On the Baire order of concentrated spaces and L_1 spaces

by

Jack B. Brown (Auburn, Ala.)

Abstract. The main purpose of this paper is to prove that the Continuum Hypothesis implies the existence of a concentrated subspace of the reals which has Baire order at least 3. It is also shown that L_1 spaces have Baire order $\leqslant 2$.

Introduction. Section 40 of Kuratowski's *Topology* I is devoted to the study of certain "singular" separable metric spaces. This paper is concerned with certain of these notions. Only separable metric spaces will be considered, and of special interest will be spaces X which have one of the following properties.

DEFINITION 1. X is a σ *space* if and only if every F_σ set in X is a G_δ set in X. X is a λ *space* if only if every countable subset of X is a G_δ set in X. X is a υ *space* if and only if every nowhere dense in X subset of X is countable. X is a *concentrated space* if and only if there is a countable subset A of X such that every open set containing A contains all but countably many points of X.

It is clear that every countable space has all of these properties, and it is shown in Section 40 of [3] that

(1) $\sigma \to \lambda \to$ always 1st category $\to$ totally imperfect, and

(2) $\upsilon \to$ concentrated $\to C'' \to C \to$ totally imperfect.

The notions "always 1st category", "totally imperfect", C'', and C are not of immediate interest in this paper and will not be defined here. It is obvious that an uncountable space cannot simultaneously be concentrated and λ.

Examples based on the Continuum Hypothesis (CH) and other set theoretic assumptions have been given to show that most of the implications (1) and (2) are not reversible. The so-called "Sierpiński set" S [8], [10], [3, p. 523] is a σ space which is not countable, and the so-called "Lusin set" L [5], [3, p. 525] is a υ space which is not countable.

EXAMPLE 1. It is easy to construct an example of a concentrated space which is not a υ space. Let $C_1, C_2, \ldots$ be a sequence of disjoint Cantor sets with union dense in the reals. Then, for each positive integer i, use Lusin's technique (and CH) to construct an uncountable subset C_i' of C_i which is uncountably dense in C_i but such that no nowhere dense in C_i subset of C_i contains uncountably many points of C_i'. Then $L' = C_1' \cup C_2' \cup \dots$ is clearly not a υ-space, but it is easily seen to be concentrated about the countable set $A = A_1 \cup A_2 \cup \dots$, where for each i, A_i is a countable subset of C_i' which is dense in C_i'.

It is also known that CH implies the existence of λ spaces which are not σ spaces. The original argument [6], [3, p. 524] for the existence of such spaces is based upon theorems concerning the possible dimensions of λ spaces and σ spaces. In particular, it is shown (using CH) that there is a 1-dimensional λ space (contained in the plane), but that there are no 1-dimensional σ spaces. More recently, Mauldin [4] has described a λ space which is not a σ space by exhibiting a subspace of the reals which is a λ space of "Baire order" ω_1 (where ω_1 is the 1st uncountable ordinal).

Indeed, much of the early interest in these singular spaces, and in the Sierpiński set and the Lusin set in particular, was due to the use of these sets in attacking the "Baire order problem" of Mazurkiewicz [7]. The Baire order of a space X is defined as follows. Let $G_0, G_1, \dots, G_\alpha, \dots$ be the usual transfinite sequence with union the Borel subset of X, where G_0 contains the open sets, G_1 contains the G_δ sets, G_2 contains the $G_{\delta\sigma}$ sets, etc. (for details see [3, p. 345]). Of course, if $\alpha = \omega_1$, then $G_\alpha = G_{\alpha+1} = \dots$ consists of the Borel sets. The *Baire order of X* is the first ordinal α such that $G_\alpha = G_{\alpha+1}$. This notion can be defined equivalently in terms of the corresponding classes $B_0, B_1, \dots, B_\alpha, \dots$ of Baire functions. The reals have Baire order ω_1 [3, Section 30], and countable spaces have Baire order 0 or 1 [9]. The "Baire order problem" [7] is to determine whether it is true that for each countable ordinal α there is a space with Baire order α. The Sierpiński set [10], [3, p. 523] was the first example of an uncountable space with Baire order 1, as it was shown that S was an uncountable σ space with Baire order $\geqslant 1$ and *every σ space has Baire order* $\leqslant 1$. So Mauldin [4] has shown in a very strong way that this last theorem on the Baire order of σ spaces does not extend to λ spaces. Mauldin did not assume CH, but instead assumed a certain "generalized rectangles hypothesis" which is known to follow from CH.

The Lusin set is the only known example of a space with Baire order 2 [7] [3, p. 526], as it follows from a theorem concerning subsets with the "Baire property" of υ spaces that *every uncountable υ space has Baire order* 2. It is the main purpose of this paper to show that this theorem about the Baire order of υ spaces does not extend to concentrated spaces because of the following:

THEOREM 1. CH *implies that there exists a subspace X of the reals such that X is concentrated about the rationals but in which there exists an $F_{\sigma\delta}$ set which is not a $G_{\delta\sigma}$ set.*

The space X of Theorem 1 will clearly be a concentrated space which is not a υ space, as is L' of Example 1. Notice that L', while not a υ space, is the union of countably many υ spaces. Such spaces, called L_1 *spaces* in [2], were found to be essential in [1] and [2] in characterizing the spaces in which certain variations of Blumberg's theorem hold. For example (assuming CH), if X is a subspace of the reals, then in order for it to be true that for every real function defined on X there

exist an uncountably dense in itself subset W of X and a dense in W subset D of W such that $f|W$ is differentiable on D (infinite derivatives are allowed), it is necessary and sufficient that X not be an L_1 space. It is clear that property L_1 fits into the list of implications (2) as follows:

(3) $\upsilon \rightarrow L_1 \rightarrow$ concentrated.

Example 1 shows that $\upsilon \not\leftarrow L_1$, and Theorem 1 will show that $L_1 \not\leftarrow$ concentrated, as it will be shown that the theorem [3, p. 526] concerning sets with the Baire property in υ spaces extends to some extent to L_1 spaces, and L_1 spaces have Baire order $\leqslant 2$.

Proof of Theorem 1. In constructing the desired space X, certain notions will be defined as they are needed. The notions of a *scattered set* and a *kernel* are used in the usual sense [3, p. 78]. The statement that a number set J *contains a rational kernel* R_0 means that there exists a set $R \leqslant J$ of rational numbers, R dense in itself, and R_0 is the union of all such sets R. If G is a collection of sets, G^* denotes the union of the sets in G.

Now, assume CH, and well order the Cantor sets which contain no rationals into a sequence $C(1), C(2), \dots$ indexed on the countable ordinals, and well order the Cantor sets which contain a rational kernel into a similar sequence $K(1), K(2), \dots$ Now, a certain subsequence $C(\alpha_1^1), C(\alpha_1^2), C(\alpha_2^2), C(\alpha_1^3), C(\alpha_2^3), \dots$ of the first sequence will be defined inductively. α_1^1 is the first ordinal such that $C(\alpha_1^1)$ is a subset of $K(1)$. Then suppose γ is an ordinal such that $C(\alpha_\delta^\lambda)$ has been defined for every $1 \leqslant \delta \leqslant \lambda < \gamma$. Then, α_1^γ is the first ordinal such that $C(\alpha_1^\gamma)$ is a subset of $K(1) - \{C(\beta) |$ for some $1 \leqslant \delta \leqslant \lambda < \gamma,\ \beta \leqslant \alpha_\delta^\lambda\}^*$. Then proceed inductively so that for $1 < \delta \leqslant \gamma$, α_δ^γ is the first ordinal such that $C(\alpha_\delta^\gamma)$ lies in $K(\delta) - \{C(\beta) |$ for some $1 \leqslant \lambda < \delta,\ \beta \leqslant \alpha_\lambda^\gamma\}^*$.

Each step in this process can be completed because it only requires the existence of a Cantor set containing no rationals lying inside any set of the form $K - W^*$, where K is a Cantor set containing a rational kernel, and W is a countable collection of Cantor sets which contain no rationals.

Now, $\{C(\alpha_\beta^\gamma) |\ 1 \leqslant \beta \leqslant \gamma < \omega_1\}$ is a collection of mutually exclusive Cantor sets. Let X contain the rationals and just one point from each set in this collection and no other points. It is clear from the construction that X has the following two properties: (i) if K is a Cantor set which contains a rational kernel, then $K \cap X$ is uncountable, and (ii) if C is a Cantor set which contains no rationals, then $C \cap X$ is at most countable. It is because of this second property that X *is concentrated about the rational numbers.*

Now, certain facts concerning the category of X will be established, and certain definitions will be needed. By a *right endpoint* of a Cantor set C is meant a point of C which is not a limit point of C from the right (analogously for *a left endpoint*). The *extreme endpoints* of a Cantor set are its maximum and minimum elements. By a *section* A of a Cantor set C is meant a set of the form $C \cap I$, where I is an interval having as right endpoint some right endpoint of C and having as left endpoint some left endpoint of C.

If J is an interval, then $J \cap X$ is second category in J; indeed, if C is a Cantor set having only rational endpoints, then $C \cap X$ is second category in C. The second, less apparent claim will be proved. Let C be a Cantor set having only rational endpoints, and suppose $C \cap X = N_1 \cup N_2 \cup \ldots$, where each N_i is nowhere dense in C. Let R_1 consist of the extreme endopoints a and b of C, and let $Q_1 = \{\ldots, A_2, A_1, B_1, B_2, \ldots\}$ be a collection of sections of C which do not intersect N_1 such that $\{a\} \ll A_{n+1} \ll A_n \ll B_n \ll B_{n+1} \ll \{b\}$ for each n, where "$\ll$" means "lies entirely to the left of", and such that a and b are both limit points of Q_1^*. Let R_2 consist of all the extreme endpoints of elements of Q_1. For each element D of Q_1 having extreme endpoints a' and b', let $Q_D = \{\ldots, A'_2, A'_1, B'_1, B'_2, \ldots\}$ be a collection of sections of D which do not intersect N_2 such that $\{a'\} \ll A'_{n+1} \ll A'_n \ll B'_n \ll B'_{n+1} \ll \{b'\}$ for each n, and a' and b' are both limit points of Q_D^*. Let $Q_2 = \{Q_D |\ D \in Q_1\}^*$. Let R_3 denote the collection of all extreme endpoints of elements of Q_2. Continue this process. Then, let $M = \mathrm{Cl}(R_1 \cup R_2 \cup \ldots)$. M is a Cantor subset of C, and M contains no elements of $N_1 \cup N_2 \cup \ldots$ except possibly for elements of $R_1 \cup R_2 \cup \ldots$, which is countable. But M contains a rational kernel, so M contains uncountably many elements of X. This is a contradiction, so $X \cap C$ is second category in C. The proof that X is second category in every interval is similar.

Now it will be shown that there is an $F_{\sigma\delta}$ set in X which is not a $G_{\delta\sigma}$ set in X. Let G_1 be a countable collection of disjoint Cantor sets, each having only rational endpoints, such that every rational number is an endpoint of some element of G_1. Then produceed inductively as follows. For each $C \in G_{n-1}$, let G_C be a countable collection of disjoint Cantor subsets of C, each having only rational endpoints, such that each endpoint of C is an endpoint of some element of G_C. Then let $G_n = \{g |\ g \in G_C$ for some set $C \in G_{n-1}\}$. Notice that $M = G_1^* \cap G_2^* \cap \ldots$ is an $F_{\sigma\delta}$ in the reals E which is not a $G_{\delta\sigma}$ in E. Now for each n and each $C \in G_n$, let $C' = C \cap X$, and let $G'_n = \{C' |\ C \in G_n\}$, and let $M' = M \cap X$. M' is an $F_{\sigma\delta}$ in X. Suppose M' is a $G_{\delta\sigma}$ in X. Then $M' = H_1 \cup H_2 \cup \ldots$, where each H_n is a G_δ in X. Since $H_1 \subseteq G_1^*$, it is first category in E. Indeed, H_1 is nowhere dense in E, for suppose there is an interval J in which H_1 is dense. There exists a sequence $O_1, O_2, \ldots$ of open in E sets such that $H_1 = X \cap O_1 \cap O_2 \cap \ldots$ Then $J-(O_1 \cap O_2 \cap \ldots)$ is first category in J, so since X is second category in J, there must be an element of $X \cap O_1 \cap O_2 \cap \ldots$ which is not in H_1. This is a contradiction, so H_1 is nowhere dense in E. Let R_1 be the set whose only elements are 0 and 1, and let $Q_1 = \{\ldots, A_2, A_1, B_1, B_2, \ldots\}$ be a collection such that for each n, A_n and B_n are sections of elements of G_1, $\{0\} \ll A_{n+1} \ll A_n \ll B_n \ll B_{n+1} \ll \{1\}$, A_n and B_n do not intersect H_1, and 0 and 1 are both limit points of Q_1^*. Let R_2 be the collection of all extreme endpoints of elements of Q_1. Now, consider an element D of Q_1 with extreme endpoints a and b. $H_2 \cap D$ is a subset of $G_2^* \cap D$, which is first category in D, and since H_2 is a G_δ set in X and $X \cap D$ is second category in D, it follows that H_2 is nowhere dense in D. So let $Q_D = \{\ldots, A'_2, A'_1, B'_1, B'_2, \ldots\}$ be a collection such that for each n, $A'_n \subseteq D$ and $B'_n \subseteq D$ are sections of elements of G_2, A'_n and B'_n do not intersect H_2, $\{a\} \ll A'_{n+1} \ll A'_n \ll B'_n \ll B'_{n+1} \ll \{b\}$, and a and b are both limit points of Q_D^*. Let $Q_2 = \{Q_D |\ D \in Q_1\}^*$. Let R_3 denote the collection of all extreme endpoints of elements of Q_2. Continue this process. Then $W = \mathrm{Cl}(R_1 \cup R_2 \cup \ldots)$ is a Cantor set containing a rational kernel, so it contains uncountably many elements of X. But W contains no points of $H_1 \cup H_2 \cup \ldots$ except possibly for the points of $R_1 \cup R_2 \cup \ldots$, which is countable. On the other hand, $W-(R_1 \cup R_2 \cup \ldots) \subseteq Q_1^* \cap Q_2^* \cap \ldots \subseteq G_1^* \cap G_2^* \cap \ldots = M$. Therefore, W contains a point of M' which is not in $H_1 \cup H_2 \cup \ldots$, which is a contradiction.

Remark. The author is unable to determine whether the Baire order of X is ω_1 or whether it is countable. If the latter is the case, then X is an example pertinent to the Baire order problem.

Baire order of L_1 spaces. A set A in a space X is said to have the *Baire property* or to be a *B set* in X if and only if $A = (G-P) \cup R$, where G is open and P and R are first category in X. It follows that A would necessarily be the union of a G_δ set and a set of first category. A is said to have the *Baire property in the restricted sense* or to be a *B_r set* if and only if it is true that for every subset P of X, $X \cap P$ is a B set relative to P. It is known that every Borel set is a B_r set [3, p. 93], and every B_r set is clearly a B set. It is known [3, p. 526] that *every B set in a υ space is the union of a G_δ set and a countable set.* If CH holds, this theorem does not extend directly to L_1 spaces, for consider the space $L' = C'_1 \cup C'_2 \cup \ldots$ of Example 1. Every subset of C'_1 is nowhere dense in L' and therefore a B set in L', but if CH holds, there are more subsets of C'_1 than there are Borel sets in L', so the theorem could not hold in L'. However, a similar theorem does hold for L_1 spaces. But first, the following theorem concerning the properties of the υ spaces which make up an L_1 space must be proved.

THEOREM 2. *If X is an L_1 space, then X is the union of countably many υ spaces, each a G_δ in X.*

Proof. Let $X = X_1 \cup X_2 \cup \ldots$, where the X_i are disjoint υ spaces, and let $O_1, O_2, \ldots$ be a countable basis of open sets for X. First, a G_δ set G_1 which is a υ space and a countable set C_1 will be constructed such that $X_1 \subseteq G_1 \cup C_1$. Let $X'_1 = \{X_j \cap O_i \cap \mathrm{Cl}(X_1) |\ i, j$ are positive integers and $X_j \cap O_i \cap \mathrm{Cl}(X_1)$ is dense in $O_i \cap \mathrm{Cl}(X_1)\}^*$. It is clear that $X_1 \subseteq X'_1 \subseteq \mathrm{Cl}(X_1) = \mathrm{Cl}(X'_1)$, so X_1 is dense in X'_1, and X'_1 is dense in $\mathrm{Cl}(X_1) = \mathrm{Cl}(X'_1)$.

X'_1 is a υ space, for suppose $N \subseteq X'_1$ is nowhere dense in X'_1. Suppose $N' = N \cap X_j \cap O_i \cap \mathrm{Cl}(X_1)$ is uncountable for some pair i, j of positive integers for which $X_j \cap O_i \cap \mathrm{Cl}(X_1)$ is dense in $O_i \cap \mathrm{Cl}(X_1)$. Since N is nowhere dense in X'_1, N' is nowhere dense in $O_i \cap X'_1$, and therefore nowhere dense in $O_i \cap \mathrm{Cl}(X_1)$. But $X_j \cap O_i \cap \mathrm{Cl}(X_1)$ is dense in $O_i \cap \mathrm{Cl}(X_1)$ so N' is nowhere dense in $X_j \cap O_i \cap \mathrm{Cl}(X_1)$. Thus, N' is an uncountable nowhere dense subset of X_j, which is a contradiction. It follows that N is countable, because it is the union of countably many countable sets N' of the form $N' = N \cap X_j \cap O_i \cap \mathrm{Cl}(X_1)$.

Now, for each $i = 2, 3, \ldots$, let $X_i' = X_i - X_1'$, which will be a υ space such that $X_i' \cap \mathrm{Cl}(X_1)$ is nowhere dense in $\mathrm{Cl}(X_1) = \mathrm{Cl}(X_1')$. Now, set

$$G_1 = \mathrm{Cl}(X_1') - \bigcup_{i=2}^{\infty} \mathrm{Cl}(\mathrm{Cl}(X_1') \cap X_i').$$

It is clear from the definition that G_1 is a G_δ set in X. G_1 is also a subset of X_1', for suppose $x \in G_1 - X_1'$. Then $x \in \mathrm{Cl}(X_1')$ and $x \in X_i'$ for some $i > 1$. So $x \in \mathrm{Cl}(\mathrm{Cl}(X_1') \cap X_i')$, but none of the points of this set are in G_1. Thus, $G_1 \subseteq X_1'$, and G_1 is a υ space.

Furthermore, the set $C_1 = X_1' - G_1$ is countable, because if $x \in X_1' - G_1$, then for some $i > 1$, $x \in X_1' \cap \mathrm{Cl}(\mathrm{Cl}(X_1') \cap X_i')$, which is nowhere dense in $\mathrm{Cl}(X_1')$ and nowhere dense in X_1', and therefore countable. Thus, since C_1 is the union of countably many such sets, it is countable.

Thus G_1 and C_1 are such that $X_1 \subseteq G_1 \cup C_1$, where G_1 is a G_δ in X and a υ space and C_1 is countable. In a similar manner, sets G_n and C_n can be constructed for each positive integer n so that G_n is a G_δ in X and a υ space, C_n is countable, and $X_n \subseteq G_n \cup C_n$. Then $C_1 \cup C_2 \cup \ldots$ can be written as $D_1 \cup D_2 \cup \ldots$, where each D_n is a single element set and therefore a G_δ and a υ space. So $X = G_1 \cup D_1 \cup G_2 \cup D_2 \cup \ldots$ is the desired decomposition of X.

EXAMPLE 2. Theorem 2 is primarily a Lemma in the process of establishing the Baire order of L_1 spaces, but it is of interest to note the degree to which it can be strengthened. First it may be noted that *the G_δ υ spaces that make up an L_1 space can be assumed to be disjoint*, for if $X = G_1 \cup G_2 \cup \ldots$, where each G_i is a G_δ in X and a υ space, then set $H_1 = G_1$ and $H_n = G_n - (G_1 \cup G_2 \cup \ldots \cup G_{n-1})$ for $n = 2, 3, \ldots$ Then for each n, H_n is a Borel set relative to G_n and therefore the union of a G_δ relative to G_n, H_n, and a countable set C_n. Then H_n is also a G_δ relative to X and of course a υ space. So if $C_1 \cup C_2 \cup \ldots$ is rewritten as $D_1 \cup D_2 \cup \ldots$, where the D_n are single element sets, then $X = H_1 \cup D_1 \cup H_2 \cup D_2 \cup \ldots$ would be a decomposition of X into *disjoint* G_δ sets which are υ spaces.

Notice that the space L' of Example 1 can actually be decomposed into *closed* υ spaces. However, this is not generally possible for L_1 spaces, for let L be the Lusin set (assume it is uncountably dense in the reals and contains no points of L') and let $Y = L \cup L'$. Now Sappose $Y = F_1 \cup F_2 \cup \ldots$, where each F_n is a υ space which is closed in Y. Let n be such that F_n contains uncountably many points of L. F_n cannot be nowhere dense in E, so there must be an interval J in which F_n is dense. Since F_n is closed in Y, it must contain all of $Y \cap J$, and for some j, $C_j' \cap J$ is uncountable and nowhere dense in F_n, which is a contradiction.

THEOREM 3. *Every B_r set in an L_1 space X is a $G_{\delta\sigma}$ in X.*

Proof. Let $X = G_1 \cup G_2 \cup \ldots$, where each G_n is a G_δ in X and a υ space. Let C be a B_r set in X. For each n, $C \cap G_n$ is a B set relative to G_n, and therefore $C \cap G_n = H_n \cup C_n$, where H_n is a G_δ relative to G_n and C_n is first category relative to G_n. C_n would necessarily be countable. Then writing $C_1 \cup C_2 \cup \ldots = D_1 \cup D_2 \cup \ldots$, where each D_n is a single element set, $C = H_1 \cup D_1 \cup H_2 \cup D_2 \cup \ldots$ is the desired decomposition.

EXAMPLE 3. Note that since the G_δ υ space $G_1, G_2 \ldots$ can be made disjoint, then *the G_δ sets that make up the arbitrary B_r set C can be made disjoint*. However, in contrast to Theorem 2 of [3, p. 526], B_r sets in L_1 spaces cannot necessarily be written as the union of a G_δ set and a countable set or even the union of a G_δ set and an F_σ set. Let Y be the space $Y = L \cup L'$ of Example 2, where $L' = C_1' \cup C_2' \cup \ldots$ as described in Example 1. Now, for each positive integer n, let C_n'' be C_n' minus a countable dense subset of C_n'. $D = C_1'' \cup C_2'' \cup \ldots$ is a $G_{\delta\sigma}$ in Y and therefore a B_r set. Suppose $D = G \cup C$, where G is a G_δ in Y and C is an F_σ in Y. C would necessarily be countable because since C is an F_σ, then for each n, $C \cap C_n'$ would be first category in C_n' (because it contains no points from a countable dense subset of C_n') and therefore countable. Since C is countable, G would have to be dense in Y. Since G is also a G_δ, it follows that $L \cup C$ is first category in Y. So $L \cup C = N_1 \cup N_2 \cup \ldots$, where each N_i is nowhere dense in Y. Since L is dense in Y, then $N_i \cap L$ is nowhere dense in L for each i and therefore countable. It follows that L is countable, which is a contradiction.

THEOREM 4. *Uncountable L_1 spaces have Baire order* 2.

Proof. Let $X = G_1 \cup G_2 \cup \ldots$, where each G_i is a υ space. It follows from Theorem 3 that X has Baire order $\leqslant 2$. For each n, let C_n be a countable dense subset of G_n. Then $C = C_1 \cup C_2 \cup \ldots$ is countable and therefore an F_σ. But C could not also be a G_δ, because if $C = O_1 \cap O_2 \cap \ldots$, where the O_n are open, then $G_i - O_j$ would necessarily be countable for each i and j, and the whole space would necessarily be countable. Therefore the Baire order of X is exactly 2.

Remark. The author would hope that the uncountable L_1 spaces are *precisely* the spaces which have Baire order 2, but has been unable to determine whether this is the case.

Added in proof. Settled in negative by R. J. Gardner.

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AUBURN UNIVERSITY

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