Morley numbers for generalized languages

by

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Abstract. Morley numbers (or Hanf numbers for omitting types) are examined for the languages $L(Q_\alpha)$ and $L_{\varkappa\omega}$. It is shown that for $L(Q_\alpha)$ the Morley number is at most $\beth(\aleph_0, (2^{|L|+\aleph_\alpha})^+)$ and for $L_{\varkappa\omega}$ the Morley number is at most $\beth(\aleph_0, (2^{|L_{\varkappa\omega}|})^+)$.

In this paper upper bounds for the Morley numbers of $L(Q_\alpha)$ and $L_{\varkappa\omega}$ are found.

By the Morley number for $L(Q_\alpha)$ we mean the least cardinal λ such that for a theory T and a set of single-variable types $\mathscr{S}$ (in $L(Q_\alpha)$), if for each cardinal ν less than λ T has a model of power at least ν which omits every member of $\mathscr{S}$, then T has arbitrarily large such models. A similar definition is used for $L_{\varkappa\omega}$.

For $L(Q_\alpha)$ we show that the Morley number is at most $\beth(\aleph_0, (2^{|L|+\aleph_\alpha})^+)$ where $|L|$ is the cardinality of the set of first-order formulae of L. For $L_{\varkappa\omega}$ the Morley number is shown to be at most $\beth(\aleph_0, (2^{|L_{\varkappa\omega}|})^+)$ where $|L_{\varkappa\omega}|$ is the cardinality of the set of well-formed formulae of $L_{\varkappa\omega}$.

The results presented here are contained in Gillam [3].

§ 1. **Preliminaries.** Throughout this paper α, β, ξ, ϱ will denote ordinals, and the remaining lower case Greek letters, with the exception of μ, φ and ψ, will denote cardinals. The cardinal $\beth(\varkappa, \xi)$ is defined by induction on ξ: $\beth(\varkappa, 0) = \varkappa$, $\beth(\varkappa, \xi+1) = 2^{\beth(\varkappa,\xi)}$, and if ξ is a limit ordinal, $\beth(\varkappa, \xi) = \sup\{\beth(\varkappa, \lambda)\colon \lambda<\xi\}$. For a set A, $|A|$ denotes the cardinality of A.

L will denote a first-order language. $L(Q_\alpha)$ is the extension of L obtained by the addition of the generalized quantifier Q_α (interpreted as "there exist at least $\aleph_\alpha$"). $L_{\varkappa\omega}$ is the usual infinitary language considered as an extension of a first-order language L. Details on the construction of these languages, and some of their properties, may be found, for example, in Bell and Slomson [1], Chapters 13 and 14. Our notation is basically that of [1].

If $\mathfrak{A}$ is a model for the language L and X is a subset of A, the universe of $\mathfrak{A}$, then $L(X)$ denotes the language obtained from L by the addition to L of constant symbols for the elements of X. Similarly, we will use $L(Q_\alpha)(X)$ and $L_{\varkappa\omega}(X)$ when dealing with $L(Q_\alpha)$ and $L_{\varkappa\omega}$ respectively.

T will denote a theory and $\mathscr{S}$ a set of types in the appropriate language (either $L(Q_\alpha)$ or $L_{\varkappa\omega}$). Here, a type is a set of formulae, not necessarily first-order, which we assume to have only the variable v_0 free. We use $|L|$, $|L(Q_\alpha)|$ and $|L_{\varkappa\omega}|$ to denote the cardinalities of the sets of formulae of L, $L(Q_\alpha)$ and $L_{\varkappa\omega}$ respectively.

We will use n_γ to denote the sup of the Morley numbers of those languages which have at most γ non-logical symbols; and m_γ for the sup of Morley numbers obtained for such languages by requiring the set $\mathscr{S}$ of types in the definition to have power at most γ. Further, $n_\gamma(Q_\alpha)$, $m_\gamma(Q_\alpha)$, $n_\gamma(\varkappa\omega)$ and $m_\gamma(\varkappa\omega)$ will be used in similar ways for $L(Q_\alpha)$ and $L_{\varkappa\omega}$ respectively.

Lastly, we note for later use the following result of Erdös and Rado.

THEOREM. *If Y is a set, $|Y| = \beth(\lambda, n-1)^+$, and $\{C_i\colon i \in I\}$ is a partition of $[Y]^{(n)}$ — the set of n-element subsets of Y — such that $|I| \leqslant \lambda$, then there is a subset X of Y of power λ^+ such that for some $j \in I$, $[X]^{(n)} \subseteq C_j$.*

§ 2. **Omitting types in $L(Q_\alpha)$.** We define a *Skolem extension* $L(Q_\alpha)^*$ of $L(Q_\alpha)$ as follows: first $L(Q_\alpha)'$ is obtained from $L(Q_\alpha)$ by the addition of function letters $f_{\varphi,\xi}$ for $\xi < \aleph_\alpha$ and for each formula φ of the form $(Q_\alpha v_0)\psi(v_0, \ldots, v_n)$, and also the usual function letters f_φ corresponding to formulae φ of the form $(\exists v_0)\psi(v_0, \ldots, v_n)$; $L(Q_\alpha)^*$ is the language obtained from $L(Q_\alpha)$ by the iteration of this procedure ω times.

For $\varphi \in L(Q_\alpha)^*$, if φ is the form $(Q_\alpha v_0)\psi(v_0, \ldots, v_n)$, for some formula ψ, put

$$R_\varphi = \{(\forall v_1)\ldots(\forall v_n)\big(\varphi \to f_{\varphi,\xi}(v_1, \ldots, v_n) \neq f_{\varphi,\zeta}(v_1, \ldots, v_n)\big)\colon \xi, \zeta < \aleph_\alpha,\ \xi \neq \zeta\} \cup$$
$$\cup \{(\forall v_1)\ldots(\forall v_n)\big(\varphi \to \psi(f_{\varphi,\xi}(v_1, \ldots, v_n), v_1, \ldots, v_n)\big)\colon \xi < \aleph_\alpha\};$$

if φ is of the form $(\exists v_0)\psi(v_0, \ldots, v_n)$ for some formula ψ, put

$$R_\varphi = \{(\forall v_1)\ldots(\forall v_n)\big(\varphi \to \psi(f_\varphi(v_1, \ldots, v_n), v_1, \ldots, v_n)\big)\}$$

and otherwise put $R_\varphi = \emptyset$.

Then, if T is a theory in $L(Q_\alpha)$, let

$$T^* = T \cup \bigcup \{R_\varphi\colon \varphi \in L(Q_\alpha)^*\}.$$

Call T^* a *Skolem theory for T.*

The next two results follow immediately from the above.

LEMMA 2.1. *If T is a theory in $L(Q_\alpha)$ then $\mathfrak{A} \models T$ if and only if there is an expansion $\mathfrak{A}^*$ of $\mathfrak{A}$ to a model of $L(Q_\alpha)^*$ such that $\mathfrak{A}^* \models T^*$.*

LEMMA 2.2. *$L(Q_\alpha)^*$ has power $|L(Q_\alpha)| + \aleph_\alpha$.*

DEFINITION 2.3. If $\langle X, <\rangle$ is an ordered set and $\mathfrak{A}$ is a model of $L(Q_\alpha)$ containing X, then X is a set of *Q_α-indiscernibles* (or, more simply, indiscernibles) in $\mathfrak{A}$, if for each two properly ordered n-tuples $\langle x_0, \ldots, x_n\rangle$ and $\langle y_0, \ldots, y_n\rangle$ in X and each formula $\varphi(v_0, \ldots, v_n)$ of $L(Q_\alpha)$,

$$\mathfrak{A} \models \varphi[x_0, \ldots, x_n] \quad \text{if and only if} \quad \mathfrak{A} \models \varphi[y_0, \ldots, y_n].$$

If $\mathfrak{A}$ is a model of $L(Q_\alpha)^*$, and $B \subseteq A$ then $\mathscr{H}(B)$ is the restriction of $\mathfrak{A}$ to the closure of B in $\mathfrak{A}$ under the functions of $L(Q_\alpha)^*$.

In the obvious way, the definitions of $\preccurlyeq$ (elementary submodel) and $\equiv$ (elementary equivalence) may be extended to corresponding notions for $L(Q_\alpha)$, denoted by $\preccurlyeq_{Q_\alpha}$ and $\equiv_{Q_\alpha}$ respectively.

The proof of the next result is an extension of the proof of the corresponding theorem for elementary logic, which proceeds by an induction on the complexity of the formulae involved.

THEOREM 2.4. *If $\mathfrak{A}$ is a model of $L(Q_\alpha)^*$ and $B \subseteq A$, then $\mathscr{H}(B) \preccurlyeq_{Q_\alpha} \mathfrak{A}$.*

The following two lemmas will be used to prove the main result (2.9) of this section; the first is proven using methods similar to those used in Section 2 of Keisler [4].

For the remainder of this section, if $\mathfrak{A}$ is a model of L and $a \in A$, then we shall use $\underline{a}$ to denote the constant letter which, when added to L, gives the language $L(\{a\})$.

LEMMA 2.5. *Suppose $\mathfrak{A}^*$ is a model, $n < \omega$ and $\langle X, <\rangle$ is a linearly ordered subset of A such that any two properly ordered n-tuples of elements of X satisfy the same atomic formulae in $\mathfrak{A}^*$. If $\langle x_1, \ldots, x_n\rangle$ and $\langle y_1, \ldots, y_n\rangle$ are properly ordered n-tuples of elements of X and $s(v_1, \ldots, v_n)$ is a term in $L(Q_\alpha)^*$ such that $s(x_1, \ldots, x_n)$ and $s(y_1, \ldots, y_n)$ are distinct elements of $\mathfrak{A}^*$, then there is an integer i, $1 \leqslant i \leqslant n$, such that, if $x_i \neq x_i'$ and $\langle x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n\rangle$ is properly ordered, $s(x_1, \ldots, x_n)$ and $s(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)$ are distinct elements of $\mathfrak{A}^*$.*

Proof. The proof proceeds by induction on n, the number of variables in the term $s(v_1, \ldots, v_n)$.

Suppose $n = 1$; if

$$\mathfrak{A}^* \models \big(s(v_0) \neq s(v_1)\big)[x_1, y_1] \quad \text{for some } x_1, y_1 \in X \text{ such that } x_1 \neq y_1,$$

then

$$\mathfrak{A}^* \models \big(s(v_0) \neq s(v_1)\big)[x_1, y_1] \quad \text{for all } x_1, y_1 \in X \text{ such that } x_1 \neq y_1.$$

Assume that the result holds for $n = k$, and consider the properly ordered $(k+1)$-tuples $\langle x_1, \ldots, x_{k+1}\rangle$, $\langle y_1, \ldots, y_{k+1}\rangle$ of X, and a term $s(v_1, \ldots, v_{k+1})$ of $L(Q_\alpha)^*$ such that $s(x_1, \ldots, x_{k+1})$ and $s(y_1, \ldots, y_{k+1})$ are distinct elements of $\mathfrak{A}^*$.

Without loss of generality, assume $y_{k+1} \leqslant x_{k+1}$; then if $s(x_1, \ldots, x_{k+1})$ and $s(y_1, \ldots, y_k, x_{k+1})$ are distinct elements of $\mathfrak{A}^*$, so also are $s(x_1, \ldots, x_k, \underline{x_{k+1}})$ and $s(y_1, \ldots, y_k, \underline{x_{k+1}})$ and the result follows from the induction hypothesis. (Applied to the model $(\mathfrak{A}, x_{k+1})^*$, the ordered set with universe $\{y \in X\colon y < x_{k+1}\}$ and the term $s(v_1, \ldots, v_k, \underline{x_{k+1}})$.)

If $s(x_1, \ldots, x_{k+1}) = s(y_1, \ldots, y_k, x_{k+1})$ then $s(y_1, \ldots, y_{k+1})$ and $s(y_1, \ldots, y_k, x_{k+1})$ are distinct, so

$$\mathfrak{A}^* \models \big(s(v_1, \ldots, v_{k+1}) \neq s(v_1, \ldots, v_k, v_{k+2})\big)[y_1, \ldots, y_{k+1}, x_{k+1}]$$

and hence also

$$\mathfrak{A}^* \models \big(s(v_1, \ldots, v_{k+1}) \neq s(v_1, \ldots, v_k, v_{k+2})\big)[z_1, \ldots, z_{k+2}]$$

for all properly ordered $(k+2)$-tuples $\langle z_1, \ldots, z_{k+2}\rangle$ in X. Thus, for properly ordered $(k+1)$-tuples $\langle x_1, \ldots, x_{k+1}\rangle$ and $\langle x_1, \ldots, x_k, x'_{k+1}\rangle$ such that $x_{k+1} \neq x'_{k+1}$,

$$s(x_1, \ldots, x_{k+1}) \neq s(x_1, \ldots, x_k, x'_{k+1}) .$$

LEMMA 2.6. *Suppose that for each $n<\omega$, Δ_n is a set of formulae in $L(Q_\alpha)^*$ with at most the variables $v_0, \ldots, v_{n-1}$ free, and*

(1) *if $\varphi(v_0, \ldots, v_{n-1}) \in L(Q_\alpha)^*$ then either $\varphi \in \Delta_n$ or $\neg\varphi \in \Delta_n$;*

(2) *if $n<m$ then $\Delta_n \subseteq \Delta_m$;*

(3) *for $n<\omega$, there is a model $\mathfrak{A}_n^*$ of $L(Q_\alpha)^*$ which contains an ordered set $\langle X(n), <\rangle$ of power $>\aleph_\alpha$ such that for each properly ordered sequence $\langle x_0, \ldots, x_{n-1}\rangle$ in $X(n)$, $\mathfrak{A}_n^* \models \varphi[x_0, \ldots, x_{n-1}]$ if and only if $\varphi \in \Delta_n$ for all $\varphi(v_0, \ldots, v_{n-1}) \in L(Q_\alpha)^*$.*

Then, for any order type τ, if $\mathrm{cf}(\alpha)>\omega$ then $L(Q_\alpha)^*$ has a model $\mathfrak{A}^*$ containing a set of indiscernibles $\langle X, <\rangle$ of order type τ such that for $x_0<\ldots<x_{n-1}$ in X and $\varphi(v_0, \ldots, v_{n-1}) \in L(Q_\alpha)^*$,

$$\mathfrak{A}^* \models \varphi[x_0, \ldots, x_{n-1}] \quad \textit{if and only if} \quad \varphi \in \Delta_n .$$

Proof. Let X be an ordered set of power greater than $\aleph_\alpha$, with a subset of order type τ, and put

$$T = \{\varphi(\underline{x}_1, \ldots, \underline{x}_n)\colon x_1, \ldots, x_n \in X,\ x_1<x_2<\ldots<x_n \text{ and}$$
$$\varphi(v_0, \ldots, v_{n-1}) \in \Delta_n \text{ for } n<\omega\} .$$

Define the equivalence relation $\sim$ on the set of constant terms of $L(Q_\alpha)^*(X)$ by: if t_1 and t_2 are constant terms in $L(Q_\alpha)^*(X)$ then $t_1 \sim t_2$ if and only if $(t_1 = t_2) \in T$. $\mathfrak{A}^*$ is a model with universe the set of $\sim$-equivalence classes of constant terms of $L(Q_\alpha)^*(X)$, such that for all atomic formulae $\varphi(v_0, \ldots, v_{n-1})$ in $L(Q_\alpha)^*$ and constant terms $t_0, \ldots, t_{n-1} \in L(Q_\alpha)^*(X)$,

$$\mathfrak{A}^* \models \varphi[t_0/\sim, \ldots, t_{n-1}/\sim] \quad \text{if and only if} \quad \varphi(t_0, \ldots, t_{n-1}) \in T.$$

Obviously $\mathfrak{A}^*$ is well-defined (using the compactness theorem and the conditions on Δ_n), and $\mathfrak{A}^* = \mathscr{H}(X)$; we show that for all formulae $\varphi(v_0, \ldots, v_{n-1}) \in L(Q_\alpha)^*$ and $t_0/\sim, \ldots, t_{n-1}/\sim \in A$,

$$\mathfrak{A}^* \models \varphi[t_0/\sim, \ldots, t_{n-1}/\sim] \quad \text{if and only if} \quad \varphi(t_0, \ldots, t_{n-1}) \in T.$$

The proof is by induction on the length of φ, the interesting case being that when φ is of the form $(Q_\alpha v_0)\psi(v_0, v_1, \ldots, v_n)$ for some ψ $L(Q_\alpha)^*$.

Suppose the result holds for ψ, then

$$\mathfrak{A}^* \models \varphi[t_1/\sim, \ldots, t_n/\sim] \quad \text{if and only if}$$
$$\mathfrak{A}^* \models \psi[t_0/\sim, \ldots, t_n/\sim] \quad \text{for at least } \aleph_\alpha \text{ distinct } t_0/\sim \in A;$$

and this holds if and only if

(*) $\psi(t_0, \ldots, t_n) \in T$ for $t_0 \in K$ where K is a set of constant terms such that $|K| \geqslant \aleph_\alpha$, and if $t, t' \in K$ then the formula $(t \neq t')$ belongs to T.

Let $\psi(v_0, t_1, \ldots, t_n) = \theta(v_0, \underline{x}_1, \ldots, \underline{x}_m)$ where $x_1, \ldots, x_m \in X$ and $\theta(v_0, \ldots, v_m) \in L(Q_\alpha)^*$.

If $(Q_\alpha v_0)\psi(v_0, t_1, \ldots, t_n) \in T$ then (*) is true since T is a Skolem theory.

Conversely, suppose that (*) holds, and

$$(Q_\alpha v_0)\psi(v_0, t_1, \ldots, t_n) \notin T; \quad \text{so} \quad \neg(Q_\alpha v_0)\psi(v_0, t_1, \ldots, t_n) \in T.$$

Since $\mathrm{cf}(\alpha)>\omega$ and (*) holds, for some $l<\omega$ there is a set K_1 of power not less than $\aleph_\alpha$, of constant terms $s(\underline{x}_1, \ldots, \underline{x}_m, \underline{z}_1, \ldots, \underline{z}_l)$ in $L(Q_\alpha)^*(X)$, which belong to distinct $\sim$-equivalence classes, such that $\theta(s, \underline{x}_1, \ldots, \underline{x}_m) \in T$ for each $s \in K_1$.

Let $\langle y_1, \ldots, y_m\rangle$ be a sequence in $X(m+l)$, order isomorphic to $\langle x_1, \ldots, x_m\rangle$, and such that, if $y_i<y_j$ for some i, j, where $1 \leqslant i, j \leqslant m$, then $\{y \in X(m+l)\colon y_i<y<y_j\}$, $\{y \in X(m+l)\colon y_i<y\}$ and $\{y \in X(m+l)\colon y_i>y\}$ all have power not less than $\aleph_0$. Such a sequence exists because $|X(m+l)|>\aleph_0$. Then

$$\langle \mathfrak{A}_{m+l}^*, x\rangle_{x \in X(m+l)} \models \neg(Q_\alpha v_0)\theta(v_0, \underline{y}_1, \ldots, \underline{y}_m)$$

by the construction of T. Let K_2 be a set of constant terms $s'(\underline{y}_1, \ldots, \underline{y}_m, \underline{w}_1, \ldots, \underline{w}_l)$ in $L(Q_\alpha)^*(X(m+l))$ which represent distinct elements of $\mathfrak{A}_{m+l}^*$ and for which

(i) $\langle \mathfrak{A}_{m+l}^*, x\rangle_{x \in X(m+l)} \models \theta(s', \underline{y}_1, \ldots, \underline{y}_m)$;

(ii) if $s'_1(\underline{y}_1, \ldots, \underline{y}_m, \underline{w}'_1, \ldots, \underline{w}'_l)$ is a constant term in $L(Q_\alpha)^*(X(m+l))$ and

$$\langle \mathfrak{A}_{m+l}^*, x\rangle_{x \in X(m+l)} \models \theta(s'_1(\underline{y}_1, \ldots, \underline{y}_m, \underline{w}'_1, \ldots, \underline{w}'_l), y_1, \ldots, y_m)$$

then

$$s'_1(\underline{y}_1, \ldots, \underline{y}_m, \underline{w}'_1, \ldots, \underline{w}'_l) \sim s'_2 \quad \text{for some constant term } s''_2 \in K_2 .$$

K_2 has power less than $\aleph_\alpha$ and there is a function $f\colon K_1 \to K_2$ such that if $s(\underline{x}_1, \ldots, \underline{x}_m, \underline{z}_1, \ldots, \underline{z}_l)$ is in K_1 then $f(s) = s(\underline{y}_1, \ldots, \underline{y}_m, \underline{w}_1, \ldots, \underline{w}_l)$ for some $\underline{w}_1, \ldots, \underline{w}_l \in X(m+l)$ where $\langle x_1, \ldots, x_m, z_1, \ldots, z_l\rangle$ and $\langle y_1, \ldots, y_m, w_1, \ldots, w_l\rangle$ are order isomorphic. Since $|K_1|>|K_2|$, there exist $s_1, s_2 \in K_1$ such that $f(s_1) = f(s_2)$ and $(s_1 \neq s_2)$ is in T. Hence, if $s_1 = s(\underline{x}_1, \ldots, \underline{x}_m, \underline{z}_1, \ldots, \underline{z}_l)$ and $s_2 = s(\underline{x}_1, \ldots, \underline{x}_m, \underline{z}_{l+1}, \ldots, \underline{z}_{2l})$ for some term $s(v_1, \ldots, v_{m+l})$ of $L(Q_\alpha)^*$, then $s(\underline{y}_1, \ldots, \underline{y}_m, \underline{w}_1, \ldots, \underline{w}_l)$ and $s(\underline{y}_1, \ldots, \underline{y}_m, \underline{w}_{l+1}, \ldots, \underline{w}_{2l})$ represent distinct elements of $\mathfrak{A}_{m+2l}^*$ if $\langle x_1, \ldots, x_m, z_1, \ldots, z_{2l}\rangle$ and $\langle y_1, \ldots, y_m, w_1, \ldots, w_{2l}\rangle$ are order isomorphic.

By 2.5, there is an $i \leqslant l$ such that

$$s(\underline{y}_1, \ldots, \underline{y}_m, \underline{w}_1, \ldots, \underline{w}_l) \quad \text{and} \quad s(\underline{y}_1, \ldots, \underline{y}_m, \underline{w}_1, \ldots, \underline{w}_{i-1}, \underline{w}'_i, \underline{w}_{i+1}, \ldots, \underline{w}_l)$$

represent distinct elements of $\langle \mathfrak{A}_{m+2l}^*, x\rangle_{x \in X(m+2l)}$ if $w_i \neq w'_i$, and

$$\langle y_1, \ldots, y_m, w_1, \ldots, w_l\rangle \quad \text{and} \quad \langle y_1, \ldots, y_m, w_1, \ldots, w_{i-1}, w'_i, w_{i+1}, \ldots, w_l\rangle$$

are order isomorphic. Since $|X(m+2l)|>\aleph_\alpha$, $\langle y_1, \ldots, y_m, w_1, \ldots, w_l\rangle$ may be re-selected so that there are at least $\aleph_\alpha$ distinct choices for w'_i.

Since $\langle \mathfrak{A}^*_{m+2l}, x\rangle_{x\in X(m+2l)} \models \theta(s_1(y_1, \ldots, y_m, w_1, \ldots, w_l)y_1, \ldots, y_m$ we have $\langle \mathfrak{A}^*_{m+2l}, x\rangle_{x\in X(m+2l)} \models (Q_\alpha v_0)\theta(v_0, y_1, \ldots, y_m)$.

Hence $(Q_\alpha v_0)\psi(v_0, t_1, \ldots, t_n)$ is in T — this is the required contradiction.

COROLLARY 2.7. *If T^* is a Skolem theory in $L(Q_\alpha)^*$ with a model containing a set of indiscernibles of power $>\aleph_\alpha$, and if $\mathrm{cf}(\alpha)>\omega$; then for any order type τ, T^* has a model containing a set of indiscernibles of type τ.*

In fact, the condition $\mathrm{cf}(\alpha)>\omega$ may be omitted in this last result (the proof then follows that of 2.6, and if K_1 is taken to be a set of representatives of $\sim$-equivalence classes, with exactly one representative from each class, then the assumption $\mathrm{cf}(\alpha)>\omega$ is not required).

COROLLARY 2.8. *If Δ_n, $\mathfrak{A}^*_n$, $X(n)$ for $n<\omega$ satisfy the conditions of 2.6, and $\mathfrak{A}^*$ is the model constructed in the proof of 2.6, then each type omitted by all the $\mathfrak{A}^*_n$ for $n<\omega$ is omitted by $\mathfrak{A}^*$.*

THEOREM 2.9. *Suppose T is a theory in $L(Q_\alpha)$, $\mathrm{cf}(\alpha)>\omega$, $\mathscr{S}$ is a set of types in $L(Q_\alpha)$, and for each cardinal $\lambda<\beth(\aleph_0, (2^{|L|+\aleph_\alpha})^+)$ T has a model of power not less than λ which omits $\mathscr{S}$; then T has arbitrarily large models which omit $\mathscr{S}$ (in fact, T has such models in every power greater than $\aleph_\alpha$).*

Proof. Put $\mu = 2^{|L|\times\aleph_\alpha}$. For each $n<\omega$ and all ordinals $\beta<\mu^+$ we construct a set of formulae Δ_n, and models $\mathfrak{A}^*_{n,\beta}$, each of which contains an ordered set $\langle X_{n,\beta}, <\rangle$ such that conditions (1), (2) and (3) of 2.6 are satisfied by Δ_n, $\mathfrak{A}^*_{n,\beta}$ and $\langle X_{n,\beta}, <\rangle$ for all $\beta<\mu^+$. Furthermore, each $\mathfrak{A}^*_{n,\beta}$ omits $\mathscr{S}$, and $X_{n,\beta}$ has power greater than $\beth(\aleph_0, \beta)$.

Suppose these have been constructed for $n\leqslant k-1$ and $\beta<\mu^+$; first construct a set of formulae $\Delta_{k,\beta}$ in $L(Q_\alpha)^*$, and a linearly ordered set $\langle X'_{k,\beta}, <\rangle$ for each β, $\mu\leqslant\beta<\mu^+$, as follows: partition $[X_{k-1,\beta+k}]^{(k)}$ into classes C_i for $i\in I$, where I is some index set, such that for any two properly ordered sequences $\langle x_0, \ldots, x_{k-1}\rangle$ and $\langle y_0, \ldots, y_{k-1}\rangle$ of $X_{k-1,\beta+k}$, $\{x_0, \ldots, x_{k-1}\}$ and $\{y_0, \ldots, y_{k-1}\}$ are in the same class C_i if and only if for each formula $\varphi(v_0, \ldots, v_{n-1})$ of $L(Q_\alpha)^*$

$$\mathfrak{A}^*_{k-1,\beta+k} \models \varphi[x_0, \ldots, x_{k-1}] \quad \text{if and only if} \quad \mathfrak{A}^*_{k-1,\beta+k} \models \varphi[y_0, \ldots, y_{k-1}].$$

There are at most μ such classes, so, by the Erdös–Rado theorem, there is $X'_{k,\beta}\subseteq X_{k-1,\beta+k}$ which is homogeneous for this partition and such that $|X'_{k,\beta}|>\beth(\aleph_0, \beta)$. Put

$$\Delta_{k,\beta} = \{\varphi(v_0, \ldots, v_{k-1}):\ \mathfrak{A}^*_{k,\beta} \models \varphi[x_0, \ldots, x_{k-1}]\}$$

for some properly ordered sequence $\langle x_0, \ldots, x_{k-1}\rangle$ in $X'_{k,\beta}\}$. For β, $\mu\leqslant\beta<\mu^+$, $\Delta_{k,\beta}$ can take at most μ values, so there is a sequence of ordinals $\langle\beta_\xi\colon \xi<\mu^+\rangle$ cofinal in μ^+ such that $\Delta_{k,\beta_\xi} = \Delta_{k,\beta_\zeta}$ for all ξ, $\zeta<\mu^+$. Let $\Delta_k = \Delta_{k,\beta_0}$; the models $\mathfrak{A}^*_{k,\beta}$ and the linearly ordered sets $X_{k,\beta}$ can now be constructed in the obvious way, using the sequence $\langle\beta_\xi\colon \xi<\mu^+\rangle$.

Applying 2.6 and 2.8, models of T of arbitrarily large powers which omit $\mathscr{S}$ may be obtained; so the theorem is proven.

COROLLARY 2.10.

$$n_\gamma(L(Q_\alpha))\leqslant\beth(\aleph_0, (2^{\gamma+\aleph_\alpha})^+).$$

The following corollary is an analogue of a result noted by Chang in [2].

COROLLARY 2.11. *If $\mathscr{S}$ is of power at most $|L|+\aleph_\alpha$ then, in the statement of 2.9, $\beth(\aleph_0, (2^{|L|+\aleph_\alpha})^+)$ may be replaced by a smaller cardinal. So, in particular,*

$$m_\gamma(L(Q_\alpha))<\beth(\aleph_0, (2^{\gamma+\aleph_\alpha})^+).$$

Proof. Let K be the set of cardinals $\varkappa$ such that there is a set of types $\mathscr{S}$ of power $\leqslant|L|+\aleph_\alpha$ and a theory T in $L(Q_\alpha)$ such that for all λ, T has a model of power at least λ which omits all the types in $\mathscr{S}$ if $\lambda<\varkappa$, and no such model exists if $\lambda\geqslant\varkappa$. Clearly $|K|\leqslant 2^{|L|+\aleph_\alpha}$, and so $\sup\varkappa<\beth(\aleph_0, (2^{\aleph_0+|L|})^+)$.

§ 3. **Omitting types in $L_{\varkappa\omega}$.** We proceed now to find an upper bound for the Morley number for $L_{\varkappa\omega}$. Where the proofs are obtained by a straight-forward reconstruction in this setting of a corresponding proof in the previous section, we simply refer to the appropriate theorem of that section for proof.

First, add Skolem function letters f_φ to $L_{\varkappa\omega}$ for each formula φ of the form $(\exists v_0)\psi(v_0, \ldots, v_n)$ of $L_{\varkappa\omega}$, and so obtain $L'_{\varkappa\omega}$; $L^*_{\varkappa\omega}$ is then obtained by ω iterations of this procedure. If T is a theory in $L_{\varkappa\omega}$, T^* is defined by

$$T^* = T\cup\bigcup\{S_\varphi\colon\ \varphi\in L^*_{\varkappa\omega}\},$$

where $S_\varphi = \{(\forall v_1)\ldots(\forall v_n)(\varphi\rightarrow\psi(f_\varphi(v_1, \ldots, v_n), v_1, \ldots, v_n))\}$ if φ is of the form $(\exists v_0)\psi(v_0, \ldots, v_n)$, and $S_\varphi = \emptyset$ otherwise.

LEMMA 3.1. *If T is a theory in $L_{\varkappa\omega}$, then $\mathfrak{A}\models T$ if and only if there is an expansion $\mathfrak{A}^*$ of $\mathfrak{A}$ to a model of $L^*_{\varkappa\omega}$ such that $\mathfrak{A}^*\models T^*$.*

DEFINITION 3.2. If φ is a formula of $L_{\varkappa\omega}$ then $l(\varphi)$, the *length of* φ, is defined by transfinite induction as follows:

(1) if φ is atomic then $l(\varphi) = 0$;

(2) if $l(\varphi)$ and $l(\psi)$ have been defined then

$$l(\neg\varphi) = l(\varphi)+1,$$

$$l(\varphi\wedge\psi) = \max\{l(\varphi), l(\psi)\}+1 \quad \text{and}$$

$$l((\exists v_\xi)\varphi) = l(\varphi)+1 \quad \text{for} \quad \xi<\varkappa;$$

(3) if ζ is an ordinal, $\zeta<\varkappa$ and $l(\varphi_\xi)$ has been defined for $\xi<\zeta$, then

$$l(\bigwedge_{\xi<\zeta}\varphi_\xi) = \sup\{l(\varphi_\xi)+1\colon\ \xi<\zeta\}.$$

LEMMA 3.3. *$L^*_{\varkappa\omega}$ has power at most $|L|^\varkappa$, where L is the first-order language with the same non-logical symbols as $L_{\varkappa\omega}$.*

Proof. Let $\Phi_\xi = \{\varphi \in L_{\varkappa\omega}: l(\varphi)<\xi\}$. Then $|\Phi_0|\leqslant|L|^\varkappa$. Suppose $|\Phi_\xi|\leqslant|L|^\varkappa$ for each $\xi<\zeta$. If $\zeta = \xi+1$ for some ξ, $\Phi_\varrho\leqslant(|L|^\varkappa)^\varkappa = |L|^\varkappa$. If $\zeta = \sup\{\xi\colon \xi<\zeta\}$ then

$$|\Phi_\zeta|\leqslant \sum_{\xi<\zeta}|\Phi_\xi|\leqslant \sum_{\xi<\zeta}|L^\varkappa| = |L^\varkappa|.$$

Hence $|L_{\varkappa\omega}|\leqslant|L^\varkappa$, so $|L'_{\varkappa\omega}| = (|L|^\varkappa)^\varkappa = |L|^\varkappa$ and $|L^*_{\varkappa\omega}| = |L|^\varkappa$.

By analogy with the definitions of the previous section, the notion of a *set of indiscernibles* may now be defined for the language $L_{\varkappa\omega}$; also, the *Skolem closure* $\mathscr{H}(B)$ of a subset B of a model $\mathfrak{A}^*$ of $L^*_{\varkappa\omega}$ and the notation $\preccurlyeq_{\varkappa\omega}$ and $\equiv_{\varkappa\omega}$.

So the following theorem is obtained; its proof is a simple extension of that for elementary logic.

THEOREM 3.4. *If $\mathfrak{A}^*$ is a model of $L^*_{\varkappa\omega}$ and $B\subseteq A$, then $\mathscr{H}(B)\preccurlyeq_{\varkappa\omega}\mathfrak{A}^*$.*

LEMMA 3.5. *Suppose that for each $n<\omega$, Δ_n is a set of formulae in $L^*_{\varkappa\omega}$ with at most $v_0, \ldots, v_{n-1}$ free, and*

(1) *if $\varphi(v_0, \ldots, v_{n-1})\in L^*_{\varkappa\omega}$ then either $\varphi\in\Delta_n$ or $(\neg\varphi)\in\Delta_n$;*

(2) *if $n<m$ then $\Delta_n\subseteq\Delta_m$;*

(3) *for $n<\omega$ there is a model $\mathfrak{A}^*_n$ of $L^*_{\varkappa\omega}$ containing an infinite linearly ordered set $\langle X_n, <\rangle$ such that, if $\langle x_0, \ldots, x_{n-1}\rangle$ is a properly ordered sequence in X_n, then $\mathfrak{A}^*_n\models\varphi[x_0, \ldots, x_{n-1}]$ if and only if $\varphi\in\Delta_n$, for all formulae $\varphi\in L^*_{\varkappa\omega}$.*

Then for any order type τ, $L^*_{\varkappa\omega}$ has a model $\mathfrak{A}^*$ containing a set of indiscernibles $\langle X, <\rangle$ of order type τ, such that for each properly ordered sequence $\langle x_0, \ldots, x_{n-1}\rangle$ in X, $\mathfrak{A}^*\models\varphi[x_0, \ldots, x_{n-1}]$ if and only if $\varphi\in\Delta_n$ for all formulae $\varphi(v_0, \ldots, v_{n-1})\in L^*_{\varkappa\omega}$.

Furthermore, any type in $L^*_{\varkappa\omega}$ omitted by each $\mathfrak{A}^*_n$ is omitted by $\mathfrak{A}^*$.

Proof. This follows the proof of 2.6, constructing a model on the constant terms of $L^*_{\varkappa\omega}(X)$ and proving the first condition on $\mathfrak{A}^*$ by induction on the length of the formulae of $L^*_{\varkappa\omega}$.

Following 2.9, 2.10 and 2.11 bounds on the Morley numbers for $L_{\varkappa\omega}$ are obtained.

THEOREM 3.6. *Suppose T is a theory in $L_{\varkappa\omega}$, $\mathscr{S}$ is a set of types in $L_{\varkappa\omega}$ and for each $\lambda<\beth(\aleph_0, (2^{|L|^\varkappa})^+)$, T has a model of power $\geqslant\lambda$ which omits $\mathscr{S}$; then T has arbitrarily large models which omit $\mathscr{S}$ (in fact, in every power $>|L|^\varkappa$).*

COROLLARY 3.7. $n_\gamma(L_{\varkappa\omega})\leqslant\beth(\aleph_0, (2^{(\gamma^\varkappa)})^+)$.

COROLLARY 3.8. *If S is a set of types in $L_{\varkappa\omega}$ of power at most $|L|^\varkappa$, then in the statement of 3.6, $(\aleph_0, \beth(2^{|L|^\varkappa})^+)$ may be replaced by a smaller cardinal. In particular,*

$$m_\gamma(L_{\varkappa\omega})<(\aleph_0, \beth(2^{(\gamma^\varkappa)})^+).$$

§ 4. Concluding remarks. Let h_γ, $h_\gamma(Q_\alpha)$, $h_\gamma(\varkappa\omega)$ and $h_\gamma(Q_\alpha, \varkappa\omega)$ denote the supremum of the Hanf numbers for single sentences of the languages L, $L(Q_\alpha)$, $L_{\varkappa\omega}$ and $L_{\varkappa\omega}(Q_\alpha)$ respectively where L is required to have at most γ symbols (here $L_{\varkappa\omega}(Q_\alpha)$ is the language $L_{\varkappa\omega}$ enriched by the addition of the quantifier Q_α). Similarly H_γ, $H_\gamma(Q_\alpha)$ etc. are the Hanf numbers of sets of sentences in the appropriate languages.

1. If $|L| = |K|\geqslant\aleph_\alpha$, where K is one of the languages mentioned above, then

$$\beth(\aleph_0, (2^{|K|+\aleph_\alpha})^+) = \beth(\aleph_0, (2^{|L|})^+)$$

and this last is the upper bound stated in [2] for n_γ if $|L| = \gamma$. It has been shown that $n_\gamma = \beth(\aleph_0, (2^\gamma)^+)$ — see Shelah [5] — so $n_\gamma(L(Q_\alpha)) = \beth(\aleph_0, (2^\gamma)^+)$ for $\gamma\geqslant\aleph_\alpha$.

PROBLEM. What is the value of $n_\gamma(L(Q_\alpha))$ for $\gamma<\aleph_\alpha$.

2. The proof used by Chang in [2] to show that $m_\gamma = h_\gamma(\gamma^+\omega)$ may easily be used to obtain $n_\gamma = H(\gamma^+\omega)$, $m_\gamma(Q_\alpha) = h(\gamma^+\omega, Q_\alpha)$ and $n_\gamma(Q_\alpha) = H(\gamma^+\omega, Q_\alpha)$.

CONJECTURE. $m_\gamma(Q_\alpha) = m_\gamma$ — at least for $\gamma\geqslant\aleph_\alpha$.

3. For $L_{\varkappa\omega}$, if $|L_{\varkappa\omega}| = \gamma\geqslant2^\varkappa$ then $\gamma^\varkappa = \gamma$, so Corollary 3.7 yields $n_\gamma(\varkappa\omega)\leqslant\beth(\aleph_0, (2^\gamma)^+)$ — again the upper bound mentioned in [2] for n_γ. So $n_\gamma(\varkappa\omega) = n_\gamma = \beth(\aleph_0, (2^\gamma)^+)$ for $\gamma\geqslant2^\varkappa$.

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