he had stated earlier the analogous theorem for sets of the second category in the space of all infinite sequences of natural numbers (see [1], p. 948).

S. Banach [2] generalized the fundamental theorem to arbitrary metric spaces in 1930 and subsequently to topological spaces (see [9]). A further extension of this theorem was obtained in [13] (see Theorem 2 below). This new generalization forms the basis of an abstract theory of Baire category, an outline of which is presented in this paper. One of the main consequences of this abstract point of view is the unification of certain analogies which have been observed between properties of Lebesgue measurable sets and sets which have the Baire property (see [5], pp. 19–22, [7], [17], [20], and [22] concerning these analogies).

1. $\mathfrak{K}$-families. Let X be a (nonempty) set. Members of any family $\mathcal{A}$ of subsets of x will be called *$\mathcal{A}$-sets.*

DEFINITION 1. A family $\mathcal{C}$ of subsets of X is called a *$\mathfrak{K}$-family* if the following axioms are satisfied.

1. For each point $x\in X$ there is a $\mathcal{C}$-set containing x; i.e. $X = \bigcup \mathcal{C}$.

2. Let A be a $\mathcal{C}$-set and let $\mathcal{D}$ be a nonempty family of disjoint $\mathcal{C}$-sets which has power less than the power of $\mathcal{C}$.

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a. If $A \cap (\bigcup \mathscr{D})$ contains a $\mathscr{C}$-set, then there is a $\mathscr{D}$-set D such that $A \cap D$ contains a $\mathscr{C}$-set.

b. If $A \cap (\bigcup \mathscr{D})$ contains no $\mathscr{C}$-set, then there is a $\mathscr{C}$-set $B \subset A$ which is disjoint from all $\mathscr{D}$-sets.

The notion of a $\mathfrak{K}$-family generalizes that of an $\mathfrak{M}$-family introduced in [13], Section 2 (see also Section 4 of the present paper).

Note that if the $\mathscr{C}$-sets containing a given point x are called neighborhoods of x, then Hausdorff's Neighborhood Axioms A and C for a topological space, given in [4], p. 213, are satisfied. As is evident from examples below, Hausdorff's Axioms B and D are not true for all $\mathfrak{K}$-families. The concept of a $\mathfrak{K}$-family is thus intermediate between that of a Fréchet (V) space (see [21], p. 3) and that of a topological space.

Unless otherwise specified, the symbol $\mathscr{C}$ will denote a $\mathfrak{K}$-family.

Definition 2 (see [13], p. 8). A set $S \subset X$ is *$\mathscr{C}$-singular* if each $\mathscr{C}$-set A contains a $\mathscr{C}$-set B which is disjoint from S. The family of all countable unions of $\mathscr{C}$-singular sets is denoted by $\mathscr{C}_{\mathrm{I}}$. The family of all subsets of X which are not $\mathscr{C}_{\mathrm{I}}$-sets is denoted by $\mathscr{C}_{\mathrm{II}}$. If X is a $\mathscr{C}_{\mathrm{II}}$-set, then the complement of a $\mathscr{C}_{\mathrm{I}}$-set is called a *$\mathscr{C}$-residual set*.

Clearly, the $\mathscr{C}$-singular sets form an ideal. Hence $\mathscr{C}_{\mathrm{I}}$ is a σ-ideal.

The following are some of the main examples of $\mathfrak{K}$-families. Additional examples are given in Section 4.

Example 1. Let $\mathscr{C}$ be the family of all nonempty open sets in a topological space. Then the $\mathscr{C}$-singular and the $\mathscr{C}_{\mathrm{I}}$-sets coincide with the nowhere dense sets and the sets of the first category, respectively.

Example 2. Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and let $\mathscr{C}$ be the family of all $\mathscr{A}$-sets of positive measure. Let $(X, \bar{\mathscr{A}}, \bar{\mu})$ be the completion of $(X, \mathscr{A}, \mu)$. In this example the $\mathscr{C}$-singular and $\mathscr{C}_{\mathrm{I}}$-sets are the same and coincide with the sets of $\bar{\mu}$-measure zero.

Remark. This example can be generalized as follows. If $\mathscr{A}$ is a σ-field of subsets of X and $\mathscr{I}$ is a proper σ-ideal in $\mathscr{A}$ such that $\mathscr{A} - \mathscr{I}$ satisfies CCC (the countable chain condition), then $\mathscr{C} = \mathscr{A} - \mathscr{I}$ is a $\mathfrak{K}$-family in which the $\mathscr{C}$-singular and $\mathscr{C}_{\mathrm{I}}$-sets coincide and

$$\mathscr{C}_{\mathrm{I}} = \{S \subset X\colon\ S \subset I \text{ for some } I \in \mathscr{I}\}.$$

Indeed, suppose $S \subset X$ is $\mathscr{C}$-singular. There is then a $\mathscr{C}$-set A such that $A \subset X - S$ and $B \notin \mathscr{C}$ whenever $B \subset X - S$ and $A \cap B = \emptyset$. Hence $S \subset X - A$ and $X - A \in \mathscr{I}$.

Example 3. Let $X = \bigcup_{\alpha} A_{\alpha}$ be a decomposition of a set into nonempty disjoint sets and let $\mathscr{C}$ consist of all the sets A_{α}. The empty set is then the only $\mathscr{C}_{\mathrm{I}}$-set.

Example 4. Let X be an uncountable, complete separable metric space and let $\mathscr{C}$ be the family of all uncountable Borel sets. Then the $\mathscr{C}$-singular and $\mathscr{C}_{\mathrm{I}}$-sets coincide (see [24], p. 25).

In Examples 2 and 4 it is easily seen that Hausdorff's Axiom B concerning the intersection of neighborhoods need not be satisfied. For the intersection of two $\mathscr{C}$-sets we do however have the following result which will be used frequently.

Theorem 1. *If A and B are $\mathscr{C}$-sets, then either $A \cap B$ contains a $\mathscr{C}$-set or $A \cap B$ is $\mathscr{C}$-singular.*

Proof. Assume $A \cap B$ contains no $\mathscr{C}$-set and let C be any $\mathscr{C}$-set. If $C \cap A$ contains a $\mathscr{C}$-set D then $D \cap B \subset A \cap B$ and consequently $D \cap B$ contains no $\mathscr{C}$-set. Therefore, there is a $\mathscr{C}$-set $E \subset D - B$ which is disjoint from $A \cap B$. On the other hand, if $C \cap A$ contains no $\mathscr{C}$-set then there is a $\mathscr{C}$-set $E \subset C - A$ which is disjoint from $A \cap B$. Thus $A \cap B$ is $\mathscr{C}$-singular.

Definition 3. A set $S \subset X$ is a *$\mathscr{C}_{\mathrm{II}}$-set on a $\mathscr{C}$-set A* if $S \cap A$ is a $\mathscr{C}_{\mathrm{II}}$-set. S is a *$\mathscr{C}_{\mathrm{II}}$-set everywhere* if S is a $\mathscr{C}_{\mathrm{II}}$-set on every $\mathscr{C}$-set. S is a *$\mathscr{C}_{\mathrm{II}}$-set everywhere on a $\mathscr{C}$-set A* if S is a $\mathscr{C}_{\mathrm{II}}$-set on every $\mathscr{C}$-set $B \subset A$.

The next two lemmas were originally formulated in [13], pp. 9–10, for $\mathfrak{M}$-families. However, as noted in a remark on p. 11 of that paper, these lemmas are also true for $\mathfrak{K}$-families. Lemma 2 below states an additional property which is easily verified.

Lemma 1. *If $\mathscr{C}$ is a $\mathfrak{K}$-family of subsets of X and $\mathscr{N}$ is a subfamily of $\mathscr{C}$ with the property that each $\mathscr{C}$-set contains an $\mathscr{N}$-set, then $\mathscr{N}$ is a $\mathfrak{K}$-family of subsets of $Y = \bigcup \mathscr{N}$ and the $\mathscr{N}$-singular sets coincide with the $\mathscr{C}$-singular subsets of Y. In addition, if U is a subset of Y and $Y - U$ is $\mathscr{N}$-singular, then $X - U$ is $\mathscr{C}$-singular.*

Lemma 2. *Each $\mathfrak{K}$-family $\mathscr{N}$ of subsets of a set Y contains a maximal subfamily, $\mathscr{M}$ of disjoint sets such that $Y - \bigcup \mathscr{M}$ is $\mathscr{N}$-singular. Moreover, $\mathscr{M}$ may be so selected that for every $\mathscr{N}$-set N there is an $\mathscr{M}$-set M such that $N \cap M$ contains an $\mathscr{N}$-set.*

The generalization of Banach's Theorem given in [13] was proven by means of a game-theoretic argument and the preceding two lemmas. It can also be established using these lemmas and a slight modification of Banach's proof (see [2] and [3], p. 132). Thus we have the following fundamental theorem.

Theorem 2 ([13], Corollary 1). *If S is a $\mathscr{C}_{\mathrm{II}}$-set, then S is a $\mathscr{C}_{\mathrm{II}}$-set everywhere on some $\mathscr{C}$-set.*

In the following, $\mathscr{E}$ will denote the family of all sets which are complements of members of $\mathscr{C}$.

Theorem 3. *If $\mathscr{C}$ satisfies* CCC *then each $\mathscr{C}$-singular set is contained in a $\mathscr{C}$-singular $\mathscr{E}_{\delta}$-set, and each $\mathscr{C}_{\mathrm{I}}$-set is contained in an $(\mathscr{E}_{\delta\sigma} \cap \mathscr{C}_{\mathrm{I}})$-set.*

Proof. Let S be $\mathscr{C}$-singular and let $\mathscr{N}$ be the family of all $\mathscr{C}$-sets disjoint from S. Apply Lemmas 1 and 2 to obtain a maximal family $\mathscr{M} = \{M_0, M_1, M_2, \ldots\}$ of disjoint $\mathscr{C}$-sets such that $X - \bigcup_k M_k$ is $\mathscr{C}$-singular. Then we have $S \subset \bigcap_k (X - M_k)$, establishing the first assertion. The second assertion about $\mathscr{C}_{\mathrm{I}}$-sets is an immediate consequence of the first.

2. The Baire property.

DEFINITION 4. A set $S \subset X$ has the Baire property with respect to $\mathscr{C}$ if for every $\mathscr{C}$-set A there is a $\mathscr{C}$-set $B \subset A$ such that either $B \cap S$ or $B \cap (X-S)$ is a $\mathscr{C}_{\mathrm{I}}$-set.

The family of sets which have the Baire property with respect to $\mathscr{C}$ will be denoted by $\mathfrak{B}(\mathscr{C})$.

In Example 1, $\mathfrak{B}(\mathscr{C})$ is the family of sets which have the Baire property in the classical sense (see [9], p. 88, condition 4); in Example 2, $\mathfrak{B}(\mathscr{C})$ is the family of sets $\mathscr{A}$; in Example 3, $\mathfrak{B}(\mathscr{C})$ consists of all those sets representable as a union of $\mathscr{C}$-sets; and in Example 4, $\mathfrak{B}(\mathscr{C})$ is the family of Marczewski sets, i.e. all sets with property (s) (see [24], 2.1).

THEOREM 4. *$\mathfrak{B}(\mathscr{C})$ is a σ-field containing all $\mathscr{C}$-sets and all $\mathscr{C}_{\mathrm{I}}$-sets.*

Proof. A simple transposition of Lebesgue's proof ([10], pp. 186–187) for the classical case shows $\mathfrak{B}(\mathscr{C})$ is a σ-field. That $\mathfrak{B}(\mathscr{C})$ contains all $\mathscr{C}$-sets is an immediate consequence of Axiom 2 of Definition 1. Obviously, all $\mathscr{C}_{\mathrm{I}}$-sets belong to $\mathfrak{B}(\mathscr{C})$.

THEOREM 5. *If $S \in \mathfrak{B}(\mathscr{C})$ and S is a $\mathscr{C}_{\mathrm{II}}$-set everywhere on a $\mathscr{C}$-set A, then $A-S$ is a $\mathscr{C}_{\mathrm{I}}$-set.*

Proof. Assume $A-S$ is a $\mathscr{C}_{\mathrm{II}}$-set. Then, by Theorem 2, $A-S$ is a $\mathscr{C}_{\mathrm{II}}$-set everywhere on some $\mathscr{C}$-set B. By Theorem 1, $A \cap B$ contains a $\mathscr{C}$-set C. The sets S and $X-S$ are then both $\mathscr{C}_{\mathrm{II}}$-sets everywhere on C. Therefore $S \notin \mathfrak{B}(\mathscr{C})$.

THEOREM 6. *If $\mathscr{C}$ satisfies* CCC *then the following statements are equivalent for a set $S \subset X$.*

(i) $S \in \mathfrak{B}(\mathscr{C})$.

(ii) *$S = (A-P) \cup R$, where A is a $\mathscr{C}_{\sigma}$-set and P, R are $\mathscr{C}_{\mathrm{I}}$-sets.*

(iii) *$S = (B-Q) \cup T$, where B is an $\mathscr{E}_{\delta}$-set and Q, T are $\mathscr{C}_{\mathrm{I}}$-sets.*

(iv) *S is the union of a $\mathscr{C}_{\sigma\delta}$-set and a $\mathscr{C}_{\mathrm{I}}$-set.*

(v) *S is the difference of an $\mathscr{E}_{\delta\sigma}$-set and a $\mathscr{C}_{\mathrm{I}}$-set.*

Proof. (i)→(ii). By a $\mathscr{C}_{\sigma}$-set we mean the union of a countable family of $\mathscr{C}$-sets. Being the union of the empty family of $\mathscr{C}$-sets, the empty set is thus a $\mathscr{C}_{\sigma}$-set. Therefore, condition (ii) is obviously satisfied whenever S is a $\mathscr{C}_{\mathrm{I}}$-set.

Suppose S is a $\mathscr{C}_{\mathrm{II}}$-set in $\mathfrak{B}(\mathscr{C})$. We shall prove by induction that there exists a countable family $\mathscr{M} = \{M_k\colon k = 1, 2, \ldots\}$ of disjoint $\mathscr{C}$-sets such that S is a $\mathscr{C}_{\mathrm{II}}$-set everywhere on each M_k and $S - (\bigcup_k M_k)$ is a $\mathscr{C}_{\mathrm{I}}$-set. According to Theorem 5, the set $\bigcup_k (M_k - S)$ is then a $\mathscr{C}_{\mathrm{I}}$-set and

$$S = [(\bigcup_k M_k) - \bigcup_k (M_k - S)] \cup [S - \bigcup_k M_k]$$

is a representation of the desired form.

Denote by Λ the smallest ordinal number whose power is equal to the power of the family $\mathscr{C}^*$ of all $\mathscr{C}$-sets on which S is a $\mathscr{C}_{\mathrm{II}}$-set everywhere and let

$$(1) \qquad B_0, B_1, \ldots, B_\alpha, \ldots, \quad \alpha < \Lambda$$

be a well-ordering of $\mathscr{C}^*$.

Set $T_0 = S$ and $A_0 = B_0$. Assume $0 < \alpha < \Lambda$ and we have already defined a family $\mathscr{D}_\alpha = \{A_\xi\colon \xi < \alpha\}$ of disjoint $\mathscr{C}^*$-sets.

If $T_\alpha = S - \bigcup \mathscr{D}_\alpha$ is a $\mathscr{C}_{\mathrm{I}}$-set then we already have the decomposition

$$S = [(\bigcup_{\xi<\alpha} A_\xi) - \bigcup_{\xi<\alpha} (A_\xi - S)] \cup [S - \bigcup_{\xi<\alpha} A_\xi]$$

in which, by virtue of CCC, α is a countable ordinal number. In this case we define $A_\beta = A_0$ for all $\beta \geqslant \alpha$.

Suppose T_α is a $\mathscr{C}_{\mathrm{II}}$-set. By Theorem 2, we know T_α is a $\mathscr{C}_{\mathrm{II}}$-set everywhere on some $\mathscr{C}^*$-set. Let B be the first such set in (1). We show first that $B \cap (\bigcup \mathscr{D}_\alpha)$ contains no $\mathscr{C}$-set. Assume, to the contrary, that $B \cap (\bigcup \mathscr{D}_\alpha)$ does contain a $\mathscr{C}$-set C. Since T_α is a $\mathscr{C}_{\mathrm{II}}$-set everywhere on B and $C \subset B$, the $\mathscr{C}$-set C is a $\mathscr{C}_{\mathrm{II}}$-set and consequently $B \cap (\bigcup \mathscr{D}_\alpha)$ is a $\mathscr{C}_{\mathrm{II}}$-set. The family $\mathscr{D}_\alpha$ being countable, it follows from Theorem 1 that there is an ordinal number $\beta < \alpha$ such that $B \cap A_\beta$ contains a $\mathscr{C}$-set D. The set T_α has the Baire property and is a $\mathscr{C}_{\mathrm{II}}$-set everywhere on D. Hence there is a $\mathscr{C}$-set $E \subset D$ such that

$$E \cap (X - T_\alpha) = [E \cap (X-S)] \cup [E \cap (\bigcup_{\xi<\alpha} A_\xi)]$$

is a $\mathscr{C}_{\mathrm{I}}$-set. But this contradicts the facts that $E \subset A_\beta$ and E is a $\mathscr{C}_{\mathrm{II}}$-set (because $E \cap T_\alpha$ is a $\mathscr{C}_{\mathrm{II}}$-set). Therefore $B \cap (\bigcup \mathscr{D}_\alpha)$ contains no $\mathscr{C}$-set. Now, the family $\mathscr{D}_\alpha$ has power less than the power of $\mathscr{C}^*$, hence has power less than the power of $\mathscr{C}$, and it follows from Axiom 2b of Definition 1 that there are $\mathscr{C}$-sets contained in $B - \bigcup \mathscr{D}_\alpha$. Moreover, such $\mathscr{C}$-sets are also $\mathscr{C}^*$-sets. Accordingly, we define A_α to be the first $\mathscr{C}^*$-set in (1) which is contained in $B - \bigcup \mathscr{D}_\alpha$.

Proceeding in this manner, we obtain a family $\mathscr{M} = \{A_\alpha\colon \alpha < \Lambda\}$ consisting of countably many distinct, disjoint $\mathscr{C}^*$-sets $M_1, M_2, \ldots, M_k, \ldots$

Assume $T = S - \bigcup_{\alpha<\Lambda} A_\alpha$ is a $\mathscr{C}_{\mathrm{II}}$-set. For each $\alpha < \Lambda$, the set T_α will then be a $\mathscr{C}_{\mathrm{II}}$-set and the sets A_α, $\alpha < \Lambda$, will all be distinct. Define, for each $\alpha < \Lambda$, $B_{\varphi(\alpha)}$ to be the first set in (1) on which T_α is a $\mathscr{C}_{\mathrm{II}}$-set everywhere. Note that, by the definition of A_α, we have $A_\alpha \subset B_{\varphi(\alpha)}$ for each α. We show φ is a strictly increasing function. Suppose $0 \leqslant \beta < \alpha < \Lambda$. The set $T_\beta \supset T_\alpha$ is a $\mathscr{C}_{\mathrm{II}}$-set everywhere on $B_{\varphi(\alpha)}$. Hence, by definition of $B_{\varphi(\beta)}$, we must have $\varphi(\beta) \leqslant \varphi(\alpha)$. But we cannot have $\varphi(\beta) = \varphi(\alpha)$. For, if this were the case, then T_α would be a $\mathscr{C}_{\mathrm{II}}$-set everywhere on $B_{\varphi(\beta)} = B_{\varphi(\alpha)}$ and, as $A_\beta \subset B_{\varphi(\beta)}$, the set $T_\alpha \cap A_\beta$ would be a $\mathscr{C}_{\mathrm{II}}$-set, which is impossible. Thus φ is defined for all $\alpha < \Lambda$ and is strictly increasing; in particular, we have $\alpha \leqslant \varphi(\alpha)$ for all $\alpha < \Lambda$. Now, we know from Theorem 2 that there is a $\mathscr{C}^*$-set in (1), say B_γ, on which T is a $\mathscr{C}_{\mathrm{II}}$-set everywhere. On B_γ, the set T_γ is then also a $\mathscr{C}_{\mathrm{II}}$-set everywhere. Consequently we must have $\varphi(\gamma) = \gamma$. But this implies $A_\gamma \subset B_\gamma$ and

$T\cap A_\gamma$ is a $\mathscr{C}_{\mathrm{II}}$-set; whereas $T\cap A_\gamma=\emptyset$. Therefore $T=S-\bigcup_{\alpha<A}A_\alpha=S-\bigcup_k M_k$ is a $\mathscr{C}_{\mathrm{I}}$-set.

The implication (i)→(ii) is thus established.

(ii)→(iv). By Theorem 3, there is an $(\mathscr{E}_{\delta\sigma}\cap\mathscr{C}_{\mathrm{I}})$-set Q containing P. Hence we have

$$S=(A-Q)\cup[R\cup(S\cap(Q-P))].$$

The implication (iv)→(i) is a simple consequence of Theorem 4.

Finally, the fact that (iii) and (v) are equivalent to (i) follows from (ii) and (iv) by considering complements.

In connection with Theorem 4, we have the following consequence of Theorem 6.

Corollary 7. *If $\mathscr{C}$ satisfies* CCC, *then $\mathfrak{B}(\mathscr{C})$ is the smallest σ-field containing all $\mathscr{C}$-sets and all $\mathscr{C}_{\mathrm{I}}$-sets.*

Theorem 8. *If $\mathscr{C}$ satisfies* CCC, *then $\mathfrak{B}(\mathscr{C})-\mathscr{C}_{\mathrm{I}}$ satisfies* CCC.

Proof. Assume the conclusion is false. Then there exists a transfinite sequence

(1) $$S_1, S_2, \ldots, S_\alpha, \ldots, \quad \alpha<\Omega$$

(where Ω is the first uncountable ordinal number) of disjoint $\mathscr{C}_{\mathrm{II}}$-sets with the Baire property. We shall define a transfinite sequence

(2) $$A_1, A_2, \ldots, A_\alpha, \ldots, \quad \alpha<\Omega$$

of disjoint $\mathscr{C}$-sets such that S_α is a $\mathscr{C}_{\mathrm{II}}$-set everywhere on A_α, for each $\alpha<\Omega$.

By Theorem 2 we know each set S_α is a $\mathscr{C}_{\mathrm{II}}$-set everywhere on some $\mathscr{C}$-set. Moreover, no two distinct sets S_α can be $\mathscr{C}_{\mathrm{II}}$-sets everywhere on the same $\mathscr{C}$-set. For, if $\beta<\alpha$ and both S_β and S_α are $\mathscr{C}_{\mathrm{II}}$-sets everywhere on the same $\mathscr{C}$-set A, then $X-S_\alpha$, which contains S_β, is also a $\mathscr{C}_{\mathrm{II}}$-set everywhere on A, contradicting the fact that $S_\alpha\in\mathfrak{B}(\mathscr{C})$.

Let A_1 be a $\mathscr{C}$-set on which S_1 is a $\mathscr{C}_{\mathrm{II}}$-set everywhere. Assume $\alpha<\Omega$ and the sets A_β have been defined for all $\beta<\alpha$. Let B_α be a $\mathscr{C}$-set on which S_α is a $\mathscr{C}_{\mathrm{II}}$-set everywhere. Then $B_\alpha\cap(\bigcup_{\beta<\alpha}A_\beta)$ contains no $\mathscr{C}$-set. For, suppose it did contain a $\mathscr{C}$-set C. From the inclusion $C\subset B_\alpha$, we know $C\cap S_\alpha$ is a $\mathscr{C}_{\mathrm{II}}$-set and hence $B_\alpha\cap$ $\cap(\bigcup_{\beta<\alpha}A_\beta)$ is also a $\mathscr{C}_{\mathrm{II}}$-set. An application of Theorem 1 shows that there exists an ordinal number $\beta<\alpha$ such that $B_\alpha\cap A_\beta$ contains a $\mathscr{C}$-set D. But then both S_β and S_α are $\mathscr{C}_{\mathrm{II}}$-sets everywhere on D. Thus, $B_\alpha\cap(\bigcup_{\beta<\alpha}A_\beta)$ does indeed contain no $\mathscr{C}$-set. We know, by the preceding paragraph, that the family $\mathscr{C}$ is uncountable. Using Axiom 2b, we define A_α to be a $\mathscr{C}$-set contained in $B_\alpha-\bigcup_{\beta<\alpha}A_\beta$. Proceeding in this manner, we establish, by transfinite induction, the existence of the transfinite sequence (2), which contradicts the hypothesis that $\mathscr{C}$ satisfies CCC.

This theorem is utilized in the proof of the next two theorems.

Theorem 9. *If $\mathscr{C}$ satisfies* CCC, *then for each set $S\subset X$ there is an $\mathscr{E}_{\delta\sigma}$-set A containing S having the property*

(*) *if $B\in\mathfrak{B}(\mathscr{C})$ and $S\subset B$ then $A-B$ is a $\mathscr{C}_{\mathrm{I}}$-set.*

Proof. By an argument similar to that of Marczewski ([23], p. 235) there is a set T in $\mathfrak{B}(\mathscr{C})$ containing S which satisfies (*). Using the equivalence of conditions (i) and (v) of Theorem 6, we have $T=A-U$ where A is an $\mathscr{E}_{\delta\sigma}$-set and U is a $\mathscr{C}_{\mathrm{I}}$-set. Clearly, A satisfies (*).

Theorem 10 (see [23]). *If $\mathscr{C}$ satisfies* CCC *then $\mathfrak{B}(\mathscr{C})$ is closed under operation ($\mathscr{A}$).*

Remark. The more general theorems about the classical Baire property (Example 1) are established in the same manner. For instance, if S has the Baire property, then

$$S=[(\bigcup_\alpha A_\alpha)-\bigcup_\alpha(A_\alpha-S)]\cup[S-\bigcup_\alpha A_\alpha].$$

In this representation, $\bigcup_\alpha A_\alpha$ is an open set, $S-\bigcup_\alpha A_\alpha$ is of the first category, and it can be shown by Banach's method (see [2]) that $\bigcup_\alpha(A_\alpha-S)$ is a set of the first category (see [8] and [22]).

We next define the notion of equivalent $\mathfrak{K}$-families, examples of which may be found in [13] and in Section 4 below.

Definition 5. Two $\mathfrak{K}$-families $\mathscr{C}$ and $\mathscr{D}$ of subsets of a set X are called *equivalent* if $\mathscr{C}_{\mathrm{I}}=\mathscr{D}_{\mathrm{I}}$ and $\mathfrak{B}(\mathscr{C})=\mathfrak{B}(\mathscr{D})$.

Remark. If $\mathscr{C}$ and $\mathscr{D}$ are $\mathfrak{K}$-families such that each $\mathscr{C}$-set contains a $\mathscr{D}$-set and conversely, then $\mathscr{C}$ and $\mathscr{D}$ are equivalent.

3. Localization.

Definition 6. A set $S\subset X$ is a *$\mathscr{C}_{\mathrm{I}}$-set at a point* $x\in X$ if for each $\mathscr{C}$-set A containing x, there is a $\mathscr{C}$-set $B\subset A$ such that $x\in B$ and $B\cap S$ is a $\mathscr{C}_{\mathrm{I}}$-set. Otherwise, S is a $\mathscr{C}_{\mathrm{II}}$-set at the point x.

As immediate consequences of Theorem 2 we have

Theorem 11 (see [2]). *A necessary and sufficient condition that a set $S\subset X$ be a $\mathscr{C}_{\mathrm{I}}$-set is that it be a $\mathscr{C}_{\mathrm{I}}$-set at every point $x\in X$.*

Theorem 12 (see [2], Section 2). *The set of all points of a set S at which S is a $\mathscr{C}_{\mathrm{I}}$-set is itself a $\mathscr{C}_{\mathrm{I}}$-set.*

Definition 7. A set $S\subset X$ has the *Baire property at a point* $x\in X$ if for each $\mathscr{C}$-set A containing x, there is a $\mathscr{C}$-set $B\subset A$ such that $x\in B$ and $B\cap S$ has the Baire property.

Theorem 13. *A necessary and sufficient condition that a set $S\subset X$ have the Baire property is that it have the Baire property at every point.*

Proof. Necessity is obvious, since $\mathscr{C}$-sets have the Baire property. For sufficiency, assume $S\notin\mathfrak{B}(\mathscr{C})$. Then there is a $\mathscr{C}$-set A on which both S and $X-S$ are

$\mathscr{C}_{\mathrm{II}}$-sets everywhere. The set $A \cap S$ being a $\mathscr{C}_{\mathrm{II}}$-set, we can choose a point $x \in A$. Suppose B is any $\mathscr{C}$-set with $x \in B$ and $B \subset A$. For every $\mathscr{C}$-set $C \subset B$, the sets $S \cap C = (S \cap B) \cap C$ and $(X-S) \cap C = [X-(S \cap B)] \cap C$ are both $\mathscr{C}_{\mathrm{II}}$-sets, so $B \cap S$ does not have the Baire property. Hence S does not have the Baire property at the point x.

4. $\mathfrak{M}$-families.

DEFINITION 8 (see [13], p. 8). A family $\mathscr{C}$ of subsets of X is called an $\mathfrak{M}$-*family* if $\mathscr{C}$ is a $\mathfrak{K}$-family and the following conditions are satisfied.

3. The intersection of any descending sequence of $\mathscr{C}$-sets is nonempty.
4. For any $x \in X$, the set $\{x\}$ is $\mathscr{C}$-singular.

Throughout this section, $\mathscr{C}$ will denote an $\mathfrak{M}$-family.

There are four basic $\mathfrak{M}$-families which illustrate and motivate the general theory. Three of them are connected with some ideas of Cantor, Baire, Borel, and Lebesgue, and the fourth has been defined by Marczewski in [24], 1.1.

EXAMPLE 5. Let X be an uncountable set and let $\mathscr{C}$ be the family of all sets whose complement is finite. The $\mathscr{C}$-singular sets, $\mathscr{C}_{\mathrm{I}}$-sets, and $\mathscr{C}_{\mathrm{II}}$-sets coincide with the finite sets, countable sets, and uncountable sets, respectively. $\mathfrak{B}(\mathscr{C})$ is the family of all sets which are countable or whose complement is countable.

EXAMPLE 6. Let (X, d) be a complete, separable metric space with no isolated points, let Q be a countable set dense in X, and let $\mathscr{C}$ be the family of all closed balls $\{x \in X\colon d(x, r) \leqslant 1/n\}$, $r \in Q$, $n = 1, 2, \ldots$ By Baire's and Cantor's theorems, $\mathscr{C}$ is an $\mathfrak{M}$-family. In view of the Remark following Definition 5, $\mathscr{C}$ is equivalent to the $\mathfrak{K}$-family of all nonempty open subsets of X (see Example 1).

Denoting by G the interior of $\bigcup_k M_k$, where the M_k are as in the proof of Theorem 3, we see that $X-G$ is a nowhere dense closed set containing S. Consequently each set of the first category is a subset of an $\mathscr{F}_\sigma$-set of the first category. Also, it follows from Theorem 6 that a set has the Baire property if and only if it is the union of a $\mathscr{G}_\delta$-set and a set of the first category.

EXAMPLE 7. Let $\bar{P}$ be the completion of a non-atomic probability measure P defined on the Borel sets of a complete separable metric space and let $\mathscr{C}$ be the family of all compact sets which have positive probability. Since P is tight, $\mathscr{C}$ is equivalent to the $\mathfrak{K}$-family of all Borel sets of positive probability (see Example 2). Hence the $\mathscr{C}$-singular and $\mathscr{C}_{\mathrm{I}}$-sets coincide with the sets of $\bar{P}$-measure zero and $\mathfrak{B}(\mathscr{C})$ is the family of all $\bar{P}$-measurable sets.

From Theorems 3 and 6 it follows that every set of $\bar{P}$-measure zero is contained in a $\mathscr{G}_\delta$-set of P-measure zero and that $\mathfrak{B}(\mathscr{C})$ coincides with the family of all sets representable as the union of an $\mathscr{F}_\sigma$-set and a set of $\bar{P}$-measure zero.

EXAMPLE 8 (see [13], Example 4). Let (X, d) be a complete separable metric space with no isolated points and let $\mathscr{C}$ be the family of all compact perfect sets. By the Alexandroff–Hausdorff Theorem, $\mathscr{C}$ is equivalent to the $\mathfrak{K}$-family of all uncountable Borel subsets of X (see Example 4). Hence the $\mathscr{C}$-singular sets are the same as the $\mathscr{C}_{\mathrm{I}}$-sets and $\mathfrak{B}(\mathscr{C})$ is the family of Marczewski sets.

Note that the assertion of Theorem 3 fails to hold for this $\mathfrak{M}$-family, since there are no uncountable $\mathscr{C}_{\mathrm{I}}$-sets which are Borel sets, while uncountable $\mathscr{C}_{\mathrm{I}}$-sets exist (see [24], 5.1(iii)). Also, the assertion of Theorem 6 does not hold. For, in this example, a Borel set which is not an $\mathscr{F}_{\sigma\delta}$-set is not representable as the union of an $\mathscr{F}_{\sigma\delta}$-set and a $\mathscr{C}_{\mathrm{I}}$-set.

The following facts are easy consequences of Axioms 3 and 4.

(i) All countable sets are $\mathscr{C}_{\mathrm{I}}$-sets.

(ii) Every $\mathscr{C}$-set is a $\mathscr{C}_{\mathrm{II}}$-set.

(iii) X is a $\mathscr{C}_{\mathrm{II}}$-set at every point.

(iv) The intersection of a $\mathscr{C}_{\mathrm{II}}$-set and a $\mathscr{C}$-residual set is a $\mathscr{C}_{\mathrm{II}}$-set.

(v) The converse of Theorem 8 holds; i.e. if $\mathfrak{B}(\mathscr{C})-\mathscr{C}_{\mathrm{I}}$ satisfies CCC, then $\mathscr{C}$ satisfies CCC.

Two decomposition theorems of Ulam can be easily generalized to the case of an arbitrary $\mathfrak{M}$-family.

THEOREM 14 (see [27], Satz I). *If $S \subset X$ is a $\mathscr{C}_{\mathrm{II}}$-set of power $\mathfrak{m}$ and if there is no weakly inaccessible cardinal number $\leqslant \mathfrak{m}$, then S can be decomposed into an uncountable family of disjoint $\mathscr{C}_{\mathrm{II}}$-sets.*

THEOREM 15 (see [27], Satz II). *If $S \subset X$ is a set of power $\mathfrak{m}$ which does not have the Baire property and if there is no weakly inaccessible cardinal number $\leqslant \mathfrak{m}$, then S can be decomposed into an uncountable family of disjoint sets, none of which has the Baire property.*

Using Sierpiński's generalization ([19], p. 214) of a theorem of Ulam ([26], p. 145), one can easily establish the following result concerning the existence of sets which do not have the Baire property.

THEOREM 16. *If $\mathscr{C}$ satisfies* CCC, *X has power $\mathfrak{m}$, and there is no weakly inaccessible cardinal number $\leqslant \mathfrak{m}$, then there is a set which does not have the Baire property with respect to $\mathscr{C}$.*

DEFINITION 9. A set $S \subset X$ is said to *have property* (L) *with respect to* $\mathscr{C}$ if S is uncountable and has at most countably many points in common with each $\mathscr{C}_{\mathrm{I}}$-set; equivalently, if S is uncountable and every uncountable subset of S is a $\mathscr{C}_{\mathrm{II}}$-set.

The proof of the existence of sets having property (L) was first given in the case of Baire category by P. Mahlo (see [12], Aufgabe 5, pp. 294–295) and, shortly thereafter, by N. Lusin (see [11], Théorème I). Reasoning by analogy, W. Sierpiński ([18], pp. 184–185) established the existence of sets with property (L) in the case of Lebesgue measure. The existence of sets having property (L) with respect to the $\mathfrak{M}$-family of all complements of countable unions of finite-dimensional Borel sets in Hilbert space is due to W. Hurewicz [6]. In all cases, the Continuum Hypothesis is assumed. More generally, we have

THEOREM 17. *Assume the Continuum Hypothesis. If $\mathscr{C}$ has power at most $2^{\aleph_0}$ and satisfies* CCC, *then every $\mathscr{C}_{II}$-set contains a set of power $2^{\aleph_0}$ which has property* (L).

Proof. Under the given hypotheses, the family $\mathscr{J}$ of all $(\mathscr{E}_{\delta\sigma} \cap \mathscr{C}_I)$-sets has power $\leqslant 2^{\aleph_0}$ and, by Theorem 3, each $\mathscr{C}_I$-set is contained in a $\mathscr{J}$-set. The conclusion then follows from Proposition P_8 of [20].

5. Concluding remarks. After establishing in [22] that the classical Baire property (in the wide sense) is the appropriate category analogue of Lebesgue measurability, E. Marczewski considered the question: What is the measure analogue of the Baire property in the restricted sense? In [25] he suggested that it was the concept of absolute measurability. Indeed, this is the case, and one can unify the notions of absolute measurability and the Baire property in the restricted sense within the above theoretical framework. For the real line, this unification may be accomplished in two different ways; by means of order isomorphic mappings of the real line into itself (see [15]) or by means of homeomorphic mappings of the set of irrational numbers into the real line (see [16]). The latter method generalizes to complete separable metric spaces with no isolated points.

There are also theorems involving translation invariance of Lebesgue measure and Baire category which can be unified (see [14]).

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