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Two closed categories of filters

by

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Abstract. Two categories mentioned in the title each have as objects all filters. The morphisms from D to E in one of the categories are just the maps sending D to a filter extending E; The morphisms in the other category are equivalence classes of such maps modulo "equality on a set in D". After discussing some elementary relationships between these categories and some pairs of adjoint functors between them and the category of sets, we showt hat the first of our categories is both left and right complete. The second category is finitely left complete and has coproducts, but we give examples showing that it lacks coequalizers and infinite products. We also show that each of two categories of filters is a closed monoidal (but not symmetric) category, in the sense of Eilenberg and Kelly, and we briefly discuss some examples of categories over these closed categories.

A useful methodological principle in modern mathematics is that, when one defines a type of mathematical structure, one should also define the notion of a morphism (or map) between two structures of that type. For a long time, this principle was ignored in the case where the structures are filters. As far as I know, the first published definition of a morphism of filters is in [6], where it is only briefly mentioned. A different definition was proposed, and the resulting category of filters investigated, in [7]. (This definition was also used, but only for ultrafilters, in [1].) The purpose of the present paper is to investigate the categories $\mathscr{F}$ and $\mathscr{G}$ arising from these two definitions of morphisms.

In Section 1, we define two categories and develop their most elementary properties, including various functors between them and the category of sets and various adjunctions between these functors. This section also contains a discussion of the heuristic meaning of the difference between the definitions of morphisms in $\mathscr{F}$ and $\mathscr{G}$. In Section 2, we prove a number of results about the existence or non-existence of various sorts of limits in our categories. The neatest of these results is that $\mathscr{F}$ is both left and right complete, but perhaps more interesting are some of the counterexamples to completeness in $\mathscr{G}$. In Section 3, we amplify the discussion, at the end of Section 1, of the relation between $\mathscr{F}$ and $\mathscr{G}$ by showing that $\mathscr{G}$ can be obtained from $\mathscr{F}$ as a category of fractions with respect to a very natural class of morphisms. We further show that this class admits a calculus of right fractions; this provides an alternate proof of some results in Section 2. Finally, in

Section 4, we point out that each of our categories is in fact a closed monoidal category. We briefly discuss and give examples of categories enriched over $\mathscr{F}$ and $\mathscr{G}$.

1. The categories $\mathscr{F}$ and $\mathscr{G}$. A *filter* F on a set A is, by definition, a nonempty family of subsets of A such that, for any $X, Y \subseteq A$, the intersection $X \cap Y$ is in F if and only if both X and Y are in F. (For the elementary properties of filters used without proof below, see [2] or any of numerous other general topology books.) Notice that we allow a filter to contain the empty set, so that on each set A there is the filter $\mathbf{N}A$ consisting of all subsets of A and often called the *improper* filter on A. Notice also that a filter F uniquely determines the set A on which it is a filter, for A is the largest set in F (or the union of all the sets in F). When F is a filter on A, we define $\mathbf{U}F$ (the *universe* of F) to be A. (We are reluctant to call $\mathbf{U}F$ the underlying set of F, because a different notion of underlying set will be needed in Section 4.)

If F is a filter on A and $f\colon A \to B$ is any function, then the subsets X of B such that $f^{-1}(X) \in F$ constitute a filter $f(F)$ on B. If G is a filter on B, then we say that f maps F into G, and we write $f\colon F \to G$, if $f(F) \supseteq G$. It is easy to verify that we obtain a category $\mathscr{F}$ if we take as objects all filters, as morphisms from F to G all functions $f\colon F \to G$, and as composition ordinary composition of functions. The identity morphism of F is just the identity function on $\mathbf{U}F$.

The "universe" operator $\mathbf{U}$ from filters to sets becomes a functor from $\mathscr{F}$ to the category $\mathscr{S}$ of sets if we define $\mathbf{U}f$ to be f for all morphisms f of $\mathscr{F}$. The "improper filter" operator $\mathbf{N}$ also becomes a functor if we define $\mathbf{N}f$ to be f for all f.

THEOREM 1. *$\mathbf{U}$ is faithful. $\mathbf{N}$ is a full embedding. $\mathbf{N}$ is left adjoint to $\mathbf{U}$.*

Proof. The first two assertions are obvious. For the third, notice that, if $f\colon A \to B$ and G is a filter on B, then $f(\mathbf{N}A) = \mathbf{N}B \supseteq G$, so $f\colon \mathbf{N}A \to G$. It immediately follows that $\mathscr{F}(\mathbf{N}A, G)$ and $\mathscr{S}(A, \mathbf{U}G)$ are not only naturally isomorphic but identical. ■

We shall need two other functors between $\mathscr{F}$ and $\mathscr{S}$. For any set A, there is not only a largest filter $\mathbf{N}A$ on A but also a smallest, namely $\mathbf{P}A = \{A\}$. (P stands for "principal".) We obtain a functor $\mathbf{P}\colon \mathscr{S} \to \mathscr{F}$ by defining $\mathbf{P}f = f$ on morphisms.

THEOREM 2. *$\mathbf{P}$ is a full embedding and right adjoint to $\mathbf{U}$.*

Proof. Since $f(F) \supseteq \mathbf{P}B$ for all $f\colon A \to B$ and all filters F on A, we have that $\mathscr{S}(\mathbf{U}F, B)$ and $\mathscr{F}(F, \mathbf{P}B)$ are naturally identical. ■

By Theorems 1 and 2, we may think of $\mathscr{S}$ as embedded in $\mathscr{F}$ either as a full reflective subcategory by (by $\mathbf{P}$) or as a full coreflective subcategory (by $\mathbf{N}$).

For any filter F, we define its *core* $\mathbf{C}F$ to be the intersection of all the sets in F. We make $\mathbf{C}$ a functor from $\mathscr{F}$ to $\mathscr{S}$ by defining $\mathbf{C}f$ to be the restriction of f to $\mathbf{C}F$ whenever $f\colon F \to G$ is a morphism in $\mathscr{F}$.

THEOREM 3. *$\mathbf{C}$ is right adjoint to $\mathbf{P}$.*

Proof. For any set A and filter F, $\mathscr{F}(\mathbf{P}A, F)$ consists of those functions f from $A = \mathbf{UP}A$ to $\mathbf{U}F$ for which

$$\begin{aligned} F &\subseteq f(\mathbf{P}A) \\ &= \{X \subseteq \mathbf{U}F \mid f^{-1}(X) \in \mathbf{P}A = \{A\}\} \\ &= \{X \subseteq \mathbf{U}F \mid f(A) \subseteq X\}, \end{aligned}$$

which means $f(A) \subseteq \mathbf{C}F$. But these functions are just the elements of $\mathscr{S}(A, \mathbf{C}F)$. ■

From Theorems 1, 2, and 3, we have the following sequence of adjunctions:

$$\mathbf{N} \dashv \mathbf{U} \dashv \mathbf{P} \dashv \mathbf{C}.$$

This chain cannot be extended in either direction. $\mathbf{N}$ has no left adjoint because it maps the terminal object 1 of $\mathscr{S}$ to a nonterminal object in $\mathscr{F}$. At the other end, if $\mathbf{C}$ had a right adjoint $\mathbf{Q}$, we could obtain a contradiction by calculating, for any set B, that

$$\mathbf{UQ}B \cong \mathscr{S}(1, \mathbf{UQ}B) \cong \mathscr{F}(\mathbf{CN}1, B) = \mathscr{S}(\emptyset, B) \cong 1$$

while

$$\mathbf{CQ}B \cong \mathscr{S}(1, \mathbf{CQ}B) \cong \mathscr{F}(\mathbf{CP}1, B) = \mathscr{S}(1, B) \cong B;$$

when B has more than one element this contradicts the obvious fact that $\mathbf{C}F \subseteq \mathbf{U}F$ for all filters F.

Before turning to the definition of our second category of filters $\mathscr{G}$, we apply some of the adjunctions obtained above to characterize various kinds of morphisms in $\mathscr{F}$. These characterizations will be useful when we discuss the relationship between $\mathscr{F}$ and $\mathscr{G}$.

THEOREM 4. (a) *$f\colon F \to G$ is a monomorphism in $\mathscr{F}$ if and only if f is one-to-one (as a function from $\mathbf{U}F$ to $\mathbf{U}G$).*

(b) *$f\colon F \to G$ is an epimorphism in $\mathscr{F}$ if and only if f maps $\mathbf{U}F$ onto $\mathbf{U}G$.*

(c) *$f\colon F \to G$ is an isomorphism in $\mathscr{F}$ if and only if f maps $\mathbf{U}F$ one-to-one onto $\mathbf{U}G$ and $f(F) = G$.*

Proof. Having a left adjoint $\mathbf{N}$ and a right adjoint $\mathbf{P}$ the functor $\mathbf{U}$ must preserve monomorphisms and epimorphisms; being faithful, $\mathbf{U}$ also reflects them. This proves (a) and (b). The "if" part of (c) is easy because $f^{-1}\colon \mathbf{U}G \to \mathbf{U}F$ satisfies $f^{-1}(G) = f^{-1}f(F) = F$, so f^{-1} maps G into F and serves as the inverse of f in $\mathscr{F}$. As for the "only if" part, f must be a bijection and its inverse in $\mathscr{F}$ must be its inverse f^{-1} in $\mathscr{S}$ because $\mathbf{U}$ is a functor. For f^{-1} to map G into F, it is necessary that $f(F) \subseteq G$. The converse inclusion holds because $f\colon F \to G$, so $f(F) = G$. ■

Notice that $\mathscr{F}$ is not balanced; that is, a morphism can be both monic and epic without being an isomorphism. The simplest example is the unique morphism from $\mathbf{N}1$ to $\mathbf{P}1$.

We turn now to the definition of $\mathscr{G}$. Essentially, it will be a quotient of $\mathscr{F}$, two functions being identified if they agree "almost everywhere". More precisely, let F be a filter on A, let B be any set, and consider functions whose domains are subsets X of A with $X \in F$ and whose ranges are subsets of B (partial functions from A to B, defined almost everywhere). Call two such functions equal modulo F (written $f = g \bmod F$) if their restrictions to some set in F are the same. It is easy to see that equality modulo F is an equivalence relation; the equivalence class $[f]_F$ of a function f is called its F-germ (see [2], I. 6). Notice that any $f\colon X \to B$ with $X \in F$ is equal modulo F to any extension of f to a total function $f'\colon A \to B$. Thus, each germ contains a total function unless B is empty. And when B is empty there are no germs at all unless F is improper, in which case there is a unique germ. This observation usually allows us to assume (with at most a slight loss of generality) that any germ under consideration contains a total function.

The definition of $f(F)$ makes sense when f is a partial function of the sort considered above, if we let $f^{-1}(X)$ mean $\{x \in \mathrm{Domain}(f) | f(x) \in X\}$. Furthermore, $f(F)$ depends only on $[f]_F$, for, if f and g are equal modulo F, we have a set $X \in F$ on which f and g are defined and equal, so for any $Y \subseteq B$, $f^{-1}(Y) \cap X = g^{-1}(Y) \cap X$. Since $Y \in f(F)$ means $f^{-1}(Y) \in F$ or, equivalently, $f^{-1}(Y) \cap X \in F$, it follows immediately that $f(F) = g(F)$ as claimed. It therefore makes sense to speak of the image of F under an F-germ: $[f]_F(F) = f(F)$.

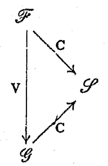
We define $\mathcal{G}$ to be the category whose objects are all the filters, whose morphisms from F to G are all the F-germs $[f]_F$ (with $f\colon X \to \mathbf{U}G$, $X \in F$) such that $[f]_F(F) \supseteq G$, and whose composition rule assigns to $[f]_F\colon F \to G$ and $[g]_G\colon G \to H$ the germ $[gf]_F\colon F \to H$. We leave to the reader the easy task of verifying that composition is well-defined and $\mathcal{G}$ is a category.

There is a forgetful functor $\mathbf{V}$ from $\mathcal{F}$ to $\mathcal{G}$ which is the identity on objects and sends each $\mathcal{F}$-morphism to its germ. By composing the functors $\mathbf{N}$ and $\mathbf{P}$ with $\mathbf{V}$ we obtain functors from $\mathcal{S}$ to $\mathcal{G}$ which we still call $\mathbf{N}$ and $\mathbf{P}$. The former is, however, rather uninteresting as $\mathbf{N}A$ is an initial object of $\mathcal{G}$, no matter what set A is; the unique $\mathcal{G}$-morphism from $\mathbf{N}A$ to F is the $\mathbf{N}A$-germ of the empty partial function. On the other hand, $\mathbf{P}$ retains in $\mathcal{G}$ many of the nice properties it had in $\mathcal{F}$.

THEOREM 5. (a) *For any set A and any filter F, $\mathcal{G}(\mathbf{P}A, F)$ and $\mathcal{F}(\mathbf{P}A, F)$ are naturally (with respect to both A and F) isomorphic (the isomorphism being $\mathbf{V}$).*

(b) *$\mathbf{P}$ is a full embedding of $\mathcal{S}$ into $\mathcal{G}$.*

(c) *$\mathbf{P}$ has a right adjoint $\mathbf{C}\colon \mathcal{G} \to \mathcal{S}$ which is the unique functor making the diagram*



commute.

Proof. (a) This is trivial once we notice that the $\mathbf{P}A$-germ of f consists of just f (and f is total).

(b) This is immediate from (a) and Theorem 2.

(c) The existence of a functor $\mathbf{C}\colon \mathcal{G} \to \mathcal{S}$ making the diagram commute follows from the observation that the F-germ of a function f completely determines the restriction of f to the core of F. Its uniqueness follows from the fact that $\mathbf{V}$ maps $\mathcal{F}$ onto all of $\mathcal{G}$ except for the morphisms $\mathbf{N}A \to \mathbf{N}\varnothing$ and a trivial verification to take case of these exceptions (recall $\mathbf{CN}A = \varnothing$). Finally, we have

$$\mathcal{G}(\mathbf{P}A, F) \cong \mathcal{F}(\mathbf{P}A, F) \cong \mathcal{S}(A, \mathbf{C}F)$$

naturally in both A and F, by (a) and Theorem 3. ■

Unlike $\mathbf{C}$ the functor $\mathbf{U}\colon \mathcal{F} \to \mathcal{S}$ does not factor through $\mathbf{V}$, because the germ of f does not determine f uniquely. Thus, the long chain of adjoint functors ($\mathbf{N} \dashv \mathbf{U} \dashv \mathbf{P} \dashv \mathbf{C}$) that we had between $\mathcal{F}$ and $\mathcal{S}$ is broken and we have only $\mathbf{P} \dashv \mathbf{C}$ between $\mathcal{G}$ and $\mathcal{S}$. We shall see in Section 2 that $\mathbf{P}\colon \mathcal{S} \to \mathcal{G}$ does not preserve infinite products, so it cannot have a left adjoint. Nor can $\mathbf{C}$ have a right adjoint $\mathbf{Q}\colon \mathcal{S} \to \mathcal{G}$. Although the argument we gave for the analogous statement about $\mathcal{F}$ is not directly applicable (for $\mathbf{U}$ is no longer available), half of that argument is still correct and yields $\mathbf{CQ}A \cong A$. Now let F be any proper filter with empty core (for example, the filter of cofinite subsets of some infinite set) and let A have at least two elements. Then there are at least two distinct constant functions from $\mathbf{U}F$ into $\mathbf{CQ}A$. Their F-germs are distinct (because F is proper) $\mathcal{G}$-morphisms from F to $\mathbf{Q}A$. Since $\mathbf{Q}$ is supposed to be right adjoint to $\mathbf{C}$, there must be two distinct maps from $\mathbf{C}F$ to A. As $\mathbf{C}F$ is empty, this is absurd.

Thus, for $\mathcal{G}$, the adjunction $\mathbf{P} \dashv \mathbf{C}$ is the best we can do. (It is true that $\mathbf{N}$ has a right adjoint $\mathcal{G} \to \mathcal{S}$, mapping all filters to the terminal object 1, but this is of no importance.) Incidentally, $\mathbf{V}$ has no adjoint on either side, for it preserves neither infinite products nor coequalizers, as we shall see in Section 2.

Parts (a) and (b) of the following theorem are due to Koubek and Reiterman [7]. It is the analog for $\mathcal{G}$ of Theorem 4 for $\mathcal{F}$.

THEOREM 6. (a) *$[f]_F\colon F \to G$ is a monomorphism in $\mathcal{G}$ if and only if there is a set $X \in F$ such that, on X, f is defined and one-to-one.*

(b) *$[f]_F\colon F \to G$ is an epimorphism in $\mathcal{G}$ if and only if $f(F) = G$.*

(c) *$\mathcal{G}$ is balanced; that is, $[f]_F\colon F \to G$ is an isomorphism if and only if it is both a monomorphism and an epimorphism.*

Proof. For parts (a) and (b), see [7], and verify that our use of germs of partial functions (rather than total functions as in [7]) makes no difference. The "only if" part of (c) is, of course, true in any category. For the converse, assume that f is defined and one-to-one on $X \in F$ and $f(F) = G$. As $f^{-1}f(X) \supseteq X \in F$, we have $f(X) \in f(F) = G$. Let $g\colon f(X) \to X$ be the inverse of the restriction of f to X, and notice that $[g]_G$ is an inverse for $[f]_F$ in $\mathcal{G}$. ■

Note that part (c) of the theorem would be false if we did not allow partial functions in defining $\mathcal{G}$.

By comparing Theorems 4 (c) and 6 (c), we find that in $\mathcal{F}$ a filter F should be thought of as being on a particular set $\mathbf{U}F$, while in $\mathcal{G}$ the same filter is, in some sense, no longer associated to any particular set. For example, if a filter F on a uncountable set A contains a countable subset B of A, then its restriction to B,

$$F \upharpoonright B = \{X \in F | \; X \subseteq B\} = \{X \cap B | \; X \in F\}$$

is essentially the same as F in $\mathcal{G}$ (more precisely, the inclusion of B into A induces an isomorphism $F \upharpoonright B \cong F$) despite the change in underlying set, whereas in $\mathcal{F}$ these two filters are not isomorphic. Heuristically, an object of $\mathcal{G}$ is thought of as an "abstract" filter, where the abstraction consists of ignoring the universe, whereas an object in $\mathcal{F}$ is a "concrete" filter. (Compare the fact that $\mathcal{F}$ is almost trivially a concrete category — it has the faithful functor $\mathbf{U}$ to $\mathcal{S}$ — with the more difficult proof in [7] that $\mathcal{G}$ is concrete — the forgetful functor in [7] is far more complex than $\mathbf{U}$.) This heuristic idea can be made precise by noticing that from any object F of $\mathcal{F}$ we get not only the object $\mathbf{V}F(=F)$ of $\mathcal{G}$ but also the object $\mathbf{PU}F$ of $\mathcal{G}$ and a morphism (natural in F) η_F: $\mathbf{V}F \to \mathbf{PU}F$ (the F-germ of the identity map on $\mathbf{U}F$, or, if you prefer, the $\mathbf{V}$ image of the adjunction $F \to \mathbf{PU}F$ in $\mathcal{F}$). Altogether, we have a functor from $\mathcal{F}$ to the comma category $(\mathcal{G}, \mathbf{P})$ (see [8] for information on the comma construction) which can be shown to be an equivalence from $\mathcal{F}$ to the full subcategory of $(\mathcal{G}, \mathbf{P})$ generated by those objects of the comma category which, as morphisms in $\mathcal{G}$, are monic. In less abstract terminology, this means that, if we identify $\mathcal{S}$ with its image under $\mathbf{P}$ in $\mathcal{G}$ (as we may, by Theorem 5 (b)), then an object F of $\mathcal{F}$ (a conrete filter) may be identified with the triple $(\mathbf{V}F, \mathbf{U}F, \eta_F)$ consisting of the abstract filter F, the ambient set $\mathbf{U}F$, and the embedding of F into this ambient set.

One can draw an analogy between the present situation of abstract vs. concrete filters and the situation in topology where usually one considers abstract spaces and homeomorphism is the appropriate equivalence relation, but in some areas (e.g. knot theory) one must consider spaces as embedded in some ambient space and the appropriate equivalence relation (e.g. ambient isotopy) involves the ambient space.

It appears that most uses of filters in mathematics do not depend on the availability of an ambient set; in other words the essential properties of filters are invariant under $\mathcal{G}$-isomorphism. For example, if F is a filter on A and f is a function from A into a topological space, then all questions about limits or adherent points of f with respect to F depend only on the F-germ of f (in fact most such questions depend only on $f(F)$). For another example, if $[f]_F$: $F \to G$ is a $\mathcal{G}$-isomorphism and $(S_a | \; a \in A)$ is a family of structures indexed by $A = \mathbf{U}G$, then the reduced product (see [4]) $\prod_{a \in A} S_a/G$ is isomorphic to $\prod_{b \in B} S_{f(b)}/F$, where $B = \mathbf{U}F$. (If f is partial, some of the factors $S_{f(b)}$ are undefined, but they do not affect the reduced product anyway; almost all the factors are defined, and that is all we need.) Furthermore, if we view the indexed family of structures as a function $(a \mapsto S_a)$ on A, the reduced product depends only on the G-germ of this function. (Incidentally, the reduced power of a set S with respect to a filter F is just $\mathcal{G}$ $(F, \mathbf{P}S)$.)

In view of examples like these, showing that abstract filters suffice for most purposes, I suggest that $\mathcal{G}$ deserves to be called the category of filters. To indicate that an object of $\mathcal{F}$ is viewed as an abstract filter plus an ambient set, I will call $\mathcal{F}$ the category of filtered sets. (It is perhaps characteristic of the category-theoretic point of view that we assign different names to the objects of $\mathcal{G}$ and those of $\mathcal{F}$, although they are the same objects — remember that $\mathcal{F}$ and $\mathcal{G}$ differ only in their morphisms. The true nature of a thing is determined not by what it "really is" (as a set) but by how it is related to similar things surrounding it.)

2. Completeness properties. It turns out that the category $\mathcal{F}$ of filtered sets behaves much better than the category $\mathcal{G}$ of filters with regard to both left and right limits. Accordingly, we begin our study of completeness by considering $\mathcal{F}$.

THEOREM 7. *$\mathcal{F}$ is both left and right complete.*

Proof. The construction of both limits and colimits in $\mathcal{F}$ is facilitated by the observation that they are preserved by $\mathbf{U}$, because $\mathbf{U}$ has adjoints on both sides.

Consider first the case of limits (= left limits). Let $\mathbf{D}$: $\mathcal{I} \to \mathcal{F}$ be a diagram in $\mathcal{F}$, i.e. a functor from a small category $\mathcal{I}$ into $\mathcal{F}$. By the preceding paragraph, we know that $\varprojlim \mathbf{D}$, if it exists, must be a filter F on the set $A = \varprojlim \mathbf{UD}$. For each object I of $\mathcal{I}$, let p_I: $A \to \mathbf{UD}I$ be the canonical projection. The fact that $\mathbf{U}$ is faithful and preserves limits shows that the canonical projection from F to $\mathbf{D}I$ must be p_I. But for p_I to even be a morphism in $\mathcal{F}$ requires that, for each $X \in \mathbf{D}I$, $p_I^{-1}(X) \in F$. Not wishing to put any unnecessary sets into F (for we shall need lots of morphisms into F to show that it is $\varprojlim \mathbf{D}$), we define F to be the filter on A generated by all sets of the form $p_I^{-1}(X)$ with $I \in \mathcal{I}$ and $X \in \mathbf{D}I$. The above discussion shows that we have $\mathcal{F}$-morphisms p_I: $F \to \mathbf{D}I$. These form a cone p: $F \to \mathbf{D}$; that is, they commute with the maps $\mathbf{D}f$, where f is a morphism of $\mathcal{I}$. To prove the commutativity, just observe that the diagrams in question become commutative when $\mathbf{U}$ is applied to them, and $\mathbf{U}$ is faithful. If G is another filter and q: $G \to \mathbf{D}$ is a cone, we claim there is a unique $\mathcal{F}$-morphism r: $G \to F$ such that $q = p \circ r$. We observe first that $\mathbf{U}q$: $\mathbf{U}G \to \mathbf{U}D$ is a cone from $\mathbf{U}G$ to the diagram $\mathbf{UD}$ whose limit is $\mathbf{U}F = A$ with the cone p. Hence, there is a unique map (in $\mathcal{S}$) r: $\mathbf{U}G \to \mathbf{U}F$ such that $q = p \circ r$. Since $\mathbf{U}$ is faithful, this r is the only possible choice for the required r in $\mathcal{F}$, and we need only check that r maps G into F. This means, we need $r^{-1}(Y) \in G$ for all $Y \in F$; by the definition of F, we need only check this when $Y = p_I^{-1}(X)$ for some $I \in \mathcal{I}$ and some $X \in \mathbf{D}I$. But then $r^{-1}(Y) = r^{-1}p_I^{-1}(X) = q_I^{-1}(X) \in G$ because q_I maps G into $\mathbf{D}I$. This proves that F, with the projections p_I, is $\varprojlim \mathbf{D}$.

Turning to colimits (= right limits), we again let D: $\mathcal{I} \to \mathcal{F}$ be any diagram. Let $A = \varinjlim \mathbf{UD}$ in $\mathcal{S}$, with injections j_I: $\mathbf{UD}I \to A$. Arguing as before, we see that $\varinjlim \mathbf{D}$, if it exists, must be a filter F on A and that the injections $\mathbf{D}I \to F$ must be j_I. For these injections to be morphisms in $\mathcal{F}$, we need $F \subseteq j_I(\mathbf{D}I)$. We want F to be as large as possible (for we shall need lots of morphisms out of F), so we define F to be $\bigcap_I j_I(\mathbf{D}I)$, the intersection being over all objects I of $\mathcal{I}$. It is then easy to check that $j\colon \mathbf{D}\to F$ is a cocone, the required commutativities being clear after one applies the faithful functor $\mathbf{U}$. If $i\colon \mathbf{D}\to G$ is any other cocone from $\mathbf{D}$, we claim there is a unique $\mathcal{F}$-morphism $k\colon F\to G$ such that $i = k\circ j$. Since $i = \mathbf{U}i\colon \mathbf{UD}\to \mathbf{U}G$ is a cocone and $\mathbf{U}F = \varinjlim \mathbf{UD}$ with injections j, there is a unique $k\colon \mathbf{U}F\to\mathbf{U}G$ such that $i = k\circ j$. As $\mathbf{U}$ is faithful, this k is the only possible choice for the required k in $\mathcal{F}$, and we need only check that it maps F into G. For each $Y\in G$, we must prove $k^{-1}(Y)\in F$, which, by definition of F, means $j_I^{-1}k^{-1}(Y)\in \mathbf{D}I$ for all I. But $j_I^{-1}k^{-1}(Y) = i_I^{-1}(Y)\in\mathbf{D}I$ because i_I maps $\mathbf{D}I$ into G. Therefore, F with the injections j_I, is $\varinjlim\mathbf{D}$. ■

Our next theorem is that $\mathcal{G}$ inherits certain kinds of limits and colimits from $\mathcal{F}$ via the forgetful functor $\mathbf{V}$. In the proof of this result, and again in Section 3, we shall need the following simple lemma.

LEMMA 8. *Let f and g be $\mathcal{F}$-morphisms from F to G, and suppose $\mathbf{V}f = \mathbf{V}g$ in $\mathcal{G}$. Then there is a set $B\in F$ such that the inclusion map $i\colon F\restriction B\to F$ satisfies $f\circ i = g\circ i$ in $\mathcal{F}$. For any such i, $\mathbf{V}i$ is an isomorphism in $\mathcal{G}$.*

Proof. The first assertion just restates the definition of equality modulo F used in defining $\mathcal{G}$. The second assertion follows immediately from our characterization of the isomorphisms of $\mathcal{G}$ in Theorem 6(c). ■

THEOREM 9. (a) *$\mathcal{G}$ has coproducts and $\mathbf{V}$ preserves them.*

(b) *$\mathcal{G}$ has finite products and $\mathbf{V}$ preserves them.*

(c) *$\mathcal{G}$ has equalizers and $\mathbf{V}$ preserves them.*

Proof. (a) Let a set $\{F_m|\ m\in M\}$ of filters be given, and let G be their coproduct in $\mathcal{F}$, with injections $j_m\colon F_m\to G$. We must show that G is also the coproduct of the F_m's in $\mathcal{G}$ with injections $[j_m] = \mathbf{V}j_m$. Suppose we are given some filter H and $\mathcal{G}$-morphisms $[f_m]\colon F_m\to H$; we claim that there is a unique $\mathcal{G}$-morphism $[g]\colon G\to H$ such that $[f_m] = [g]\circ[j_m]$ for all m. Suppose first that the morphisms $[f_m]$ contain total functions f_m, so $[f_m] = \mathbf{V}f_m$. Then, in $\mathcal{F}$, there is a (unique) $g\colon G\to H$ such that $f_m = g\circ j_m$ for all m; then $[g]\colon G\to H$ is a $\mathcal{G}$-morphism as required. Before proving its uniqueness (and disposing of the exceptional case where $[f_m]$ contains no total function), we remark that the preceding argument proved the following general fact.

PROPOSITION 10. *A full functor that is surjective on objects preserves weak coproducts.* ■

(Weak coproducts are defined like coproducts except that the factorization is not required to be unique. See [8].)

Returning to the proof of Theorem 9(a), we note that, when some $[f_m]$ contains no total function, H must be $\mathbf{N}\varnothing$. Since $\mathbf{N}\varnothing$ is isomorphic in $\mathcal{G}$ to $\mathbf{N}A$ for any other A, we can reduce this case to the one already treated.

To show that the factorization is unique, suppose we also had $[g']\colon G\to H$ with $[f_m] = [g']\circ[j_m]$ for all m. This means $f_m = g'\circ j_m \bmod F_m$, say $g\circ j_m = f_m = g'\circ j_m$ on $X_m\in F_m$. Then $g = g'$ on $j_m(X_m)$ for each m. Therefore, g and g' agree on $\bigcup_m j_m(X_m)$, which is in G (see the construction of $\mathcal{F}$-coproducts in the proof of Theorem 7). So $[g] = [g']$ as required.

(b) Let $\{F_m|\ m\in M\}$ be a finite set of filters, and let G be their $\mathcal{F}$-product with projections p_m. Unless some F_m is $\mathbf{N}\varnothing$, G is a weak $\mathcal{G}$-product of the F_m's with projections $[p_m]$, by the dual of Proposition 10. The exceptional case ($F_m = \mathbf{N}\varnothing$) is easily handled by a direct argument. Thus, given morphisms $[f_m]\colon H\to F_m$ there exists $[g]\colon H\to G$ with $f_m = p_m\circ g \bmod H$ for all m. We must show that if g' has the same properties then $[g]_H = [g']_H$. Since $p_m\circ g = f_m = p_m\circ g' \bmod H$, let $X_m\in H$ be such that $p_m\circ g = p_m\circ g'$ on X_m. Let $X = \bigcap_m X_m$. As M is finite, $X\in H$. And, since $p_m\circ g = p_m\circ g'$ on X for all m, we have $g = g'$ on X (for the p_m are collectively monic in $\mathcal{S}$).

(c) Let $[f]$ and $[g]$ be $\mathcal{G}$-morphisms from F to G. If either morphism contains no total function, then $F = \mathbf{N}A$ for some A and $f = g \bmod F$, so the identity map of F serves as equalizer. Henceforth, we assume both f and g are total. Let $e\colon E\to F$ be their equalizer in $\mathcal{F}$; we show that $[e]$ is the equalizer of $[f]$ and $[g]$ in $\mathcal{G}$. Suppose $[h]\colon H\to F$ satisfies $f\circ h = g\circ h \bmod H$. We may assume h is total (for, if $[h]$ contains no total function, H is initial in $\mathcal{G}$). Applying Lemma 8 to $f\circ h$ and $g\circ h$, we find $i\colon H\restriction B\to H$ with $f\circ h\circ i = g\circ h\circ i$ in $\mathcal{F}$ and $\mathbf{V}i$ an isomorphism in $\mathcal{G}$. Then $h\circ i$ factors through e. Applying $\mathbf{V}$ and composing with $(\mathbf{V}i)^{-1}$, we find that $[h]$ factors through $[e]$. To prove the uniqueness of the factorization, we observe that, by Theorems 4(a) and 6(a), $\mathbf{V}$ preserves monomorphisms; since e is monic in $\mathcal{F}$, $[e]$ must be monic in $\mathcal{G}$. ■

We will obtain an alternative proof of parts (b) and (c) of Theorem 9 in Section 3.

The rest of this section will be devoted to examples showing that we cannot strengthen Theorem 9 by changing "coproducts" to "colimits" in (a) or by removing the finiteness hypothesis in (b).

EXAMPLE 11. A pair of $\mathcal{G}$-morphisms with no coequalizer. Let ω be the set of natural numbers, let i be its identity map and s the successor map ($n\mapsto n+1$), and let F be the filter on ω consisting of those sets whose complements are finite. Then the pair of morphisms $[i], [s]\colon F\to\mathbf{P}\omega$ has no coequalizer. To see this, consider any morphism $[p]\colon \mathbf{P}\omega\to H$ with $p\circ i = p\circ s \bmod F$. We shall show that $[p]$ is not the desired coequalizer. Let $X\in F$ be such that $p\circ i = p\circ s$ on X. Being in F, the set X must contain all integers from some n on; thus $p(k) = p(k+1)$ for all $k\geqslant n$, so there exists $c\,(\in \mathrm{C}H)$ with $p(k) = c$ for all $k\geqslant n$. Let $q\colon \omega\to\omega$ be defined by $q(k) = k$ if $k\leqslant n+1$ and $q(k) = n+1$ if $k > n+1$. Then $[q]\colon \mathbf{P}\omega\to\mathbf{P}\omega$ in $\mathcal{G}$, and $q\circ i = q\circ s \bmod F$ because $q(k) = q(k+1)$ for all $k\geqslant n+1$. If $[p]$ were the desired coequalizer, there would be a morphism $[f]\colon H\to\mathbf{P}\omega$ such that $q = f\circ p \bmod \mathbf{P}\omega$. Then $q = f\circ p$ as functions on ω (see Theorem 5(a)), so

$$n = q(n) = f(p(n)) = f(c) = f(p(n+1)) = q(n+1) = n+1,$$

a contradiction.

EXAMPLE 12. A set of filters with no product in $\mathscr{G}$. For each $m \in \omega$, let $F_m = \mathbf{P}2$, where $2 = \{0, 1\}$. Suppose F, with projections $[p_m]$: $F \to \mathbf{P}2$, were their product in $\mathscr{G}$. Having a left adjoint, the core functor $\mathbf{C}$ preserves products, so $\mathbf{C}F$, with projections $\mathbf{C}p_m = p_m \restriction \mathbf{C}F$, is a product of ω copies of $\mathbf{CP}2 = 2$. Let G be the filter of cofinite subsets of ω (the F of Example 11), and define functions f_m, g_m from ω to 2 as follows. $f_m(k) = 1$ for all k and all m. $g_m(k) = 0$ for $k<m$, $g_m(k) = 1$ for $k \geqslant m$. Then $f_m = g_m \bmod G$ for each m; let $\alpha_m = [f_m] = [g_m]$: $G \to \mathbf{P}2$ (actually, α_m is independent of m). Since $\mathbf{C}F$ is the product of ω copies of 2, we have (unique) functions f, g: $\omega \to \mathbf{C}F \subseteq \mathbf{U}F$ with $p_m \circ f = f_m$ and $p_m \circ g = g_m$ for all m. Notice that therefore $f(k) \neq g(k)$ for all $k \in \omega$. Therefore, $[f]$ and $[g]$ are distinct morphisms from G to F whose composites with $[p_m]$ are α_m. This contradicts the uniqueness clause in the definition of products.

These two examples show that $\mathscr{G}$ is neither left nor right complete. They also show, as promised in Section 1, that $\mathbf{V}$ preserves neither coequalizers nor infinite products, so it has neither a right nor a left adjoint. Also, Example 12 shows that $\mathbf{P}$: $\mathscr{F} \to \mathscr{G}$ has no left adjoint for it fails to preserve infinite products.

Although these examples show that limits and colimits do not exist in $\mathscr{G}$ in general, they leave open the possibility that $\mathscr{G}$ might have certain special types of limits or colimits, for example directed ones, not covered by Theorem 9. We show next that $\mathscr{G}$ has neither directed limits nor directed colimits. The former is easy, given Theorem 9 and Example 12, for if $\mathscr{G}$ had directed limits then we could construct infinite products as limits of finite partial products:

$$\prod_{m \in M} F_m = \varprojlim_{M'} \prod_{m \in M'} F_m,$$

where M' ranges over finite subsets of M, directed by inclusion, and the morphisms of the inverse system are just projections. The nonexistence of directed colimits is harder.

EXAMPLE 13. A directed system of filters with no colimit. Let A be the set of those infinite sequences s of natural numbers such that $s_n = 0$ for all but finitely many n. Let p: $A \to A$ be the function defined by $p(s)_n = s_{n+1}$. Thus p is the operator deleting the first term of any sequence; its iterate p^k deletes the first k terms. The subsets $X \subseteq A$ such that, for some k, $p^k(A-X) \subseteq \{0\}$ (where 0 is the identically zero sequence in A) clearly form a filter F on A, and p maps F into F. We shall show that the direct system $\mathbf{D}$:

$$F \xrightarrow{[p]} F \xrightarrow{[p]} F \xrightarrow{[p]} \dots$$

(indexed by ω) has no colimit in $\mathscr{G}$.

Suppose G, with injections $[j_k]$, $k \in \omega$, were a colimit of this system; we may assume that each j_k is total. For each k, we have $j_0 = j_k p^k \bmod F$, so, by definition of F, there exists $n(k) \in \omega$ such that $j_0(s) = j_k p^k(s)$ whenever $p^{n(k)}(s) \neq 0$. In other words, for sequences s with non-zero terms beyond the $n(k)$-th, $j_0(s)$ is independent of the first k terms of s.

Choose a strictly monotone function m: $\omega \to \omega$ such that $n(k) < m(k)$ for all k. Define i_0: $A \to A$ by setting $i_0(s) = p^k(s)$ whenever $p^{m(k)}(s) \neq 0$ but $p^{m(k+1)}(s) = 0$. Thus $i_0(s)$ is independent of the first k terms of s whenever s has non-zero terms beyond the $m(k)$-th. More explicitly, let z_n: $A \to A$ be the function which puts n zeros at the beginning of a sequence

$$z_n(s)_k = \begin{cases} 0 & \text{if} \quad k<n, \\ s_{k-n} & \text{if} \quad k \geqslant n, \end{cases}$$

and let $i_n = i_0 \circ z_n$. Then $i_0(s) = i_n p^n(s)$ whenever $p^{m(n)}(s) \neq 0$. Thus, $i_0 = i_n p^n \bmod F$, and the $\mathscr{G}$-morphisms $[i_n]$ form a cocone from our direct system $\mathbf{D}$ to $\mathbf{P}A$. Let $[f]$: $G \to \mathbf{P}A$ be the unique factorization of this cocone through the direct limit G of $\mathbf{D}$. Then $i_0 = f \circ j_0 \bmod F$, so, by definition of F and the monotonicity of m, we can find $k \in \omega$ so large that $i_0(s) = f(j_0(s))$ whenever $p^{m(k)-1}(s) \neq 0$. If we restrict attention to those $s \in A$ such that $p^{m(k)-1}(s) \neq 0$ but $p^{m(k)}(s) = 0$, then, in addition to $i_0(s) = f(j_0(s))$, we have $i_0(s) = p^{k-1}(s)$ by definition of i_0 and $j_0(s) = j_k p^k(s)$ by choice of $n(k)$ and $m(k)-1 \geqslant n(k)$. Combining these equations, we find $p^{k-1}(s) = f j_k p^k(s)$, for the s under consideration. But this is absurd, for the left side depends on s_{k-1} while the right side does not.

We leave to the reader the verification of the details of the following example.

EXAMPLE 14. A colimit in $\mathscr{G}$ not reflected by $\mathbf{V}$. For each $n \in \omega$, let F_n be the filter on $\omega \times \omega$ consisisting of sets X such that (a) $\{k\} \times \omega \subseteq X$ for all but finitely many $k \in \omega$, and (b) if $m<n$, then $(m, p) \in X$ for all but finitely many $p \in \omega$. Clearly $F_n \supseteq F_{n+1}$, so the identity function on $\omega \times \omega$ maps F_n into F_{n+1}. The proof of Theorem 7 shows that in $\mathscr{F}$ the direct system $\mathbf{D}$;

$$F_0 \xrightarrow{1} F_1 \xrightarrow{1} F_2 \xrightarrow{1} \dots$$

has as its colimit the filter $\bigcap_n F_n$ on $\omega \times \omega$, the injections all being the identity function.

The image in $\mathscr{G}$ of the diagram $\mathbf{D}$ has, however, an entirely different colimit which can be described as follows. Its underlying set is the disjoint union of two copies of $\omega \times \omega$. A set belongs to the filter F iff (a) it includes $\{k\} \times \omega$ in the first copy of $\omega \times \omega$ for all but finitely many k, and (b) for each $m \in \omega$ it contains (m, p) in the the second copy of $\omega \times \omega$ for all but finitely many p. The injection of F_n into F is the germ of the map taking (m, p) to the point (m, p) in the first (resp. second) copy of $\omega \times \omega$ if $m \geqslant n$ (resp. $m<n$).

3. A calculus of fractions. In Section 1, when we discussed the relationship between $\mathscr{F}$ and $\mathscr{G}$, we showed that a concrete filter $F \in \mathscr{F}$ can be identified with the triple consisting of the abstract filter $\mathbf{V}F \in \mathscr{G}$, the ambient set $\mathbf{U}F$, and the embedding $\mathbf{V}F \to \mathbf{PU}F$. Thus, $\mathscr{F}$ can be defined (up to equivalence) in terms of $\mathscr{G}$ (and $\mathbf{P}$) as the full subcategory of $(\mathscr{G}, \mathbf{P})$ generated by the monomorphisms. In this section, we shall prove a result in the other direction, showing that one can obtain $\mathscr{G}$ from $\mathscr{F}$ simply by inverting certain morphisms. We let Σ be the class of all morphisms in $\mathscr{F}$ of the form $i\colon F\restriction B\to F$, where $B\in F$, $F\restriction B=\{X\in F|\ X\subseteq B\}$, and i is the inclusion map from B to $\mathbf{U}F$. Such an i can be viewed as changing the underlying set without affecting the abstract filter. Each $i\in\Sigma$ becomes an isomorphism in $\mathscr{G}$, and our brief discussion in Section 1 of the "abstraction" involved in passing from $\mathscr{F}$ to $\mathscr{G}$ indicates that the invertibility of the morphisms in Σ is essentially the only difference between $\mathscr{F}$ and $\mathscr{G}$. The following theorem gives a precise statement of this heuristic idea.

THEOREM 15. *$\mathscr{G}$, with the forgetful functor $\mathbf{V}$, is the category of fractions $\mathscr{F}[\Sigma^{-1}]$.*

Proof. By Lemma 8, $\mathbf{V}$ is a functor taking morphisms in Σ to isomorphisms. Suppose $\mathbf{S}\colon\mathscr{F}\to\mathscr{C}$ is any functor taking all the morphisms in Σ to isomorphisms; we shall show that there is a unique functor $\mathbf{T}\colon\mathscr{G}\to\mathscr{C}$ with $\mathbf{S}=\mathbf{T}\circ\mathbf{V}$. Since $\mathbf{V}$ is the identity on objects, we have no choice but to set $\mathbf{T}F=\mathbf{S}F$ for all filters F. If a $\mathscr{G}$-morphism $[f]\colon F\to G$ contains a total function f, then we are forced to define $\mathbf{T}[f]=\mathbf{S}f$. This is well-defined, for if $[f]=[g]$ then Lemma 8 provides an $i\in\Sigma$ such that $f\circ i=g\circ i$; then $\mathbf{S}(f)\circ\mathbf{S}(i)=\mathbf{S}(g)\circ\mathbf{S}(i)$ and, because $\mathbf{S}(i)$ is an isomorphism, $\mathbf{S}(f)=\mathbf{S}(g)$. We must still define $\mathbf{T}[f]$ when $[f]$ contains no total function. This happens only when $[f]\colon\mathbf{N}A\to N\emptyset$ is the inverse in $\mathscr{G}$ of $[i]\colon\mathbf{N}\emptyset\to\mathbf{N}A$, where i is the unique map $\emptyset\to A$. As $i\in\Sigma$, we can (indeed, we must) define $\mathbf{T}[f]$ to be the inverse in $\mathscr{C}$ of $\mathbf{T}[i]=\mathbf{S}i$. This proves the uniqueness of $\mathbf{T}$; existence will follow once we check that $\mathbf{T}$ is a functor (for $\mathbf{S}=\mathbf{T}\circ\mathbf{V}$ is obvious). Since this verification is trivial, we leave it to the reader. ∎

THEOREM 16. *Σ admits a calculus of right fractions.*

Proof. We recall the definition [5] of admitting a calculus of right fractions. It means that

(a) Σ contains all the identity morphisms of $\mathscr{F}$.

(b) Σ is closed under composition,

(c) Given morphisms $F\xrightarrow{f}G\xleftarrow{s}G'$ in $\mathscr{F}$ with $s\in\Sigma$, there is a commutative square

$$\begin{array}{ccc} F' & \xrightarrow{f'} & G' \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle s} \\ F & \xrightarrow{f} & G \end{array}$$

in $\mathscr{F}$ with $s'\in\Sigma$.

(d) If f,g are $\mathscr{F}$-morphisms such that $sf=sg$ for some $s\in\Sigma$, then $fi=gi$ for some $i\in\Sigma$.

Claims (a) and (b) are trivial, and (d) follows immediately from Lemma 8, so it remains to prove (c). Suppose $B\subseteq\mathbf{U}G$ is the set in G such that $G'=G\restriction B$ and s is the inclusion $B\to\mathbf{U}G$. As f maps F into G, the set $A=f^{-1}(B)$ is in F. Let F' be $F\restriction A$ and let s' be the inclusion $A\to\mathbf{U}F$. Finally, let f' be the restriction of f to A. ∎

These theorems give a new, more abstract, proof of part of Theorem 9.

COROLLARY 17. *$\mathscr{G}$ has, and $\mathbf{V}$ preserves, finite limits.*

Proof. Immediate from Theorems 7, 15, and 16 by the dual of [5] Proposition I. 3.1. ∎

4. Closed monoidal structure. It is well-known [6] that, in addition to the usual product of filters (as in Theorems 7 and 9), there is another notion of product suitable for reducing iterated limits in topology to ordinary (single) limits and iterated reduced products in model theory to single reduced products. This product is defined by

$$F\otimes G=\{X\subseteq\mathbf{U}F\times\mathbf{U}G|\ \{a\in\mathbf{U}F|\ \{b\in\mathbf{U}G|\ (a,b)\in X\}\in G\}\in F\};$$

thus, $X\in F\otimes G$ iff, for F-almost all a it is the case that for G-almost all b, $(a,b)\in X$. Note that $\otimes$ is, in general, not symmetric; there is no isomorphism between $F\otimes G$ and $G\otimes F$. It is however, associative; the obvious bijection from $(\mathbf{U}F\times\mathbf{U}G)\times\mathbf{U}H$ to $\mathbf{U}F\times(\mathbf{U}G\times\mathbf{U}H)$ is an isomorphism from $(F\otimes G)\otimes H$ to $F\otimes(G\otimes H)$. We make $\otimes$ a bifunctor by defining $f\otimes g$, when $f\colon F\to F'$ and $g\colon G\to G'$, to be the cartesian product map $f\times g\colon\mathbf{U}F\times\mathbf{U}G\to\mathbf{U}F'\times\mathbf{U}G'$; it is trivial that this maps $F\otimes G$ into $F'\otimes G'$. It is also trivial that the associativity isomorphism described above is natural, that there are natural isomorphisms

$$\mathbf{P}(1)\otimes F\cong F\cong F\otimes\mathbf{P}(1),$$

and that all these isomorphisms are coherent, so that $\mathscr{F}$ is a (non-symmetric) monoidal category with the product $\otimes$ [3].

If $f=f'\bmod F$ and $g=g'\bmod G$, then $f\otimes g=f'\otimes g'\bmod F\otimes G$. We can therefore also define $\otimes$ as a bifunctor on $\mathscr{G}$, and we verify easily that $\mathscr{G}$ is also a monoidal category (with respect to $\otimes$) and that $\mathbf{V}$ is a monoidal functor. (Strictly speaking the monoidal functor is not just $\mathbf{V}$ but $\mathbf{V}$ together with the identity natural transformation from $\mathbf{V}F\otimes\mathbf{V}G$ to $\mathbf{V}(F\otimes G)$ and the identity map from $\mathbf{P}1$ to $\mathbf{VP}1$; see [3]. Henceforth, we omit such technicalities.) It is also easy to see that $\mathbf{U}\colon\mathscr{F}\to\mathscr{S}$, $\mathbf{C}\colon\mathscr{G}\to\mathscr{S}$, and $\mathbf{C}\colon\mathscr{F}\to\mathscr{S}$ are monoidal functors. In fact, $\mathbf{C}$ is (up to natural isomorphism) the "underlying set" functor for both $\mathscr{F}$ and $\mathscr{G}$, because (by Theorem 5)

$$\mathscr{F}(\mathbf{P}1,F)\cong\mathscr{G}(\mathbf{P}1,F)\cong\mathscr{S}(1,\mathbf{C}F)\cong\mathbf{C}F.$$

It is this fact that made us reluctant in Section 1 to call $\mathbf{U}F$ the underlying set of F.

The operation $\otimes$ deserves this "tensor product" notation because, for each F, the functor $-\otimes F$ has a right adjoint, an internal Hom-functor. (Note that the ordinary product $-\times F$, where F is the filter of cofinite subsets of ω, has no right adjoint for it does not preserve coproducts, not even the coproduct of countably many copies of $\mathbf{P}1$.) The right adjoints for $\otimes$ are defined as follows. In $\mathscr{F}$, $\mathrm{Hom}_{\mathscr{F}}(F,G)$ has as its underlying set the set $\mathscr{S}(\mathbf{U}F,\mathbf{U}G)$ of all maps from $\mathbf{U}F$ into $\mathbf{U}G$, and it is generated by the subsets

$$\{f|\ f^{-1}(X)\in F\}=\{f|\ X\in f(F)\}\quad\text{for}\quad X\in G.$$

(Thus $\mathbf{C}\,\mathrm{Hom}_{\mathscr{F}}(F, G) = \mathscr{F}(F, G)$.) In $\mathscr{G}$, the underlying set of $\mathrm{Hom}_{\mathscr{G}}(F, G)$ is the set of all F-germs of maps from sets in F into $\mathbf{U}G$, and a basis for $\mathrm{Hom}_{\mathscr{G}}(F, G)$ consists of the sets $\{[f]_F|\ X \in f(F)\}$ for $X \in G$. (So again $\mathbf{C}\mathrm{Hom}_{\mathscr{G}}(F, G) = \mathscr{G}(F, G)$.) We can describe $\mathrm{Hom}_{\mathscr{F}}(F, G)$ as the filter on $\mathscr{F}(F, \mathbf{PU}G)$ generated by the sets $\mathscr{F}(F, \mathbf{P}X)$ for $X \in G$; then we can describe $\mathrm{Hom}_{\mathscr{G}}(F, G)$ by just changing all $\mathscr{F}$'s to $\mathscr{G}$'s in this formulation. We leave it to the reader to verify that the functors we have described give right adjoints for $\otimes$ and that both $\mathscr{F}$ and $\mathscr{G}$ thus become closed monoidal categories [3].

We close with a brief discussion of categories over $\mathscr{F}$; with obvious modifications, the same remarks apply to categories over $\mathscr{G}$, especially since every $\mathscr{F}$-category induces a $\mathscr{G}$-category via $\mathbf{V}$ [3]. An $\mathscr{F}$-structure on a category $\mathscr{C}$ assigns to each pair C, D of objects of $\mathscr{C}$ a filter $\mathrm{Hom}_{\mathscr{C}}(C, D)$, on some set $A = \mathbf{U}\mathrm{Hom}_{\mathscr{C}}(C, D)$, with core $\mathscr{C}(C, D)$, the ordinary Hom-set. The elements of A can be thought of as "approximate $\mathscr{C}$-morphisms" from C to D, and the sets in the filter $\mathrm{Hom}_{\mathscr{C}}(C, D)$ correspond to conditions on the degree to which an element of A resembles an actual morphism (so that all these conditions together imply that the element is an actual morphism).

For example, the category $\mathscr{M}$ of metric spaces and uniformly continuous maps admits the following $\mathscr{F}$-structure. $\mathbf{U}\mathrm{Hom}_{\mathscr{M}}(C, D)$ is the set of all functions from C to D. $\mathrm{Hom}_{\mathscr{M}}(C, D)$ is generated by the sets

$$A_\varepsilon = \{f\colon C \to D|\ (\exists \delta > 0)(\forall x, y \in C)\big(d_C(x, y) < \delta \to d_D(f(x), f(y)) < \varepsilon\big)\}$$

for $\varepsilon > 0$. Thus, $\mathbf{C}\mathrm{Hom}_{\mathscr{M}}(C, D) = \bigcap_{\varepsilon > 0} A_\varepsilon = \mathscr{M}(C, D)$ as required. The composition operation of $\mathscr{M}$ lifts to the map $\mathrm{Hom}_{\mathscr{M}}(C, D) \otimes \mathrm{Hom}_{\mathscr{M}}(B, C) \to \mathrm{Hom}_{\mathscr{M}}(B, D)$ given by ordinary composition of functions, so $\mathscr{M}$ is indeed an $\mathscr{F}$-category. It is clear that, for example, the category of uniform spaces has an analogous $\mathscr{F}$-structure.

For another example, consider the category $\mathscr{A}$ of abelian groups and homomorphisms. Again, we let $\mathbf{U}\mathrm{Hom}_{\mathscr{A}}(A, B)$ consist of all functions from A to B. The filter $\mathrm{Hom}_{\mathscr{A}}(A, B)$ is generated by the sets

$$X_F = \{f\colon A \to B|\ (\forall x, y \in F)\, f(x+y) = f(x) + f(y)\}$$

where F ranges over finite subsets of A. The core of this filter consists precisely of the homomorphisms from A to B. The composition operation of $\mathscr{A}$ can be lifted to the ordinary composition operation $\mathrm{Hom}_{\mathscr{A}}(A, B) \otimes \mathrm{Hom}_{\mathscr{A}}(B, C) \to \mathrm{Hom}_{\mathscr{A}}(A, C)$. Note that the order of factors on the left is the reverse of that required in the definition [3] of an $\mathscr{F}$-category. Since $\mathscr{F}$ is not symmetric, this reversal cannot be circumvented. Thus, what we have shown is that the dual of $\mathscr{A}$, not $\mathscr{A}$ itself, admits an $\mathscr{F}$-structure. A similar discussion provides an $\mathscr{F}$-structure for the dual of any algebraic category.

Finally, we remark that for $\mathscr{F}$-categories (but not for $\mathscr{G}$-categories) the objects and approximate morphisms also form a category (induced by $\mathbf{U}$ from the given category). In the preceding examples this was the category of sets or its dual, but it would be easy to give an $\mathscr{F}$-structure for the dual of the category of rings such that the approximate morphisms are just the homomorphisms of the underlying additive groups.

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