# The real closure of a commutative regular $f$-ring *

by

**L. Lipshitz** (West Lafayette, Ind.)

**Abstract.** In this note it is shown that every commutative regular $f$-ring has a unique, prime real closure and a unique prime extension to a model of the theory of real closed commutative regular $f$-rings without minimal idempotents, which is the model completion of the theory of commutative regular $f$-rings (cf. A. Macintyre, *Model-completeness for sheaves of structures*, Fund. Math. 81 (1973)).

In [4] and [5] it has been shown that the theory, $K^*_{\overline{\mathrm{CRF}}}$ of real closed commutative regular $f$-rings without minimal idempotents is the model completion of the theory $K_{\mathrm{CRF}}$ of commutative regular $f$-rings. In this note it is shown that every commutative regular $f$-ring has a unique prime real closure (i.e. a prime extension to a model of the theory $K_{\overline{\mathrm{CRF}}}$ of real closed commutative regular $f$-rings) and a unique prime extension to a model of $K^*_{\overline{\mathrm{CRF}}}$. (See definitions below.) These results are in contrast to the results of [3] where necessary conditions were given for a commutative regular ring $R$ to have a prime integral closure. The following example illustrates the contrast. Let $R_1$ be the subring of $Q^\omega$ of all sequences which are constant except on finite sets. Let $e_i$ be the function $e_i(j) = \delta_{ij}$ and let $R = R_1[\sqrt{2}e_i|\ i \in \omega]$. Then $R$, viewed as a commutative regular ring has no prime extension to an integrally closed commutative regular ring (see [3]), but $R$, viewed as a commutative regular $f$-ring has a prime real closure. The reason is as follows: In $R$ (as a model of $K_{\mathrm{CR}}$) there is no canonical way to adjoin a global square root of 2. In $R$ (as a model of $K_{\mathrm{CRF}}$) there is, viz. the largest square root of 2.

DEFINITIONS. (i) $K_{\mathrm{CR}}$ is the theory of commutative regular rings with unit (cf. [3], [4] or [5]).

(ii) $K_{\overline{\mathrm{CR}}} = K_{\mathrm{CR}} \cup$ {every monic polynomial has a root} is the theory of integrally closed commutative regular rings.

(iii) An *l-ring* is a ring which in addition is a lattice under the operations $\wedge$ and $\vee$, such that, if $\leqslant$ is the lattice partial order, then $x \geqslant 0$ and $y \geqslant 0 \Rightarrow xy \geqslant 0$, and $x \geqslant y \Rightarrow z+x \geqslant z+y$.

---

* This research supported in part by N.S.F. grant GP 43749.

(iv) An $f$-*ring* is an $l$-ring in which $a \wedge b = 0$ and $c \geqslant 0 \Rightarrow ca \wedge b = ac \wedge b = 0$ (see [1] and [4]).

(v) $K_{\mathrm{CRF}}$ is the theory (in $\langle +, \cdot, \vee, \wedge, 0, 1\rangle$) of commutative regular $f$-rings, with unit.

(vi) $K_{\overline{\mathrm{CRF}}} = K_{\mathrm{CRF}} \cup$ {every monic polynomial of odd degree has a root} $\cup$ $\cup$ $\{a \geqslant 0 \Rightarrow x^2 - a$ has a root$\}$.

(vii) $K^*_{\mathrm{CRF}} = K_{\mathrm{CRF}} \cup$ {there are no ($\neq 0$) minimal idempotents}.

(viii) If $T_1 \subset T_2$ are two theories and $\mathfrak{A} \models T_1$, $\mathfrak{A} \subset \mathfrak{L} \models T_2$ then $\mathfrak{L}$ is called a *prime extension* of $\mathfrak{A}$ to a model of $T_2$ if whenever $\mathfrak{L}' \models T_2$ and $\varphi\colon \mathfrak{A} \to \mathfrak{L}'$ is an embedding, then $\varphi$ extends to an embedding of $\mathfrak{L}$ into $\mathfrak{L}'$. If $R \models K_{\mathrm{CR}}$ ($K_{\mathrm{CRF}}$) then a prime extension of $R$ to a model of $K_{\overline{\mathrm{CR}}}$ ($K_{\overline{\mathrm{CRF}}}$) is called an *integral* (*real*) *closure* of $R$.

For other definitions the reader is referred to [3] and [4]. We shall need the following elementary facts. For proofs the reader is referred to [3] and [4] § 6 and the references therein.

(a) If $R \models K_{\mathrm{CRF}}$ then every maximal ideal of $R$ is an $l$-ideal and $R$ can be realized as a ring (as global sections of a sheaf of rings) of functions on the maximal ideal space $S_R$ of $R$. $S_R$ is the Stone space of the Boolean algebra of idempotents $B_R$ of $R$.

(b) If $R_1, R_2 \models K_{\mathrm{CRF}}$ and $\varphi\colon R_1 \to R_2$ is an embedding of $l$-rings then $\varphi^*\colon S_{R_2} \to S_{R_1}$ defined by $\varphi^*(s) = \varphi^{-1}(s)$, for $s \in S_{R_2}$, is a continuous mapping of $S_{R_2}$ onto $S_{R_1}$. If $\varphi^{*-1}(s)$ contains more than one point of $S_{R_2}$ we say that $s (\in S_{R_1})$ splits in $R_2$. If no $s \in S_{R_1}$ splits in $R_2$ then $\varphi^*$ is a homeomorphism.

(c) If $R \models K_{\mathrm{CRF}}$ and $\psi(a_1, \ldots, a_n)$ is a variable free formula $(a_1, \ldots, a_n \in R)$ then the set of points $s \in S_R$ such that $\psi$ holds in $R/s$ (i.e. in the stalk at $s$) is a clopen subset of $S_R$ and hence corresponds to some idempotent of $R$. If $p(x), q(x) \in R[x]$ then $(p, q)(x) \in R[x]$, i.e. there is a polynomial $(p, q)(x)$ which is the g.c.d. of $p(x)$ and $q(x)$ at each point of $S_R$. If $p(x) \in R[x]$ then using the definition of a Sturm sequence given in [2] p. 281 there exists a Sturm sequence $p_0(x), \ldots, p_n(x)$ for $p(x)$ in $R[x]$; $p_0, \ldots, p_n$ is a Sturm sequence for $p$ at each point of $S_R$.

The following lemma was proved in [3].

LEMMA 1. *If $R \subset R_1$ are models of $K_{\mathrm{CR}}$ and $\alpha \in R_1$ satisfies $p(\alpha) = 0$ for some $p(x) \in R[x]$, then $R[\alpha] \models K_{\mathrm{CR}}$ and every maximal ideal of $R[x]$ is of the form $(s \cup \{q(\alpha)\})$ where $s \in S_R$ and $q(x)$ is irreducible at $s$ and $q(x) | p(x)$ at $s$.*

LEMMA 2. *Let $R \models K_{\mathrm{CRF}}$, $p(x) \in R[x]$ be a monic polynomial of odd degree or a polynomial of the form $x^2 - a$ $(a \geqslant 0)$ and let $R \subset \overline{R} \models K_{\overline{\mathrm{CRF}}}$ and let $\alpha$ be the largest root of $p$ in $\overline{R}$, then $R[\alpha] \models K_{\mathrm{CRF}}$ and no point of $S_R$ splits in $R[\alpha]$ ($R[\alpha]$ is the subring (not sub-$l$-ring) of $\overline{R}$ generated by $R$ and $\alpha$).*

Proof. At each point $s \in S_{\overline{R}}$, $p$ has a largest root $\alpha_s$. $\alpha_s$ is the largest root of $p$ in some neighborhood of $s$ and hence using the compactness of $S_{\overline{R}}$, $\alpha$ exists. First we show that no $s \in S_R$ splits in $R[\alpha]$. Let $(s \cup \{q_1(\alpha)\}) \in S_{R[\alpha]}$ and let $p(x) = q_1^{n_1}(x) \ldots q_k(x)^{n_k}$ at $s$, with the $q_i$ irreducible at $s$, and $(q_i, q_j) = 1$ at $s$, for $i \neq j$.



Hence at $s$ there is no common root of $q_1$ and $q_i$ $(i > 1)$. Let $q_i(x) = c_{ik_i} x^{k_i} + \ldots$ $\ldots + c_{i0}$.

Using the elimination of quantifiers for real closed fields there exists a quantifier free formula $\psi(y_1, \ldots, y_m)$ $(m = \prod(k_i + 1))$ such that $\psi(c_{10}, c_{11}, \ldots, c_{kn_k})$ holds at a point $s'$ of $S_{\overline{R}}$ if and only if the largest root of $q_1(x)$ at $s'$ is larger than all the roots of $q_2(x), \ldots, q_k(x)$ at $s'$. By (c) above $\psi$ holds on some clopen neighborhood $N$ of $s$ (in $S_R$), corresponding to idempotent $e$ of $R$. Hence $\alpha$ satisfies $q_1(\alpha) = 0$ on $e$ (in $\overline{R}$) and hence since $(q_1, q_i) = 1$ at $s$, $1 \in (s \cup \{q_i(\alpha)\})$ for $i > 1$ and so $s$ does not split in $R[\alpha]$. Hence $S_{R[\alpha]} = S_R$.

To show that $R[\alpha] \models K_{\mathrm{CRF}}$ it is sufficient to show that if $q(\alpha) \in R[\alpha]$ then $q(\alpha) \vee 0 \in R[\alpha]$ $(\subset \overline{R})$. For this it is sufficient to show that the set of points $E_q = \{s \in S_R |\ q(\alpha) \geqslant 0$ at $s\}$ is clopen in $S_R = S_{R[\alpha]}$, and hence corresponds to some idempotent $e_q$ of $R$. Then $q(\alpha) \vee 0 = q(\alpha) e_q$. From Lemma 1 we know that $E_{q,0} = \{s \in S_R |\ q(\alpha) = 0$ at $s\}$ is clopen in $S_R$. Hence it is sufficient to show that $E_{q,+} = \{s \in S_R |\ q(\alpha) > 0$ at $s\}$ is open in $S$, for then by symmetry $E_{q,-} = \{s \in S_R |\ q(\alpha) < 0$ at $s\}$ is open and the result follows. Again, using the elimination of quantifiers for real closed fields there is a quantifier free formula $\varphi$ (in the coefficients of $p(x)$ and $q(x)$) such that $\varphi$ holds at $s' \in S_R$ if and only if $q(\alpha) > 0$. Since $q(\alpha) > 0$ at $s$ then again by (c) $\varphi$ holds on an idempotent $f$ containing $s$ and hence $q(\alpha) > 0$ on some neighborhood of $s$.

I would like to thank L. van den Dries for pointing out an error in an earlier proof of Lemma 2.

LEMMA 3. *If $R \subset R_1, R_2$; $R_1, R_2 \models K_{\overline{\mathrm{CRF}}}$; $R \models K_{\mathrm{CRF}}$ and $p$ is as in Lemma 2 and $\alpha_1, \alpha_2$ are the largest roots of $p$ in $R_1, R_2$ respectively, then $R[\alpha_1] \simeq R[\alpha_2]$.*

Proof. From Lemma 2 we see that $R[\alpha_1]$ and $R[\alpha_2]$ are both rings of functions on $S_R$ and that $\alpha_1$ and $\alpha_2$ are the same function, considering $R[\alpha_i]$ as a sub-$f$-ring of $\prod_{s \in S_R} \overline{(R/s)}$, where $\overline{F}$ is the real closure of field $F$. Hence $R[\alpha_1] \simeq R[\alpha_2]$.

THEOREM 1. *If $R \models K_{\mathrm{CRF}}$, then $R$ has a unique real closure.*

Proof. Let $\overline{R}$ be obtained from $R$ by repeatedly adjoining the largest root of a monic polynomial of odd degree or of a polynomial of the form $x^2 - |a|$, until we have a real closed $f$-ring. Then Lemma 3 shows that $\overline{R}$ is prime over $R$. The minimality of $\overline{R}$ follows in exactly the same way as the corresponding result in [3] (for definition of minimal see [3]). Hence $\overline{R}$ is unique.

THEOREM 2. *If $R \models K_{\mathrm{CRF}}$ then $R$ has a prime extension to a model of $K^*_{\mathrm{CRF}}$ and also a prime extension to a model of $K^*_{\overline{\mathrm{CRF}}}$.*

Proof. The proofs are very similar to those in § 5 of [3], and so we omit them.

## References


[1] G. Birkhoff, *Lattice Theory*, A.M.S. Colloquium Publications 25, Providence 1967.

[2] N. Jacobson, *Lectures in Abstract Algebra*, Vol. III, *Theory of Fields and Galois Theory*, Princeton 1964.

[3] L. Lipshitz, *Commutative regular rings with integral closure*, Trans. Amer. Math. Soc. 211 (1975), pp. 161–170.
[4] A. Macintyre, *Model-completeness for sheaves of structures*, Fund. Math. 81 (1973), pp. 73–89.
[5] V. Weispfenning, *Model-completeness and elimination of quantifiers for subdirect products of structures.*

PURDUE UNIVERSITY

*Accepté par la Rédaction le 12. 12. 1974*

# Cozero and Baire maps on products of uniform spaces

by

**Gloria J. Tashjian** (Hamilton, N. Y.)

**Abstract.** The main result is this: THEOREM. *A uniform Baire function from a product of uniform spaces into a metric space depends on countably many coordinates.* The proof uses a set-theoretic result: let $X$ be a cartesian product and let $\Sigma$ denote the collection of subsets of $X$ which depend on countably many coordinates. LEMMA. *If $\mathcal{U}$ is a family of subsets of $X$ such that* $\bigcup \mathcal{V} \in \Sigma$ *whenever* $\mathcal{V} \subset \mathcal{U}$, *then $\mathcal{U}$ depends on countably many coordinates.* Corollaries describe the cozero and Baire-fine coreflections of a product of uniform spaces in terms of its countable subproducts, and it follows in particular that metric-fine coreflections of products of metric spaces and measurable coreflections of products of complete metric spaces are proximally fine. (The various classes of uniform spaces mentioned above are discussed in recent papers of A. W. Hager, Z. Frolík, and M. D. Rice.)

In this paper we study cozero and Baire-measurable functions defined on uniform products and the cozero-fine and Baire-fine uniformities derived from these functions. The basic result is that these mappings, with metric range, depend on countably many coordinates. We then prove that the cozero-fine, Baire-fine, and proximally-fine coreflections of a product of uniform spaces are generated by the cozero maps, Baire maps, or proximally continuous maps when the coreflections on each countable subproduct are so generated.

**1. Basic result.** We begin with some definitions. Let $uX$ be a uniform space where $\mathcal{U}$ denotes a family of covers of $X$ satisfying the usual axioms for a uniformity, as in [11]. A *cozero set* in $uX$ is a set of the form $\{x \in X: f(x) \neq 0\}$ for some uniformly continuous real-valued function $f: uX \to R$. Let $\mathrm{Coz}(uX)$ denote the family of all cozero sets in $uX$, and let Baire $(uX)$ be the $\sigma$-algebra on the set $X$ generated by $\mathrm{Coz}(uX)$. Baire $(uX)$ is classified in the same way that the Borel sets of a metric space are classified, replacing open set by cozero set, in [5], [9], or [13]. Then, a family $\mathcal{U}$ of Baire sets is said to be of bounded class if there exists an ordinal $\alpha < w_1$ such that each element of $\mathcal{U}$ belongs to additive or multiplicative Baire class $\leqslant \alpha$. Given uniform spaces $uX$ and $vY$, a function $f: uX \to vY$ is a cozero or Baire map if $f^{-1}(U)$ is a cozero or Baire set in $uX$ for every cozero or Baire set $U$ in $vY$.

Now let $X = \prod_{a \in A} X_a$ be a cartesian product of sets. A function $f: X \to Y$ to another set $Y$ depends on the index set $I \subseteq A$ if $f(x) = f(y)$ whenever $\pi_I(x) = \pi_I(y)$.