Monotone decompositions of continua

by

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Abstract. The aim of the paper is to prove that for every metric continuum X and for every class $\mathscr{A}$ of metric continua there exists a unique upper semi-continuous monotone decomposition of X which is minimal among all upper semi-continuous monotone decompositions of X, each of which has the property that each subcontinuum of X in $\mathscr{A}$ is contained in some element of the decomposition. The results are applied to continua irreducible about a finite subset.

1. Introduction. A continuum is understood to mean a compact connected metric space. By a mapping we mean a continuous function. If X is a continuum, then by a decomposition of X we mean a family $\mathscr{D}$ of mutually disjoint closed subsets of X the union of which is the whole X. The reader is referred to [3] and [4] for the definitions of terms not defined here. In this paper it is proved that for every continuum X and for every class $\mathscr{A}$ of continua there exists a unique upper semi-continuous monotone decomposition of X which is minimal among all upper semi-continuous monotone decompositions of X, each of which has the property that each subcontinuum of X belonging to $\mathscr{A}$ is contained in some element of the decomposition. The structure of this minimal decomposition is shown in the third section. The fourth section contains investigations of the decomposition space of an $\mathscr{A}$-admissible decomposition (see below) of a continuum. The results are used to generalize J. M. Russell's results concerning monotone decompositions of continua irreducible about a finite subset.

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2. Admissible decompositions. Let X be a continuum and let $\mathscr{A}$ be an arbitrary class of continua. A decomposition $\mathscr{D}$ of X is said to be *$\mathscr{A}$-admissible* if

1^0 $\mathscr{D}$ is upper semi-continuous,

2^0 $\mathscr{D}$ is monotone (i.e., each element of $\mathscr{D}$ is a continuum),

3^0 every subcontinuum of X which belongs to $\mathscr{A}$ is contained in some element of $\mathscr{D}$.

For every class $\mathscr{A}$ of continua, every continuum X has an $\mathscr{A}$-admissible decomposition, for instance the trivial one, i.e., such that the whole X is the only element of the decomposition.

Let X be a continuum. Consider the family $\mathscr{L}$ of all layers of all irreducible subcontinua of X. Putting $\mathscr{L}$ for $\mathscr{A}$ in the definition we obtain the admissible decomposition in the sense of [2], p. 115.

If $\mathscr{D}$ and $\mathscr{E}$ are upper semi-continuous monotone decompositions of a continuum X, then $\mathscr{D}\leqslant\mathscr{E}$ means that every element of $\mathscr{D}$ is contained in some element of $\mathscr{E}$, i.e., $\mathscr{D}$ *refines* $\mathscr{E}$. Clearly $\leqslant$ defines a partial ordering on the family of upper semi-continuous monotone decompositions of X.

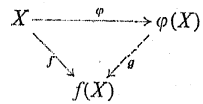
THEOREM 1. *For every continuum X and for every class $\mathscr{A}$ of continua there exists a unique $\mathscr{A}$-admissible decomposition of X which is minimal among all $\mathscr{A}$-admissible decompositions of X.*

Proof (cf. [6], the proof of Theorem 3, p. 8 and [2], the proof of Theorem 3, p. 118). Let $\{\mathscr{D}_\alpha\colon \alpha\in A\}$ be a chain of $\mathscr{A}$-admissible decompositions of X, and for $z\in X$ and $\alpha\in A$ let Z_α be an element of $\mathscr{D}_\alpha$ containing z. For fixed $z\in X$ $\{Z_\alpha\colon \alpha\in A\}$ is a chain of continua and we denote by Z the intersection of this chain. Denoting by $\mathscr{D}_0$ the collection $\{Z\colon z\in X\}$ we see that $\mathscr{D}_0$ is a decomposition of X into continua. Let K be a subcontinuum of X containing z and belonging to $\mathscr{A}$. The decompositions $\mathscr{D}_\alpha$, $\alpha\in A$ are $\mathscr{A}$-admissible, and thus we have $K\subset Z_\alpha$ for each $\alpha\in A$; hence $K\subset\bigcap\{Z_\alpha\colon \alpha\in A\}=Z$. Therefore $\mathscr{D}_0$ satisfies condition 3^0. To prove the upper semi-continuity of $\mathscr{D}_0$ suppose that U is an open subset of X containing Z which belongs to $\mathscr{D}_0$. For some $\alpha\in A$ we have $Z_\alpha\subset U$ and since $\mathscr{D}_\alpha$ is upper semi-continuous, some open subset V of U contains Z and is the union of elements of $\mathscr{D}_\alpha$. Thus V contains Z and is the union of elements of $\mathscr{D}_0$. Therefore $\mathscr{D}_0$ is upper semi-continuous according to [3], § 19, II, Theorem 4, p. 185. Thus $\mathscr{D}_0$ is $\mathscr{A}$-admissible. Since $\mathscr{D}_0$ refines each $\mathscr{D}_\alpha$, it is a lower bound for the chain. Applying the Kuratowski–Zorn lemma we conclude that there exists a minimal $\mathscr{A}$-admissible decomposition of X. Let $\mathscr{D}$ and $\mathscr{E}$ be two $\mathscr{A}$-admissible decompositions of X, and suppose that some element of $\mathscr{E}$ meets two different elements of $\mathscr{D}$. Further, let $\mathscr{E}'$ be a decomposition of X into components of the non-empty intersections $D\cap E$, where $D\in\mathscr{D}$ and $E\in\mathscr{E}$. We show that $\mathscr{E}'$ satisfies condition 3^0. In fact, if K is a subcontinuum of X which belongs to $\mathscr{A}$, then there are elements D and E in $\mathscr{D}$ and $\mathscr{E}$ respectively such that $K\subset D\cap E$. Since K is a connected set it is contained in a component of $D\cap E$, i.e., in an element of $\mathscr{E}'$. The decomposition $\mathscr{E}'$ is upper semi-continuous (see [2], Lemma 3, p. 118) and monotone. Thus $\mathscr{E}'$ is admissible. Since $\mathscr{E}'$ refines $\mathscr{E}$, $\mathscr{E}$ is not minimal. It follows that a minimal $\mathscr{A}$-admissible decomposition of X refines every $\mathscr{A}$-admissible decomposition of X, and thus the uniqueness is established. This completes the proof.

Another form of Theorem 1 is the following

THEOREM 2. *For every continuum X and for every class $\mathscr{A}$ of continua there exists a unique monotone mapping φ of X onto $\varphi(X)$ such that for each monotone mapping f of X onto $f(X)$ with the property that each subcontinuum of X belonging to $\mathscr{A}$ is mapped onto a point under f, there exists a unique mapping g of X onto $f(X)$ such that the diagram*

(2.1)



commutes and g is monotone.

Proof. Consider the minimal $\mathscr{A}$-admissible decomposition of X described in Theorem 1 and denote by φ the quotient mapping of X onto the induced decomposition space. Taking an arbitrary point $z\in X$, we infer that $f(\varphi^{-1}(z))$ is a point. Denote this point by $g(z)$. If $z=\varphi(x)$, then $g(z)=f(x)$; thus $g(\varphi(x))=f(x)$ for every $x\in X$, i.e., diagram (2.1) commutes. We infer that g is continuous, unique and monotone in the same way as in the proof of Theorem 7 in [1], p. 30.

Let $\mathscr{A}$ be a class of continua. A continuum M is said to be $\mathscr{A}$-*monostratic* if the minimal $\mathscr{A}$-admissible decomposition of M is trivial, i.e., the whole M is the only element of the decomposition.

THEOREM 3. *Let $\mathscr{A}$ be a class of continua. Every $\mathscr{A}$-monostratic subcontinuum of a continuum X is contained in some element of the minimal $\mathscr{A}$-admissible decomposition.*

Proof. Suppose that there exist two different elements D' and D'' of the minimal $\mathscr{A}$-admissible decomposition $\mathscr{D}$ of X such that $D'\cap M\neq\emptyset\neq D''\cap M$. Therefore the decomposition $\mathscr{D}'$ of M into components of the non-empty intersections $D\cap M$, where $D\in\mathscr{D}$, is upper semi-continuous (see [2], Corollary 1, p. 117) and not trivial. Since $\mathscr{D}$ is $\mathscr{A}$-admissible, if K is a subcontinuum of M belonging to $\mathscr{A}$, then there exists an element D in $\mathscr{D}$ such that $K\subset D$. The continuum K is a connected set, hence it is contained in a component of $D\cap M$, i.e., in an element of $\mathscr{D}'$. Therefore $\mathscr{D}'$ is $\mathscr{A}$-admissible. Hence M is not $\mathscr{A}$-monostratic and the proof is complete.

COROLLARY 1. *Let $\mathscr{A}$ be a class of continua. If every element of the minimal $\mathscr{A}$-admissible decomposition of a continuum X has an empty interior (with respect to X), then every $\mathscr{A}$-monostratic subcontinuum of X has an empty interior.*

The conversion of Corollary 1 is not true (see [2], the example on p. 128).

3. The structure of the minimal $\mathscr{A}$-admissible decomposition. We use the basic ideas employed in [2] to describe elements of the canonical decomposition of a continuum, and earlier in [1] to describe elements of the canonical decomposition of a λ-dendroid.

Let a continuum X and a class $\mathscr{A}$ of continua be established. Firstly, for $x\in X$, we define (by transfinite induction) an increasing sequence of continua $A_\alpha(x)$ each of which contains the point x.

Let $x \in X$. Consider all subcontinua $K(x)$ of X belonging to $\mathscr{A}$ such that $x \in K(x)$. Put

(3.1) $$A_0(x) = \overline{\{x\} \cup \bigcup K(x)}$$

where the union on the right side of the equality runs over all continua $K(x)$ belonging to $\mathscr{A}$ and such that $x \in K(x) \subset X$. Now suppose that the sets $A_\beta(x)$ are defined for $\beta<\alpha$, and put

(3.2) $$A_\alpha(x) = \begin{cases} \bigcup \{\mathrm{Ls}\, A_\beta(x_n)\colon \lim x_n \in A_\beta(x)\}, & \text{if} \quad \alpha = \beta+1, \\ \overline{\bigcup A_\beta(x)}, & \text{if} \quad \alpha = \lim\limits_{\beta<\alpha} \beta, \end{cases}$$

where, in the case $\alpha = \beta+1$, the union is taken over all convergent sequences of points $x_n \in X$ with $\lim x_n \in A_\beta(x)$. So the sets $A_\alpha(x)$ are well defined for $\alpha<\Omega$. The sequence $A_\alpha(x)$ is increasing, i.e.,

(3.3) $$x \in A_0(x) \subset A_1(x) \subset \ldots \subset A_\alpha(x) \subset \ldots$$

Indeed, $x \in A_0(x)$ by (3.1). Assume

$$x \in A_0(x) \subset A_1(x) \subset \ldots \subset A_\beta(x) \quad \text{for all } \beta<\alpha.$$

If $\alpha = \beta+1$ then putting $x_n = x$ in (3.2) we have $\lim x_n \in A_\beta(x)$ and $\mathrm{Ls}\, A_\beta(x_n) = A_\beta(x)$; hence $A_\beta(x) \subset A_\alpha(x)$. In the case $\alpha = \lim\limits_{\beta<\alpha} \beta$ the last inclusion follows immediately from (3.2). Therefore (3.3) is established.

Now we shall prove that

(3.4) The sets $A_\alpha(x)$ are continua.

Apply transfinite induction. If $\alpha = 0$, then we see that $\{x\} \cup \bigcup K(x)$ is a connected set because each $K(x)$ is a connected set and contains the point x; hence $A_0(x)$ is a continuum by (3.1). If $\alpha>0$, then the proof of (3.4) runs exactly as the corresponding part of the proof of Lemma 1 in [1], p. 19. So (3.4) follows.

Thus $\{A_\alpha(x)\}$ is an increasing sequence of continua. Since the space is separable as a metric continuum, there exists a countable ordinal ξ such that

(3.5) $$\text{If} \quad \xi<\eta<\Omega, \quad \text{then} \quad A_\xi(x) = A_\eta(x)$$

and we put

(3.6) $$S(x) = A_\xi(x).$$

Repeating sentence by sentence the proofs of Lemmas 2 and 3 and of Theorem 2 in [1], pp. 22–24, writing "a subcontinuum K of X belonging to $\mathscr{A}$" instead of "a tranche T of an irreducible subcontinuum of X", we can prove the following properties of the sets $S(x)$

(3.7) $$\text{If} \quad \lim x_n = x, \quad \text{then} \quad \mathrm{Ls}\, S(x_n) \subset S(x).$$

(3.8) $$\text{If} \quad S(x) \cap S(y) \neq \emptyset, \quad \text{then} \quad S(x) = S(y).$$

Therefore for various x the sets $S(x)$ are either disjoint or identical. Since they are continua by (3.4) and (3.6) we have defined a monotone decomposition of X into the sets $S(x)$. Just as in [1], Theorem 3, p. 25 we can obtain

(3.9) The decomposition of X into the sets $S(x)$ is upper semi-continuous.

The main result of this section is

THEOREM 4. *The decomposition* $X = \bigcup \{S(x)\colon x \in X\}$ *coincides with the minimal* $\mathscr{A}$*-admissible decomposition of* X.

Proof. Since for each point $x \in X$ and for each subcontinuum $K(x)$ containing x and belonging to $\mathscr{A}$ the continuum $K(x)$ is contained in the set $S(x)$ by (3.1), (3.3) and (3.6), condition 3^0 holds for the decomposition of X into the sets $S(x)$, and so this decomposition is $\mathscr{A}$-admissible. Let $\mathscr{D}$ be an arbitrary $\mathscr{A}$-admissible decomposition of X and let $D \in \mathscr{D}$. We shall prove the following

(3.10) $$\text{If} \quad x \in D, \quad \text{then} \quad A_\alpha(x) \subset D \quad \text{for every } \alpha<\Omega.$$

Apply transfinite induction. Let $\alpha = 0$. Taking a point $x \in X$, let $K(x)$ denote a subcontinuum of X containing x and belonging to $\mathscr{A}$. Since the decomposition $\mathscr{D}$ is $\mathscr{A}$-admissible, the condition $x \in D$ implies $K(x) \subset D$ by 3^0. This leads to $\bigcup K(x) \subset D$, where the union is taken over all members of $\mathscr{A}$ such that $x \in K(x) \subset X$. The element D of $\mathscr{D}$ is closed, and hence $\overline{\{x\} \cup \bigcup K(x)} \subset D$, which means $A_0(x) \subset D$ by (3.13). If $\alpha>0$ the proof of (3.10) is identical to the corresponding part of the proof of Lemma 4 in [1], p. 28. Thus the proof of (3.10) is complete. Therefore, if $x \in D$, then — in particular — $A_\xi(x) \subset D$, where ξ is an ordinal for which (3.5) holds. According to Definition (3.6) we see that $x \in D$ implies $S(x) \subset D$ and the theorem is proved.

4. The decomposition space of an $\mathscr{A}$-admissible decomposition. In this section we assume that the class $\mathscr{A}$ of continua has the following property:

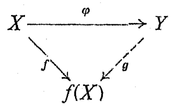
(4.1) If X is a continuum, a mapping f of X onto $f(X)$ is monotone and if the continuum $f(X)$ belongs to $\mathscr{A}$, then there exists a continuum M in X belonging to $\mathscr{A}$ such that $f(M) = f(X)$.

THEOREM 5. *If* X *is a continuum and the class* $\mathscr{A}$ *of continua satisfies condition* (4.1), *then the decomposition space of an* $\mathscr{A}$*-admissible decomposition of* X *contains no non-degenerate continua belonging to* $\mathscr{A}$.

Proof. Let $\mathscr{D}$ be an $\mathscr{A}$-admissible decomposition of X and let q denote the quotient mapping. If K is a subcontinuum of $q(X)$ which belongs to $\mathscr{A}$, then the partial mapping $q|q^{-1}(K) = f$ is monotone and $f(f^{-1}(K)) = K$. Since $\mathscr{A}$ satisfies (4.1) there exists a continuum $M \subset f^{-1}(K)$ belonging to $\mathscr{A}$ such that $f(M) = K$. Clearly $M \subset X$ and $q(M) = K$. The decomposition $\mathscr{D}$ is $\mathscr{A}$-admissible, and hence K is a point.

A number of families of continua satisfy condition (4.1). In particular, it follows from [4], § 48, V, Theorem 4, p. 208 that the class of all indecomposable continua satisfies (4.1). Therefore Theorems 2 and 5 lead to

COROLLARY 2. *For every continuum X there exists a unique monotone mapping φ of X onto a hereditarily decomposable continuum Y such that if a mapping f of X onto $f(X)$ is monotone and each indecomposable subcontinuum of X is mapped onto a point under f, then there exists a unique mapping g of Y onto $f(X)$ such that the diagram*

(4.3)


commutes and g is monotone.

5. Applications to continua irreducible about a finite subset. A continuum X is called *irreducible about a set A* if X contains A and no proper subcontinuum of X contains A. A *dendrite* is a hereditarily unicoherent and locally connected continuum.

LEMMA 1. *If a continuum X irreducible about a finite subset is mapped onto a hereditarily arcwise connected continuum Y under a monotone mapping f, then each nowhere dense subcontinuum of X is mapped onto a point under f.*

Proof. If a non-trivial continuum X is irreductible about a finite subset, then there exists a natural $n\geqslant 2$ such that X is irreducible about a set of n, but no fewer, of its points, say $a_1, a_2, \ldots, a_n$. Clearly Y is irreducible about points $f(a_1) = b_1$, $f(a_2) = b_2, \ldots, f(a_n) = b_n$. Let K be a nowhere dense subcontinuum of X, i.e., such that

(5.1) $$X = \overline{X\setminus K}.$$

Obviously for each b_i we have either $b_i \in f(K)$ or $b_i \notin f(K)$. We can assume without loss of generality that

(5.2) $$b_1, b_2, \ldots, b_{k-1} \in f(K) \quad \text{and} \quad b_k, b_{k+1}, \ldots, b_n \notin f(K)$$

for some integer $1\leqslant k\leqslant n+1$, where in the case $k = 1$ we assume that no points $b_1, \ldots, b_n$ are in $f(K)$ and similarly, in the case $k = n+1$, that no points $b_1, \ldots, b_n$ are out of $f(K)$. Since Y is hereditarily arcwise connected, for each i with $k\leqslant i\leqslant n$ there exists an arc b_ic_i such that

(5.3) $$b_ic_i \cap f(K) = \{c_i\}.$$

It follows from (5.2) that $K\cap f^{-1}(b_i) \neq \emptyset$ for each $i = 1, 2, \ldots, k-1$; analogously, it follows from (5.3) that $K\cap f^{-1}(b_ic_i) \neq \emptyset$ for each $i = k, k+1, \ldots, n$. Thus since the mapping f is monotone, the union

$$K\cup f^{-1}(b_1)\cup f^{-1}(b_2)\cup \ldots \cup f^{-1}(b_{k-1})\cup f^{-1}(b_kc_k)\cup \ldots \cup f^{-1}(b_nc_n)$$

is a continuum. Therefore

$$X = \overline{X\setminus K}\subset f^{-1}(b_1)\cup f^{-1}(b_2)\cup \ldots \cup f^{-1}(b_{k-1})\cup f^{-1}(b_kc_k)\cup \ldots \cup f^{-1}(b_nc_n)$$

by (5.1). Hence

$$Y = \{b_1, b_2, \ldots, b_{k-1}\}\cup b_kc_k \cup \ldots \cup b_nc_n.$$

It follows that

$$f(K) = f(K)\cap Y = \{b_1, b_2, \ldots, b_{k-1}, c_k, c_{k+1}, \ldots, c_n\}$$

by (5.2) and (5.3). Since $f(K)$ is a connected set, we have $b_1 = b_2 = b_3 = \ldots = b_{k-1} = c_k = c_{k+1} = \ldots = c_n$, which completes the proof.

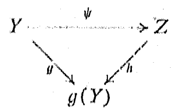
LEMMA 2. *Let X, Y and f be as in Lemma 1. If M is an indecomposable subcontinuum of X, then the image $f(M)$ is a point.*

Proof. Consider a composant C of some point p in M. If x is an arbitrary point of C, then by the definition of a composant there exists a proper subcontinuum K of M, which contains both p and x. The continuum K has an empty interior as a subcontinuum of C, which has an empty interior itself (see [4], § 48, VI, Theorem 6, p. 212). Applying Lemma 1, we conclude that $f(K)$ is a point, hence $f(x) = f(p)$. Since x is an arbitrary point of C, it follows that $f(C) = f(p)$. Finally $f(M) = f(\overline{C}) \subset \overline{f(C)} = \{f(p)\}$ and the proof is finished.

LEMMA 3. *Let X, Y and f be as in Lemma 1. If T is a layer of an irreducible subcontinuum of X, then T is mapped onto a point under f.*

Proof. If T is a layer of an irreducible subcontinuum of X, then T is the union of a (finite or infinite) sequence of nowhere dense continua and indecomposable continua (see [4], § 48, VII, Theorem 4, p. 216). Therefore by Lemmas 1 and 2 the image $f(T)$ is the union of a sequence of points. Since $f(T)$ is a connected set, it is a point.

LEMMA 4. *For every hereditarily decomposable continuum Y which is irreducible about a set of n, but no fewer, of its points, where $n\geqslant 2$, there exists a unique monotone mapping ψ of Y onto a dendrite Z such that if a mapping g of Y onto a dendrite $g(Y)$ is monotone, then there exists a unique mapping h of Z onto $g(Y)$ such that the diagram*

(5.4)


commutes and h is monotone.

Proof. It follows from [5], Theorem 2.4 and Corollary 2.5, pp. 260–262, that there exists a unique upper semi-continuous monotone decomposition $\mathscr{D}$ of Y, with a dendrite as the decomposition space, which is minimal among all monotone upper semi-continuous decompositions of Y having a dendrite as the decomposition space. Denoting by ψ the quotient mapping of Y onto $Y/\mathscr{D}$, we complete the proof similarly to the proof of Theorem 2.

The following is a generalization of J. M. Russell's results stated in [5] as Theorem 2.4, p. 260 and Corollary 2.5, p. 262.

THEOREM 6. *For every continuum X irreducible about a finite subset there exists a unique decomposition $\mathscr{D}$ of X such that*

(1) *$\mathscr{D}$ is upper semi-continuous,*

(2) *$\mathscr{D}$ is monotone,*

(3) *the decomposition space $X/\mathscr{D}$ is a dendrite (possibly degenerate),*

(4) *$\mathscr{D}$ is minimal among all decompositions of X satisfying conditions* (1), (2) *and* (3).

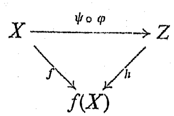
Proof. Let φ be a mapping of X onto a hereditarily decomposable continuum Y described in Corollary 2. Clearly the continuum Y is irreducible about a finite subset. Consider two cases. Firstly, let Y be degenerate. Then the trivial decomposition of X, i.e., such that the continuum X is the only element of the decomposition satisfies conditions (1), (2), (3) and (4). Secondly, if Y is not degenerate, then there exists an integer $n \geqslant 2$ such that Y is irreducible about n, but no fewer, of its points. Let ψ be a mapping of Y onto a dendrite Z described in Lemma 4. The mapping $\psi \circ \varphi$ is monotone, and thus the decomposition $\mathscr{D}$ of X into the sets $(\psi \circ \varphi)^{-1}(z)$, $z \in Z$, satisfies conditions (1), (2) and (3). To see that condition (4) holds consider a decomposition $\mathscr{E}$ of X satisfying conditions (1), (2) and (3). Let f denote the quotient mapping of X onto the decomposition space $X/\mathscr{E}$. Thus the mapping f is monotone and according to Lemma 2 has the property that each indecomposable subcontinuum of X is mapped onto a point under f. Therefore, applying Corollary 2, we conclude that there exists a unique monotone mapping g of Y onto $f(X)$ such that diagram (4.2) commutes, i.e.,

$$g(\varphi(x)) = f(x) \quad \text{for each } x \in X. \tag{5.5}$$

Further, it follows from Lemma 4 that there exists a unique monotone mapping h of Z onto $g(Y)$ such that diagram (5.4) commutes, i.e.,

$$h(\psi(y)) = g(y) \quad \text{for each } y \in Y. \tag{5.6}$$

Therefore the diagram



commutes by (5.5) and (5.6). It follows that $\mathscr{D}$ refines $\mathscr{E}$. So we have proved that $\mathscr{D}$ refines every decomposition of X satisfying conditions (1), (2) and (3). Therefore $\mathscr{D}$ satisfies condition (4) and the uniqueness is established. Thus the proof is complete.

The following is well known (see [2], Theorems 1, 2 and 3, pp. 116–118).

LEMMA 5. *For every continuum X there exists a unique decomposition $\mathscr{C}$ of X such that*

(1) *$\mathscr{C}$ is upper semi-continuous,*

(2) *$\mathscr{C}$ is monotone,*

(3) *for each irreducible subcontinuum I of X each layer of X is contained in some element of $\mathscr{C}$,*

(4) *$\mathscr{C}$ is minimal among all decompositions of X satisfying conditions* (1), (2) *and* (3).

Furthermore, the decomposition space $X/\mathscr{C}$ is hereditarily arcwise connected.

THEOREM 7. *Let X be a continuum irreducible about a finite subset. If $\mathscr{D}$ is the decomposition of X described in Theorem* 6 *and if $\mathscr{C}$ is the decomposition of X described in Lemma* 5, *then $\mathscr{D} = \mathscr{C}$.*

Proof. It follows immediately from Lemmas 3 and 5 that $\mathscr{C}$ refines $\mathscr{D}$. Further, since $X/\mathscr{C}$ is a monotone image of the continuum X irreducible about a finite subset, the continuum $X/\mathscr{C}$ is itself irreducible about a finite subset. Thus, the continuum $X/\mathscr{C}$ being hereditarily arcwise connected, it is easy to verify that $X/\mathscr{C}$ is a dendrite. Therefore $\mathscr{D}$ refines $\mathscr{C}$ by Theorem 6. Finally $\mathscr{D} = \mathscr{C}$.

References

[1] J. J. Charatonik, *On decompositions of λ-dendroids*, Fund. Math. 67 (1970), pp. 15–30.
[2] — *On decompositions of continua*, Fund. Math. 79 (1973), pp. 113–130.
[3] K. Kuratowski, *Topology* I, New York–London–Warszawa 1966.
[4] — *Topology* II, New York–London–Warszawa 1968.
[5] J. M. Russell, *Monotone decompositions of continua irreducible about a finite set*, Fund. Math. 72 (1971), pp. 255–264.
[6] E. S. Thomas, Jr., *Monotone decompositions of irreducible continua*, Dissertationes Math. 50 (1966).

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