Example 4.3 illustrates the same point when $\mathscr{X}$ and $\mathscr{Y}$ both generate compact T_2 topologies. (In this example, neither space contains singletons.)

5.10. COROLLARY. *If* $(X, \mathscr{U})$ *and* $(Y, \mathscr{V})$ *are spaces having closed subbases which* (i) *contain the singletons,* (ii) *are closed under finite unions, and* (iii) *are union isomorphic, then* $(X, \mathscr{U})$ *and* $(Y, \mathscr{V})$ *are homeomorphic.*

Proof. Let $\mathscr{X}$ and $\mathscr{Y}$ be the respective closed subbases in $(X, \mathscr{U})$ and $(Y, \mathscr{V})$, and let θ be an isomorphism of $\mathscr{X}$ onto $\mathscr{Y}$. By Lemma 5.6, the equation $\tilde{\theta}(x) = \{\theta(\{x\})\}$ defines a map of $\mathscr{X}$ to $\mathscr{Y}$, and this map is clearly a bijection.

By Theorem 5.8, we have $\tilde{\theta}(K) = \theta(K)$ for every K in $\mathscr{X}$, and the reasoning of Corollary 3.15 is applicable to showing that $\tilde{\theta}$ is a homeomorphism.

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Accepté par la Rédaction le 17. 9. 1974

On $\aleph_0$-categoricity and the theory of trees

by

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Abstract. The principal result is: every $\aleph_0$-categorical tree is decidable. This follows from a general theorem which asserts that every nuclear, $\aleph_0$-categorical structure is finitely axiomatizable. Other facts about trees are also proved. For example, the finitely axiomatizable, $\aleph_0$-categorical trees are characterized.

In this paper we investigate $\aleph_0$-categoricity, in general, and $\aleph_0$-categorical trees, in particular. The concept of a nuclear structure is introduced in § 1, where it is shown that each nuclear, $\aleph_0$-categorical structure is finitely axiomatizable. Any $\aleph_0$-categorical, linearly ordered set is easily seen to be nuclear, so we get the result of Rosenstein [4] that every $\aleph_0$-categorical linearly ordered set is finitely axiomatizable. This result is extended in § 2, using the notion of nuclearity, to show that every $\aleph_0$-categorical tree is decidable. In addition, those $\aleph_0$-categorical trees which are also finitely axiomatizable are characterized.

Our method of proof is quite different from Rosenstein's. His is based on an analysis of linear orderings similar to ones given by Erdös and Hajnal [2], or by Läuchli and Leonard [3] in their proof of the decidability of the theory of linearly ordered sets. It is hoped that our method can more easily be applied to other theories. More generally, it is hoped that the notion of a nuclear structure will lead to a classification of those $\aleph_0$-categorical theories which are finitely axiomatizable.

For a first-order theory T we denote by $\varrho(T)$ the similarity type of T. We will consider only T for which $\varrho(T)$ is finite. For convenience we assume that $\varrho(T)$ contains only relation symbols (although none of our results depends on this restriction). A theory is $\aleph_0$-*categorical* iff all of its countable (and here we include the possibility of finite) models are isomorphic, so that an $\aleph_0$-categorical theory is automatically complete. If T is a theory then an n-*type* (of T) is a maximal set of n-ary formulas (i.e. those formulas all of whose free variables are among $v_0, \ldots, v_{n-1}$) which is consistent with T. We denote the set of n-types of T by $S_n(T)$. We will often attribute to a structure $\mathfrak{A}$ a property that $\mathrm{Th}(\mathfrak{A})$ has (e.g. $\aleph_0$-categoricity, decidability).

* Many of the results contained herein were annouced in [6]. The intended proof of Theorem 1 in [6] was erroneous; however, this has no effect on the contents of this paper.

An indispensable tool in the study of $\aleph_0$-categoricity is the following theorem of Ryll-Nardzewski [5].

RYLL-NARDZEWSKI'S THEOREM. *A complete theory T is $\aleph_0$-categorical iff $S_n(T)$ is finite for each $n<\omega$.*

It is of course natural to have to construct isomorphisms in studying $\aleph_0$-categoricity. We will often use the back-and-forth technique to do this. We refer the reader to the expository article [1] for information on this technique.

§ 1. **Nuclear structures.** In the theorem of this section we will give a sufficient condition for an $\aleph_0$-categorical structure to be finitely axiomatizable.

If $p\in S_n(T)$ and $I\subseteq n$, then $p|I$ will denote the set of formulas in p whose free variables are among $\{v_i\colon i\in I\}$. Now let $\mathfrak{A}$ be a structure, and let $X\subseteq A$ be finite, and let $a\in A$. We say that a subset $Y\subseteq X$ is a *nucleus* of X for a iff the following property holds: if $X=\{a_0,\ldots,a_{m-1}\}$ and $Y=\{a_0,\ldots,a_{n-1}\}$ where $n\leqslant m$, and p is the $(m+1)$-type realized by $\langle a_0,\ldots,a_{m-1},a\rangle$, then p is the unique $(m+1)$-type extending $p|m\cup p|(n\cup\{m\})$. The structure $\mathfrak{A}$ is *n-nuclear* iff, for each $a\in A$, each finite subset has a nucleus with no more than n elements. If $\mathfrak{A}$ is n-nuclear for some $n<\omega$, then we will say that $\mathfrak{A}$ is *nuclear*. Notice that if T is $\aleph_0$-categorical and some model of T is n-nuclear, then each model of T is n-nuclear.

The prototypical nuclear structures are linearly ordered sets. Indeed, it is immediate that every linearly ordered set is 2-nuclear.

THEOREM 1.1. *Every nuclear, $\aleph_0$-categorical structure with finite similarity type is finitely axiomatizable.*

Proof. Let $\mathfrak{B}$ be a countable, nuclear, $\aleph_0$-categorical structure such that $\varrho(\mathfrak{B})$ is finite, and let $T=\mathrm{Th}(\mathfrak{B})$. Choose $n<\omega$ large enough so that $\mathfrak{B}$ is n-nuclear and that there are no r-ary relation symbols for any $r\geqslant n+2$.

From Ryll-Nardzewski's Theorem it follows that there is a finite sequence $\langle\Phi_p\colon p\in S_{n+1}(T)\rangle$ of formulas satisfying the following properties:

(0) $\Phi_p\in p$ for each $p\in S_{n+1}(T)$.

(1) For each $p\in S_{n+1}(T)$ and each $(n+1)$-ary atomic formula α, either $\models\Phi_p\rightarrow\alpha$ or $\models\Phi_p\rightarrow\neg\alpha$.

(2) If $p,q\in S_{n+1}(T)$ and $p\neq q$, then $\models\Phi_p\rightarrow\neg\Phi_q$.

We now make some definitions concerning formulas in the language of T. We call a formula φ a *Φ-formula* if φ is a conjunction of formulas of the form $\Phi_p(v_{i_0},\ldots,v_{i_n})$. Suppose that φ is an m-ary Φ-formula. We will say that φ is *complete* iff whenever $i_0,\ldots,i_n<m$, then there is a unique $p\in S_{n+1}(T)$ such that $\Phi_p(v_{i_0},\ldots,v_{i_n})$ is a conjunct of φ. Furthermore, we will say that φ is *consistent* iff $T\vdash\exists v_0,\ldots,v_{m-1}\varphi$. If $m<\omega$ and $p\in S_m(T)$, then φ_p is the m-ary Φ-formula defined by:

$$\varphi_p=\bigwedge\{\Phi_q(v_{i_0},\ldots,v_{i_n})\in p\colon i_0,\ldots,i_n<m \text{ and } q\in S_{n+1}(T)\}\,.$$

Notice that each φ_p is a complete and consistent Φ-formula.

Let σ_0 be the sentence

$$\forall v_0,\ldots,v_n\bigvee\{\varphi_p\colon p\in S_{n+1}(T)\}\,.$$

It is obvious that σ_0 is a consequence of T. The models of σ_0 are just those structures $\mathfrak{A}$ for which each m-tuple $\langle a_0,\ldots,a_{m-1}\rangle$ of elements of A satisfies a unique complete (although not necessarily consistent) m-ary Φ-formula.

Now let $\mathfrak{A}$ be a model of σ_0, and let $X\subseteq A$ be finite and let $a\in A$. We say that a subset $Y\subseteq X$ is a *φ-nucleus* of X for a iff the following property holds: if $X=\{a_0,\ldots,a_{m-1}\}$ and $Y=\{a_0,\ldots,a_{r-1}\}$ where $r\leqslant m$, and if $\theta_1,\theta_2,\theta_3$ are complete Φ-formulas which are satisfied by $\langle a_0,\ldots,a_{m-1}\rangle$, $\langle a_0,\ldots,a_{r-1},a\rangle$ and $\langle a_0,\ldots,a_{m-1},a\rangle$ respectively, then

$$\mathfrak{A}\models\forall v_0,\ldots,v_m[\theta_1(v_0,\ldots,v_{m-1})\wedge\theta_2(v_0,\ldots,v_{r-1},v_m)\rightarrow\theta_3(v_0,\ldots,v_m)]\,.$$

We now come to the definition of the sentence σ which is an axiomatization of the theory T. Let σ be the conjunction of σ_0 and each of the following sentences:

(3) $\forall v_0,\ldots,v_{2n-1}(\varphi_p\rightarrow\exists v_{2n}\varphi_q)$, where $q\in S_{2n+1}(T)$ and $p=q|2n$.

(4) $\forall v_0,\ldots,v_{n-1}(\varphi_p\rightarrow\forall v_n\bigvee\{\varphi_q\colon q\in S_{n+1}(T) \text{ and } p=q|n\})$, where $p\in S_n$.

(5) Every set with $\leqslant 2n$ elements has a Φ-nucleus with $\leqslant n$ elements.

(6) $\forall v_0,\ldots,v_{2n}(\varphi\rightarrow\psi)$, where φ and ψ are $(2n+1)$-ary Φ-formulas and $T\vdash\varphi\rightarrow\psi$.

It is clear that σ is consistent, for indeed $T\vdash\sigma$. Let $\mathfrak{A}$ be a countable model of σ. We will show that $\mathfrak{A}\cong\mathfrak{B}$. We do this by the back-and-forth technique, which we apply here by proving the following two conditions:

(B) Suppose that $a_0,\ldots,a_{m-1}\in A$ and $b_0,\ldots,b_m\in B$ are such that $\langle b_0,\ldots,b_m\rangle$ realizes the $(m+1)$-type $p\in S_{m+1}(T)$ and $\langle a_0,\ldots,a_{m-1}\rangle$ satisfies the complete m-ary Φ-formula $\varphi_{p|m}$. Then there is $a_m\in A$ such that $\langle a_0,\ldots,a_m\rangle$ satisfies the formula φ_p.

(F) Suppose that $b_0,\ldots,b_{m-1}\in B$ and $a_0,\ldots,a_m\in A$ are such that for some $p\in S_m(T)$, $\langle a_0,\ldots,a_{m-1}\rangle$ satisfies the complete m-ary Φ-formula φ_p and $\langle b_0,\ldots,b_{m-1}\rangle$ realizes the type p. Then there is $b_m\in B$ such that $\langle b_0,\ldots,b_m\rangle$ realizes the $(m+1)$-type q and $\langle a_0,\ldots,a_m\rangle$ satisfies the formula φ_q.

Having verified (B) and (F) we can see, because of property (1), that an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ can be constructed.

To verify (B), let us assume that the hypothesis of (B) is satisfied. Let $X=\{b_0,\ldots,b_{m-1}\}$ and let Y be a nucleus of X for b_m with at most n elements, where $Y=\{b_{i_0},\ldots,b_{i_{n-1}}\}$. Let $q\in S_{n+1}(T)$ be the type realized by $\langle b_{i_0},\ldots,b_{i_{n-1}},b_m\rangle$. Sentence (3) implies the existence of an $a_m\in A$ such that $\langle a_{i_0},\ldots,a_{i_{n-1}},a_m\rangle$ satisfies the Φ-formula φ_q. We now claim that $\langle a_0,\ldots,a_m\rangle$ satisfies the formula φ_p. For suppose that $r\in S_{n+1}(T)$ and $j_0,\ldots,j_{n-1}<m$ are such that $\varphi_r(v_{j_0},\ldots,v_{j_{n-1}},v_m)\in p$. Let $s\in S_{2n}(T)$ be the type realized by $\langle b_{i_0},\ldots,b_{i_{n-1}},b_{j_0},\ldots,b_{j_{n-1}}\rangle$. Thus the sentence

$$\forall v_0,\ldots,v_{2n}[\varphi_s(v_0,\ldots,v_{2n-1})\wedge\varphi_q(v_0,\ldots,v_{n-1},v_{2n})\rightarrow\varphi_r(v_n,\ldots,v_{2n})]$$

is a consequence of T. But this is a sentence of the type in (6), and hence is one of the conjuncts of σ. Thus $\langle a_{j_0}, \ldots, a_{j_{n-1}}, a_m\rangle$ satisfies the formula φ_r.

To verify (F), let us suppose that the hypothesis of (F) is satisfied. Let $X = \{a_0, \ldots, a_{m-1}\}$. We will find a Φ-nucleus Y of X for a_m with at most n elements. If $m \leqslant n$, then just take $Y = X$, so assume $m > n$. Let $X_0, \ldots, X_k$ be a list of all subsets of X with n elements. By induction we will get a sequence $Y_0, \ldots, Y_k$ of subsets of X each with n elements. Choose $Y_0 = X_0$. Suppose that we already have Y_i for some $i<k$. Then let Y_{i+1} be a Φ-nucleus of $Y_i \cup X_i$ for a_m with n elements. (The existence of such a Φ-nucleus is guaranteed by (5).) We then set $Y = Y_k$, and then claim that Y is a Φ-nucleus of X for a_m. This last claim is easily verified.

Now let $Y = \{a_{i_0}, \ldots, a_{i_{n-1}}\}$, where $i_0, \ldots, i_{n-1} < m$. Let φ be the complete $(n+1)$-ary Φ-formula which is satisfied by $\langle a_{i_0}, \ldots, a_{i_{n-1}}, a_m\rangle$. By (4) the formula φ is a consistent Φ-formula, so that $\varphi = \varphi_r$ for some $r \in S_{n+1}(T)$. Choose $b_m \in B$ such that $\langle b_{i_0}, \ldots, b_{i_{n-1}}, b_m\rangle$ realizes r. (Such a b_m exists because $\mathfrak{B}$ is homogeneous.)

Let q be the $(m+1)$-type realized by $\langle b_0, \ldots, b_m\rangle$. It remains to show that $\langle a_0, \ldots, a_m\rangle$ satisfies the formula φ_q. To show this suppose that $j_0, \ldots, j_{n-1} < m$ and that $\langle b_{j_0}, \ldots, b_{j_{n-1}}, b_{i_0}, \ldots, b_{i_{n-1}}, b_m\rangle$ realizes the type s. Let $t = s|2n$. Then from (3) the sentence

$$\forall v_0, \ldots, v_{2n-1}(\varphi_t \rightarrow \exists v_{2n}\varphi_s)$$

is a conjunct of σ. But then, since Y is a Φ-nucleus of X, it must be that

$$\langle a_{j_0}, \ldots, a_{j_{n-1}}, a_{i_0}, \ldots, a_{i_{n-1}}, a_m\rangle$$

satisfies the Φ-formula φ_s. Thus $\langle a_0, \ldots, a_m\rangle$ satisfies φ_q. ■

§ **2. Trees.** In this section we will apply Theorem 1.1 to the theory of trees. A *tree* $(T, <)$ is a partially ordered set such that the set of predecessors of any element is linearly ordered by $<$. (Notice that we do not require a tree to be well-founded, nor that it be rooted.)

A *component* X of T is a minimal non-empty subset of T satisfying: if $x, y, z \in T$ are such that $x \in X$ and $y \leqslant x, z$, then $z \in X$. The set of components of a tree form a partition of the tree. If $x, y \in T$, then the set

$$T_{xy} = \{z \in T\colon \text{ whenever } w \leqslant x, y, \text{ then } w < z\}.$$

For each $n < \omega$ we will say that a tree $(T, <)$ is *n-branching* iff whenever $x, y \in T$, then $(T_{xy}, <)$ has at most n components. A tree is 1-branching iff it is a linear ordering. Notice that for each $n < \omega$ there is an $\forall\exists$ sentence whose models are just the n-branching trees. A tree is *finite-branching* iff for some $n < \omega$ it is n-branching.

The theorems which we will prove in this section concerning $\aleph_0$-categorical trees are the following two.

THEOREM 2.1. *Every $\aleph_0$-categorical tree is decidable.*

THEOREM 2.2. *An $\aleph_0$-categorical tree is finitely axiomatizable iff it is finite-branching.*

Proof of Theorem 2.1. Let $(T, <)$ be a countable $\aleph_0$-categorical tree. The proof will consist of finding an expansion of $(T, <)$ which is nuclear and $\aleph_0$-categorical.

If $x, y \in T$ and X is a component of T_{xy}, let us say that the *set-type* of X is $\mathrm{Th}(T, <, X)$. By Ryll-Nardzewski's Theorem, there are only a finite number of set-types. Let $\Sigma_0, \ldots, \Sigma_{n-1}$ be those set-types which are realized in $(T, <)$ by an infinite number of components of some T_{xy}. For each $x, y \in T$ and $i < n$, let $<^i_{xy}$ be a linear ordering of those components of T_{xy} which realize Σ_i. Each relation $<^i_{xy}$ is chosen in such a way that $<^i_{xy} = <^i_{wz}$ if $T_{xy} = T_{wz}$, and the order type of $<^i_{xy}$ is that of the rationals. Now define the 4-ary relations R_i, for $i < n$, as follows: $\langle x, y, a, b\rangle \in R_i$ iff there are components X, Y of T_{xy} each realizing Σ_i such that $a \in X$, $b \in Y$ and $X \leqslant^i_{xy} Y$. It is easy to see that $(T, <, R_0, \ldots, R_{n-1})$ is $\aleph_0$-categorical and nuclear (cf. Lemma 2.3). Thus it is finitely axiomatizable, so that $(T, <)$ is decidable. ■

We now will prove Theorem 2.2. To show that every finite-branching $\aleph_0$-categorical tree is finitely axiomatizable, it suffices to prove the following lemma.

LEMMA 2.3. *If $4 \leqslant n < \omega$, then every n-branching tree is $(n-1)$-nuclear.*

Proof. Let $4 \leqslant n < \omega$ and let $(T, <)$ be an n-branching tree. We will show that $(T, <)$ is $(n-1)$-nuclear. Let $X \subseteq T$ be finite and $a \in T$. Let $X_0 = \{x \in X\colon a \leqslant x\}$. We will consider two cases which depend on whether X_0 is empty or not.

First suppose $X_0 = 0$. Let $x \in X$ be such that whenever $y \in X$, then $T_{ax} \subseteq T_{ay}$. If Y is the component of T_{ax} which contains a, then $Y \cap X = 0$. Thus there is a set $A \subseteq X$, $\mathrm{card}(A) \leqslant n-1$, such that A meets every component of T_{ax} that X does. By a back-and-forth argument it is easy to see that A is a nucleus of X for a.

Now suppose $X_0 \neq 0$. Let $x, y \in X_0$ be such that whenever $u, v \in X_0$, then $T_{uv} \subseteq T_{xy}$. Let $z \in X - X_0$ be such that whenever $w \in X - X_0$, then $T_{az} \subseteq T_{aw}$. Again, by a back-and-forth argument, it is easy to see that $\{x, y, z\}$ is a nucleus for a. ■

Remark. Every 1-branching tree is 2-nuclear; every 2- or 3-branching tree is 3-nuclear. These results and Lemma 2.3 are easily seen to be optimal.

To prove the other direction of Theorem 2.2 it will be convenient to introduce some definitions. First we recall the notion of quantifier-rank of a sentence. Let us define the *rank* of a formula φ as the number of distinct variables occurring in φ. Thus the rank of a sentence is the same as its quantifier-rank. Let us denote by Q_n the set of formulas which have rank $\leqslant n$. Notice that any subformula of a formula in Q_n is also in Q_n. For each n, the set Q_n is finite (up to logical equivalence). Let $(T_1, <)$ and $(T_2, <)$ be trees. Define $(T_1, <) \equiv_n (T_1, <)$ iff whenever σ is a sentence in Q_n, then $(T_1, <) \models \sigma$ iff $(T_2, <) \models \sigma$. Certainly $\equiv_n$ is an equivalence relation with only a finite number of equivalence classes. Define $(T_1, <) \prec_n (T_2, <)$ iff $(T_1, <) \subseteq (T_2, <)$ and whenever $\varphi(v_0, \ldots, v_{k-1}) \in Q_n$ is a k-ary formula and $a_1, \ldots, a_{k-1} \in T_1$ are such that $(T_2, <) \models \exists v_0 \varphi(v_0, a_1, \ldots, a_{k-1})$, then there is

$a_0 \in T_1$ such that $(T_2, <) \models \varphi(a_0, \ldots, a_{k-1})$. It is easily shown (as in the proof of Tarski's Criterion in [7]) that $(T_1, <) \prec_n (T_2, <)$ implies $(T_1, <) \equiv_n (T_2, <)$.

Let us say that a subset $A \subseteq T$ is *n-able* iff whenever $x, y \in T$ and X is a component of T_{xy}, then the number of components Y of T_{xy} such that $(X, <) \equiv_n (Y, <)$ and $Y \cap A \neq 0$ is at most n.

LEMMA 2.4. *Let $(T, <)$ be a tree and $A \subseteq T$ a finite n-able subset. Suppose that $\varphi(v_0, \ldots, v_{k-1}) \in Q_n$ and $a_1, \ldots, a_{k-1} \in A$ are such that $(T, <) \models \exists v_0 \varphi(v_0, a_1, \ldots, a_{k-1})$. Then there exists $a_0 \in T$ such that $(T, <) \models \varphi(a_0, \ldots, a_{k-1})$ and $A \cup \{a_0\}$ is n-able.*

Proof. Let $b_0 \in T$ be such that $(T, <) \models \varphi(b_0, a_1, \ldots, a_{k-1})$, and set $A_0 = A$. Let us suppose, by way of contradiction, that there is no $a_0 \in T$ such that $(T, <) \models \varphi(a_0, \ldots, a_{k-1})$ and $A_0 \cup \{a_0\}$ is n-able. Then there are $x, y \in T$ and distinct components $X_0, \ldots, X_n$ of T_{xy} such that if $m = 0$ and $i < j \leqslant n$, then

(1) $(X_i, <) \equiv_n (X_j, <)$,

(2) $X_i \cap A \neq 0$,

(3) $b_m \in X_n$.

Notice that $X_n \cap A = 0$ since A is n-able. Now let $i < n$ be such that $X_i \cap \{a_1, \ldots, a_{k-1}\} = 0$. Then there is $b_1 \in X_i$ such that $(T, <) \models \varphi(b_1, a_1, \ldots, a_{k-1})$. Let $A_1 = X_i \cap A$ and $Y_1 = X_i$.

We proceed by induction. Suppose we already have b_m, A_m, Y_m where Y_m is a component of some T_{xy}, $A_m = Y_m \cap A$, $b_m \in Y_m$, $(T, <) \models \varphi(b_m, a_1, \ldots, a_{k-1})$ and $\{a_1, \ldots, a_{k-1}\} \cap Y_m = 0$. Thus, there are $x, y \in Y_m$ and distinct components $X_0, \ldots, X_n$ of T_{xy} such that if $i < j \leqslant n$, then (1)-(3) above hold. Choose $b_{m+1} \in X_0$ so that $(T, <) \models \varphi(b_{m+1}, a_1, \ldots, a_{k-1})$. Then set $Y_{m+1} = X_0$ and $A_{m+1} = Y_{m+1} \cap A$.

We then get a sequence $A_0 \supset A_1 \supset A_2 \supset \ldots$ But A is finite; hence this is a contradiction, completing the proof of the lemma. ■

LEMMA 2.5. *Let $(T, <)$ be a tree and let $n < \omega$. Then there is a finite-branching $(T_1, <) \prec_n (T, <)$.*

Proof. By the Downward Löwenheim–Skolem Theorem, we can assume that T is countable. Let $\langle \varphi_i(v_0, a_1^i, \ldots, a_{k_i-1}^i) \colon i < \omega \rangle$ be a list of all unary formulas in Q_n which are satisfiable in T (with parameters from T), and let $\langle b_i \colon i < \omega \rangle$ be a list of the elements of T. By induction on j, define n-able subsets $A_j \subseteq T$ as follows. Let $A_0 = 0$. Now suppose A_j has already been defined. If j is even, then choose the least $i < \omega$ such that $b_i \notin A_j$, but $b_i < a$ for some $a \in A_j$. Then set $A_{j+1} = A_j \cup \{b_i\}$, which is still n-able. (If no such i exists, then the set $A_{j+1} = A_j$.) If j is odd, then choose the least $i < \omega$ such that

$$a_1^i, \ldots, a_{k_i-1}^i \in A_j \quad \text{and} \quad (T, <) \models \neg \varphi_i(a, a_1^i, \ldots, a_{k_i-1}^i)$$

for each $a \in A_j$. By Lemma 2.4 there is $b \in T$ such that $(T, <) \models \varphi_i(b, a_1^i, \ldots, a_{k_i-1}^i)$ and $A_j \cup \{b\}$ is n-able. Then the set $A_{j+1} = A_j \cup \{b\}$. Now setting $T_1 = \bigcup \{A_j \colon j < \omega\}$ it is easy to see that $(T_1, <) \prec_n (T, <)$. Furthermore, if $x < y \in T_1$, then $x \in T_1$. It is easy to see, then, that $(T_1, <)$ is n-branching. ■

Proof of Theorem 2.2. Suppose that $(T, <)$ is a finitely axiomatizable, $\aleph_0$-categorical tree. If $(T, <)$ is axiomatized by a sentence $\sigma \in Q_n$, then by Lemma 2.5 there is a finite-branching $(T_1, <) \prec_n (T, <)$. But then $(T_1, <) \models \sigma$ so that $(T_1, <) \equiv (T, <)$. Thus $(T, <)$ is finite-branching. ■

Lemma 2.5 has the following interesting corollary.

COROLLARY 2.6. *If a sentence σ is true in some tree, then it is true in some finite-branching tree.*

There is a result analogous to Corollary 2.6 for $\aleph_0$-categorical trees. We first need to prove a lemma.

LEMMA 2.7. *Let $(T, <)$ be an $\aleph_0$-categorical tree. Let $n < \omega$ have the property that whenever $x, y \in T$ and X, Y are components of T_{xy} such that $(X, <) \equiv_n (Y, <)$, then $(X, <) \equiv (Y, <)$. Let $A \subseteq T$ be a finite n-able subset. Finally, suppose that $a_0, \ldots, a_m, b_0, \ldots, b_{m-1} \in A$ are such that $\langle a_0, \ldots, a_{m-1} \rangle$ and $\langle b_0, \ldots, b_{m-1} \rangle$ realize the same m-type. Then there is $b_m \in T$ such that $\langle a_0, \ldots, a_m \rangle$ and $\langle b_0, \ldots, b_m \rangle$ realize the same $(m+1)$-type and $A \cup \{b_m\}$ is n-able.*

Proof. Suppose the hypotheses of the lemma. Let $b \in T$ be such that $\langle a_0, \ldots, a_m \rangle$ and $\langle b_0, \ldots, b_{m-1}, b \rangle$ realize the same $(m+1)$-type. If $A \cup \{b\}$ is n-able we are done, so suppose it is not. Then there are $x, y \in T$ and distinct components $X_0, \ldots, X_n$ of T_{xy} such that whenever $i < j \leqslant n$, then

(1) $(X_i, <) \equiv (X_j, <)$,

(2) $X_i \cap A \neq 0$,

(3) $b \in X_n$.

Since $\{a_0, \ldots, a_m\}$ is n-able, so is $\{b_0, \ldots, b_{m-1}, b\}$. Therefore, there is $i < n$ such that $X_i \cap \{b_0, \ldots, b_{m-1}\} = 0$. The proof can now be completed as in the proof of Lemma 2.4. ■

THEOREM 2.8. *If a sentence σ is true in some $\aleph_0$-categorical tree, then it is true in some finite-branching $\aleph_0$-categorical (and, hence, finitely axiomatizable) tree.*

Proof. Let $(T, <)$ be a countable $\aleph_0$-categorical tree. Let $n < \omega$ have the property that whenever $x, y \in T$ and X, Y are components of T_{xy} such that $(X, <) \equiv_n (Y, <)$, then $(X, <) \equiv (Y, <)$. By dove-tailing applications of Lemmas 2.4 and 2.7, we can get a finite-branching tree $(T_1, <) \prec_n (T, <)$ with the additional property that whenever $a_0, \ldots, a_m$, $b_0, \ldots, b_{m-1} \in T_1$ and $\langle a_0, \ldots, a_{m-1} \rangle$ and $\langle b_0, \ldots, b_{m-1} \rangle$ realize the same m-type (in $(T, <)$), then there is $b_m \in T_1$ such that $\langle a_0, \ldots, a_m \rangle$ and $\langle b_0, \ldots, b_m \rangle$ realize the same $(m+1)$-type (also in $(T, <)$). Now by a back-and-forth argument it is easy to see that if two m-tuples $\langle a_0, \ldots, a_{m-1} \rangle$ and $\langle b_0, \ldots, b_{m-1} \rangle$ realize the same m-type in $(T, <)$, then they realize the same m-type in $(T_1, <)$. Thus by Ryll-Nardzewski's Theorem, $(T_1, <)$ is $\aleph_0$-categorical. ■

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Accepté par la Rédaction le 8. 10. 1974

Two closed categories of filters

by

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Abstract. Two categories mentioned in the title each have as objects all filters. The morphisms from D to E in one of the categories are just the maps sending D to a filter extending E; The morphisms in the other category are equivalence classes of such maps modulo "equality on a set in D". After discussing some elementary relationships between these categories and some pairs of adjoint functors between them and the category of sets, we showt hat the first of our categories is both left and right complete. The second category is finitely left complete and has coproducts, but we give examples showing that it lacks coequalizers and infinite products. We also show that each of two categories of filters is a closed monoidal (but not symmetric) category, in the sense of Eilenberg and Kelly, and we briefly discuss some examples of categories over these closed categories.

A useful methodological principle in modern mathematics is that, when one defines a type of mathematical structure, one should also define the notion of a morphism (or map) between two structures of that type. For a long time, this principle was ignored in the case where the structures are filters. As far as I know, the first published definition of a morphism of filters is in [6], where it is only briefly mentioned. A different definition was proposed, and the resulting category of filters investigated, in [7]. (This definition was also used, but only for ultrafilters, in [1].) The purpose of the present paper is to investigate the categories $\mathcal{F}$ and $\mathcal{G}$ arising from these two definitions of morphisms.

In Section 1, we define two categories and develop their most elementary properties, including various functors between them and the category of sets and various adjunctions between these functors. This section also contains a discussion of the heuristic meaning of the difference between the definitions of morphisms in $\mathcal{F}$ and $\mathcal{G}$. In Section 2, we prove a number of results about the existence or non-existence of various sorts of limits in our categories. The neatest of these results is that $\mathcal{F}$ is both left and right complete, but perhaps more interesting are some of the counterexamples to completeness in $\mathcal{G}$. In Section 3, we amplify the discussion, at the end of Section 1, of the relation between $\mathcal{F}$ and $\mathcal{G}$ by showing that $\mathcal{G}$ can be obtained from $\mathcal{F}$ as a category of fractions with respect to a very natural class of morphisms. We further show that this class admits a calculus of right fractions; this provides an alternate proof of some results in Section 2. Finally, in