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Semilattice theory with applications to point-set topology

by

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Abstract. By means of semilattice theory, it is shown that if an intersection-preserving epimorphism θ exists between a pair of closed subbases containing the singletons, and $\theta^{-1}(\emptyset) = \{\emptyset\}$, then the generated topologies are homeomorphic. The necessity of including the singletons in this context is studied, and a similar theorem is proved for subbases which do not necessarily contain the singletons, but which generate compact T_1 topologies. These results generalize theorems of Birkhoff concerning bases or entire families of closed sets and set maps which are lattice isomorphisms.

1. Introduction. Theorems expressing conditions under which the isomorphism of the lattices of closed sets of a pair of topological spaces implies the topological equivalence of the spaces are an established part of the lattice theory literature. (See, for example, [1], [3], and [4].)

By means of general semilattice theorems, it is shown here that, for T_1 spaces, consideration of these questions can be profitably extended to *semilattices* of closed sets which are *subbases* for the topologies involved — instead of totalities of closed sets or of bases, — and to *intersection-* or *union-epimorphisms* — instead of lattice isomorphisms (e.g. for an intersection morphism, $f(A \cap B) = f(A) \cap f(B)$ for all A, B). Several counterexamples are presented to delineate the extent to which some of the hypotheses can be weakened.

In Section 3 we study meet epimorphisms between semilattices (Theorem 3.6). The results obtained are applied to show that if an intersection morphism exists from a closed subbasis containing singletons onto a second, then the generated topologies are homeomorphic (Corollary 3.15). This generalizes a theorem of Birkhoff. A counterexample shows the necessity of including the singletons, set theoretic investigations yield Corollary 3.12 which essentially concerns cardinalities, and further topological considerations are discussed which stem naturally from the study of intersection morphisms (Corollaries 3.19 and 3.32).

Section 4 has as its main result (Theorem 4.5) a generalization of another theorem of Birkhoff. It is proved that if an intersection morphism exists from one closed subbasis onto another, and if the topologies which they generate are compact and T_1, then the topologies are homeomorphic. Again, counterexamples are given to show that various hypotheses cannot be removed.

In Section 5, union morphisms are studied. Results dual to those proved earlier are first established. It is then proved that if two closed subbases containing the singletons are union isomorphic, then the topologies which they generate are homeomorphic (Corollary 5.10).

§ 2. Notation. A *poset*, or partially ordered set is a set with a reflexive anti-symmetric and transitive relation.

A *semilattice* is a set with a binary idempotent commutative and associative operation.

Given a semilattice $\langle A, \circ\rangle$, define the binary relations $\leqslant_\wedge$ and $\leqslant_\vee$ on A as follows: $a \leqslant_\wedge b$ iff $a = a\circ b$; $a \leqslant_\vee b$ iff $b = a\circ b$. Then $\langle A; \leqslant_\wedge\rangle$ ($\langle A, \leqslant_\vee\rangle$) is a poset in which $\inf\{a, b\}$ ($\sup\{a, b\}$) exists for every pair $\{a, b\}$ of elements (see, e.g. [2, p. 9]). We define $a\wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$ in these two cases.

A map f from a poset $(P, \leqslant)$ to a poset $(Q, \leqslant)$ is *isotone* if $x \leqslant y$ implies $f(x) \leqslant f(y)$.

A map f from a semilattice P to a semilattice Q is a *meet* (*join*) *morphism* if $f(a\wedge b) = f(a)\wedge f(b)$ ($f(a\vee b) = f(a)\vee f(b)$) for every (a, b) in P^2. A one-to-one meet (join) morphism will be called a meet (join) *isomorphism*. Every meet (join) morphism is isotone with respect to $\leqslant_\wedge (\leqslant_\vee)$.

Suppose that $\mathscr{X} \subseteq \exp X$ and $\mathscr{Y} \subseteq \exp Y$ are closed under finite intersections (unions). A map $f\colon \mathscr{X}\to\mathscr{Y}$ is an *intersection-* (a *union-*) *morphism* if

$$f(A\cap B) = f(A)\cap f(B) \quad (f(A\cup B) = f(A)\cup f(B))$$

for every A, B in $\mathscr{X}$. Thus, f is a meet (join) morphism from the semilattice $(\mathscr{X}, \cap)$ $[(\mathscr{X}, \cup)]$ to the semilattice $(\mathscr{Y}, \cap)$ $[(\mathscr{Y}, \cup)]$.

A *zero* of a poset P is an element 0 with $0 \leqslant x$ for all $x\in P$. Posets contain at most one zero.

By an *atom* x of a poset P, we shall mean an element which is not the zero, but which is such that if $y \leqslant x$ and $y \neq x$, then $y = 0$. An *atomic* poset is one in which every element is the supremum of its atom predecessors.

A family $\mathscr{F} \subseteq \exp X$ is a *closed subbasis* for (or *C-generates*) a topology $\mathscr{T}$ if $\mathscr{T}$ is the smallest topology whose closed sets contain $\mathscr{F}$.

By N we shall denote the set of all (strictly) positive integers, and we let $Z^+ = N \cup \{0\}$.

§ 3. Intersection morphisms I: subbases with singletons.

3.1. Remark. An element k of a semilattice S is an atom if and only if $k\wedge l = k$ or $k\wedge l = 0$ for every l in S.

3.2. Lemma. *Let S be a poset with zero, and let θ be an isotone map of S onto a poset T. Then $\theta(0)$ is the zero of T.*

Proof. For every element s of S, we have $0\leqslant s$, and so $\theta(0)\leqslant\theta(s)$. Since θ is an epimorphism, $\theta(0)$ is the zero of T.

3.3. Lemma. *Let S and T be semilattices and $\theta\colon S\to T$ a meet epimorphism. If j is an atom of $(S, \leqslant_\wedge)$ then $\theta(j)$ is either the zero or an atom of $(T, \leqslant_\wedge)$.*

Proof. For every $s\in S$, we have $s\wedge j = j$ or 0 by Remark 3.1. Thus,

$$\theta(s\wedge j) = \theta(j) \text{ or } \theta(0),$$

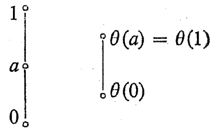
and so

$$\theta(s)\wedge\theta(j) = \theta(j) \text{ or } 0$$

for every s in S by Lemma 3.2.

Since θ is an epimorphism, we can conclude that $\theta(j)$ is an atom by Remark 3.1.

3.4. Remark. If a meet morphism θ between semilattices is one to one, and the range of θ contains 0, then $\theta^{-1}(0) = \{0\}$ by Lemma 3.2. The condition $\theta^{-1}(0) = \{0\}$, however, does not suffice to make even a lattice epimorphism one to one. For example:



Corollary 3.8 indicates a set of circumstances in whose context the condition $\theta^{-1}(0) = \{0\}$ suffices to make a meet epimorphism an isomorphism.

3.5. Definition. Let S be a semilattice. By S_a we denote the set of atoms of $(S, \leqslant_\wedge)$.

3.6. Theorem. *Let S and T be semilattices with $\leqslant_\wedge$-zeros and θ a meet-morphism of S onto T. Then*

(1) *the map θ is one to one on $S_a\sim\theta^{-1}(0)$;*

(2) *if $\theta^{-1}(0) = \{0\}$ and S is atomic, then $\theta[S_a] = T_a$;*

(3) *for every $k\in S$,*

$$\theta[\{l\in S_a\sim\theta^{-1}(0)\colon\ l\leqslant_\wedge k\}] = \{z\in\theta[S_a]\colon\ 0\neq z\leqslant_\wedge\theta(k)\}.$$

Proof. We shall use the symbol $\leqslant$ for $\leqslant_\wedge$. Let j_1 and j_2 be distinct elements of $S_a\sim\theta^{-1}(0)$. Since

$$0 = \theta(0) = \theta(j_1\wedge j_2) = \theta(j_1)\wedge\theta(j_2),$$

and since $\theta(j_1)\neq 0\neq\theta(j_2)$, we conclude that $\theta(j_1)\neq\theta(j_2)$.

(2) It follows from Lemma 3.3 that $\theta[S_a]\subseteq T_a$.

Let $t\in T_a$ with $t = \theta(s)$, and let $j\leqslant s$ where $j\in S_a$.

Since $0 \neq j \leqslant s$, and since $\theta^{-1}(0) = 0$ or $\emptyset$, we have

$$0 \neq \theta(j) \leqslant \theta(s) = t .$$

Thus, $\theta(j) = t$ because t is an atom, and so $\theta[S_a] \doteq T_a$.

(3) Let $l \in S_a \sim \theta^{-1}(0)$ with $l \leqslant k$. Thus, $\theta(l) \in \theta[S_a]$ and

$$0 \neq \theta(l) \leqslant \theta(k) .$$

Now suppose that $z = \theta(t)$ with $t \in S_a$ and $0 \neq \theta(t) = z \leqslant \theta(k)$. To prove that

$$z \in \theta[\{l \in S_a \sim \theta^{-1}(0):\ l \leqslant k\}],$$

we need only prove $t \leqslant k$.

Since

$$\theta(t \wedge k) = \theta(t) \wedge \theta(k) = \theta(t) \neq 0 ,$$

we have $t \wedge k \neq 0$. Since t is an atom, it follows that $t \leqslant k$.

3.7. Remark. The requirement of Theorem 3.6 that S and T have zeros can be removed with the obvious modifications. However, if a poset has more than one atom, then it has a zero in any case.

3.8. Corollary. *Let S and T be atomic semilattices with $\leqslant_\wedge$-zero's. If θ is a meet morphism of S onto T with $\theta^{-1}(0) = \{0\}$, then θ is a meet isomorphism.*

Proof. Let a and b be distinct elements of S, and let A (B) be the set of atom predecessors of a (b). By atomicity, $a = \sup A$ and $b = \sup B$, and since $a \neq b$, we have $A \neq B$.

Thus, since $\theta^{-1}(0) = \{0\}$,

$$\{z \in T_a:\ 0 \neq z \leqslant \theta(a)\} = \theta(A) \neq \theta(B) = \{w \in T_a:\ 0 \neq w \leqslant \theta(b)\} ,$$

with the first and third equalities following from part (3) of Theorem 3.6, and the second from part (1) of Theorem 3.6. Thus, $\theta(a) \neq \theta(b)$, and the corollary is proved.

3.9. Definition. If θ is a set valued map whose domain is a subfamily of $\exp X$ containing the singletons, then we define X_θ by

$$X_\theta = \{x \in X:\ \theta(\{x\}) \neq \emptyset\} .$$

3.10. Corollary. *Let θ be an intersection morphism of $\mathscr{X} \subseteq \exp X$ onto $\mathscr{Y} \subseteq \exp Y$ where X and Y contain at least two points, $\mathscr{X}$ and $\mathscr{Y}$ are closed under finite intersections and $\mathscr{X}$ and $\mathscr{Y}$ contain the singletons of X and Y respectively.*

Then

(1) *the map $\tilde{\theta}$: $X_\theta \to Y$, defined by $\{\tilde{\theta}(x)\} = \theta(\{x\})$ is one-to-one;*

(2) *if $\theta^{-1}(\emptyset) = \{\emptyset\}$, then $\tilde{\theta}$ maps X onto Y; and*

(3) *for every $K \in \mathscr{X}$,*

$$\tilde{\theta}[K \cap X_\theta] = \theta(K) \cap \tilde{\theta}[X_\theta] .$$

Proof. Let x_1 and x_2 be distinct elements of X. Thus, $\emptyset = \{x_1\} \cap \{x_2\} \in \mathscr{X}$. Similarly, $\mathscr{Y}$ contains $\emptyset$.

Parts (1) and (2) follow obviously from parts (1) and (2) respectively of Theorem 3.6.

Ad (3). The following statements are equivalent: $y \in \tilde{\theta}[K \cap X_\theta]$; $y = \tilde{\theta}(k)$ for some $k \in K \cap X_\theta$; $\{y\} = \theta(\{k\})$ for some $\{k\} \leqslant K$ and $\{k\} \in \mathscr{X}_a \sim \theta^{-1}(0)$; $\{y\} \in \theta[\mathscr{X}_a]$ and $0 \neq \{y\} \leqslant \theta(K)$ (by part (3) of Theorem 3.6); $y \in \theta(K) \cap \tilde{\theta}[X_\theta]$. This establishes part (3).

3.11. Remark. The hypotheses "$\mathscr{X}$ and $\mathscr{Y}$ are closed under finite intersections, $\theta^{-1}(\emptyset) = \{\emptyset\}$ and $\theta(A \cap B) = \theta(A) \cap \theta(B)$ for all A and B in $\mathscr{X}$" can, in Corollary 3.10 be replaced by "$\mathscr{X}$ and $\mathscr{Y}$ are any families, $\bigcap_{i=1}^{n} \theta(D_i) \neq \emptyset$ for every finite subfamily $\{D_1, D_2, \ldots, D_n\}$ of $\mathscr{X}$ with $\bigcap_{i=1}^{n} D_i \neq \emptyset$, and if $\{A_1, A_2, \ldots, A_m\}$ and $\{B_1, B_2, \ldots, B_n\}$ are finite subfamilies of $\mathscr{X}$ with $\bigcap_{i=1}^{m} A_i = \bigcap_{j=1}^{n} B_j$, then $\bigcap_{i=1}^{m} \theta(A_i) = \bigcap_{j=1}^{n} \theta(B_j)$". A similar replacement is possible in the case of unions (see § 5).

This follows from the observation that if S is a subset of a semilattice G_1, and θ maps S into any semilattice G_2 such that for every pair $\{a_1, a_2, \ldots, a_m\}$ and $\{b_1, b_2, \ldots, b_n\} \subseteq S$ satisfying $a_1 a_2 \ldots a_m = b_1 b_2 \ldots b_n$, we have $\theta(a_1)\theta(a_2) \ldots \theta(a_m) = \theta(b_1)\theta(b_2) \ldots \theta(b_n)$, then θ can be extended to an epimorphism of $[S]$ onto $[\theta(S)]$. (By $[T]$ we denote the subsemigroup generated by T.)

For every element $a_1 a_2 \ldots a_k$ of $[S]$, we define $\theta_1(a_1 a_2 \ldots a_k)$ by $\theta_1(a_1 a_2 \ldots a_k) = \theta(a_1)\theta(a_2) \ldots \theta(a_k)$. The hypothesis ensures that θ_1 is well defined on $[S]$. The map θ_1 extends θ because if $a_1 a_2 \ldots a_k \in S$, then since

$$a_1 a_2 \ldots a_k = (a_1 a_2 \ldots a_k)^2$$

we have, by hypothesis,

$$\theta(a_1)\theta(a_2) \ldots \theta(a_k) = [\theta(a_1 a_2 \ldots a_k)]^2 ,$$

and so

$$\theta_1(a_1 a_2 \ldots a_k) = \theta(a_1 a_2 \ldots a_k) .$$

Using these observations, we can state the following set-theoretic result of Corollary 3.10.

3.12. Corollary. *Let $\mathscr{X} \subseteq \exp X$ contain $\emptyset$ and the singletons. Let θ: $\mathscr{X} \to \exp Y$ be such that $\theta(\mathscr{X})$ contains $\emptyset$ and the singletons of Y. Suppose also that $\bigcap_{i=1}^{n} \theta(D_i) \neq \emptyset$ for every finite subfamily $\{D_1, D_2, \ldots, D_n\}$ of X with $\bigcap_{i=1}^{n} D_i \neq \emptyset$, and that for every* *pair* $\{A_1, A_2, \ldots, A_m\}$ *and* $\{B_1, B_2, \ldots, B_n\}$ *of finite subsets of* $\mathscr{X}$ *with* $\bigcap_{i=1}^{m} A_i = \bigcap_{j=1}^{n} B_j$, *we have* $\bigcap_{i=1}^{m} \theta(A_i) = \bigcap_{j=1}^{n} \theta(B_j)$.

Then there is a one-to-one map $\tilde{\theta}$ *of* X *onto* Y *for which* $\theta(F) = \tilde{\theta}(F)$ *for every* F *in* X. *(In particular,* X *and* Y *have the same cardinality.)*

3.13. Let X and Y be sets, and, $\mathscr{X}$ and $\mathscr{Y}$ be families of subsets of X and Y respectively, which are closed under finite intersections and which c-generate T_1 topologies. The following example shows, *inter alia*, that the existence of an intersection morphism θ of $\mathscr{X}$ onto $\mathscr{Y}$ for which $\theta^{-1}(\emptyset) = \{\emptyset\}$ does not necessarily imply that the topologies c-generated by $\mathscr{X}$ and $\mathscr{Y}$ are homeomorphic. Corollary 3.15 shows that if both $\mathscr{X}$ and $\mathscr{Y}$ contain the singletons, then these conditions are indeed sufficient.

The example shows, in fact, that sets X and Y exist with respective closed bases $\mathscr{X}$ and $\mathscr{Y}$ which are closed under both finite intersections and finite unions and which are lattice isomorphic, but which generate nonhomeomorphic T_1 topologies, even though $\mathscr{Y}$ contains the singletons, and $\mathscr{X}$ generates a compact topology.

These refinements show firstly that the well known result (Corollary 3.18 below) that T_1 topologies (and thus T_D topologies — see [4]) are characterized by their lattices of all closed sets, cannot be extended without modification to lattices of closed sets which form bases.

Secondly, the refinements in this example are relevant to Theorem 4.4 and Corollary 5.10 below.

3.14. Example. Let $\mathscr{X}$ ($\mathscr{Y}$) be the family consisting of all finite subsets of $\underset{\sim}{N}(Z^+)$, together with all subsets of Z^+ whose complements are finite subsets of $\underset{\sim}{N}(Z^+)$.

The families $\mathscr{X}$ and $\mathscr{Y}$ are closed under finite unions and finite intersections.

The topology $\mathscr{W}$ (say) on Z^+ for which $\mathscr{Y}$ is a closed basis, is the discrete topology. The topology $\mathscr{V}$ (say) on Z^+ for which $\mathscr{X}$ is a closed basis, is the one point compactification of $(\underset{\sim}{N}, \mathscr{H})$ — $\mathscr{H}$ being the discrete topology — with 0 as compactification point. In particular, then, $(Z^+, \mathscr{V})$ and $(Z^+, \mathscr{W})$ are nonhomeomorphic T_1 topological spaces.

For $D \subseteq \underset{\sim}{N}$, define $D_0 \subseteq Z^+$ by

$$D_0 = \{n\colon n+1 \in D\}.$$

Define $\theta\colon \mathscr{X} \to \mathscr{Y}$ by

$$\theta(Z^+ \sim J) = Z^+ \sim J_0 \quad \text{and} \quad \theta(J) = J_0$$

for every finite subset, J, of $\underset{\sim}{N}$.

The map θ is a one-to-one epimorphism.

We show that $\theta(A \cap B) = \theta(A) \cap \theta(B)$ and $\theta(A \cup B) = \theta(A) \cup \theta(B)$ for every A, B in $\mathscr{X}$.

Let S and T be finite subsets of $\underset{\sim}{N}$. The foregoing assertion is established by the following equalities:

$$\theta(S \cap T) = (S \cap T)_0 = S_0 \cap T_0 = \theta(S) \cap \theta(T);$$

$$\theta[S \cap (Z^+ \sim T)] = \theta(S \sim T) = (S \sim T)_0 = S_0 \sim T_0 = S_0 \cap (Z^+ \sim T_0)$$
$$= \theta(S) \cap \theta(Z^+ \sim T);$$

$$\theta[(Z^+ \sim S) \cap (Z^+ \sim T)] = \theta[Z^+ \sim (S \cup T)] = Z^+ \sim (S \cup T)_0 = Z^+ \sim (S_0 \cup T_0)$$
$$= (Z^+ \sim S_0) \cap (Z^+ \sim T_0) = \theta(Z^+ \sim S) \cap \theta(Z^+ \sim T);$$

$$\theta(S \cup T) = (S \cup T)_0 = S_0 \cup T_0 = \theta(S) \cup \theta(T);$$

$$\theta[S \cup (Z^+ \sim T)] = \theta[Z^+ \sim (T \sim S)] = Z^+ \sim (T \sim S)_0 = Z^+ \sim (T_0 \sim S_0)$$
$$= S_0 \cup (Z^+ \sim T_0) = \theta(S) \cup \theta(Z^+ \sim T);$$

$$\theta[(Z^+ \sim S) \cup (Z^+ \sim T)] = \theta[Z^+ \sim (S \cap T)] = Z^+ \sim (S \cap T)_0 = Z^+ \sim (S_0 \cap T_0)$$
$$= (Z^+ \sim S_0) \cup (Z^+ \sim T_0) = \theta(Z^+ \sim S) \cup \theta(Z^+ \sim T).$$

3.15. Corollary. *Let* $(X, \mathscr{U})$ *and* $(Y, \mathscr{V})$ *be spaces containing at least two elements, with respective closed subbases* $\mathscr{X}$ *and* $\mathscr{Y}$ *which contain the singletons and are closed under finite intersections. If an intersection-morphism* θ *of* $\mathscr{X}$ *onto* $\mathscr{Y}$ *exists with* $\theta^{-1}(\emptyset) = \{\emptyset\}$, *then* $(X, \mathscr{U})$ *and* $(Y, \mathscr{V})$ *are homeomorphic.*

(More generally, the subspaces X_θ *and* $\theta(X_\theta)$ *are homeomorphic for every intersection morphism* θ *of* $\mathscr{X}$ *onto* $\mathscr{Y}$.)

Proof. The map $\tilde{\theta}$ defined in Corollary 3.10 is one to one on X_θ and, by part (3), maps elements of $X_\theta \cap \mathscr{X}$ onto elements of $\tilde{\theta}[X_\theta] \cap \mathscr{Y}$.

Let $F \subseteq X$ be closed in $(X, \mathscr{U})$. Thus,

$$F = \bigcap \{L_1^\alpha \cup L_2^\alpha \cup \ldots \cup L_{n(\alpha)}^\alpha\colon \alpha \in A\}$$

for some set A, and some elements L_i^α of $\mathscr{X}$. We have

$$\tilde{\theta}(F \cap X_\theta) = \tilde{\theta}[\bigcap\{(L_1^\alpha \cap X_\theta) \cup (L_2^\alpha \cap X_\theta) \ldots (L_{n(\alpha)}^\alpha \cap X_\theta)\colon \alpha \in A\}$$
$$= \{\tilde{\theta}(L_1^\alpha \cap X_\theta) \cup \tilde{\theta}(L_2^\alpha \cap X_\theta) \cup \ldots \cup \tilde{\theta}(L_{n(\alpha)}^\alpha \cap X_\theta)\colon \alpha \in A\}$$

(since $\tilde{\theta}$ is one to one on X_θ)

$$= \{[\theta(L_1^\alpha) \cap \tilde{\theta}(X_\theta)] \cup [\theta(L_2^\alpha) \cap \tilde{\theta}(X_\theta)] \cup \ldots \cup [\theta(L_{n(\alpha)}^\alpha) \cap \tilde{\theta}(X_\theta)]\colon \alpha \in A\}$$

(by part (3) of Corollary 3.10)

$$= \theta(F) \cap \tilde{\theta}(X_\theta),$$

which is closed in $\tilde{\theta}(X_\theta)$.

Now let J be closed in $(Y, \mathscr{V})$, and let

$$J = \bigcap \{\bigcup_{i=1}^{m(\beta)} V_i^\beta\colon \beta \in D,\ V_i^\beta \in \mathscr{Y}\}.$$

For every (β, i), let J_i^β be such that $\theta(J_i^\beta) = V_i^\beta$. Since $\tilde{\theta}$ is one to one on X_θ and $\tilde{\theta}(J_i^\beta \cap X_\theta) = \theta(J_i^\beta) \cap \theta(X_\theta)$, we have

$$\tilde{\theta}^{-1}(V_i^\beta \cap \tilde{\theta}(X_\theta)) = J_i^\beta \cap X_\theta .$$

Thus,

$$\begin{aligned}\tilde{\theta}^{-1}(J \cap \tilde{\theta}(X_\theta)) &= \bigcap \{\bigcup_{i=1}^{m(\beta)} \tilde{\theta}^{-1}(V_i^\beta \cap \tilde{\theta}(X_\theta)) \colon \beta \in D\} \\ &= \bigcap \{\bigcup_{i=1}^{m(\beta)} (J_i^\beta \cap X_\theta) \colon \beta \in D\} \\ &= [\bigcap \{\bigcup_{i=1}^{m(\beta)} J_i^\beta \colon \beta \in D\}] \cap X_\theta ,\end{aligned}$$

which is closed in X_θ.

3.16. Corollary 3.15 should be contrasted with Theorem 9 of [1, p. 225] (which is generalized by Theorem 4.5 of the present paper). We assume neither compactness, nor that $\mathscr{B}$ is a basis nor that $\mathscr{B}$ is closed under union. We do assume, however, that $\mathscr{B}$ contains the singletons.

3.17. Corollary. *Let X and Y be T_1 spaces. Every intersection morphism θ of 2^X onto 2^Y induces a homeomorphism between X_θ and $\tilde{\theta}[X_\theta]$.*

3.18. Corollary (Birkhoff [1, p. 217]). *Any T_1 space is determined up to homeomorphism by the atomic dually Brouwerian lattice of all its closed sets, ordered by inclusion.*

3.19. Corollary. *A continuous map f from a T_1 space X onto a T_1 space Y is a homeomorphism if and only if $f^{-1}(2^Y) = 2^X$.*

Proof. Consider the meet epimorphism f^{-1} of 2^Y onto 2^X, and apply Corollary 3.15.

3.20. Remark. The preceding corollary is easy to see, independently of Corollary 3.15.

3.21. We shall study some topological properties of the set X where θ is a set valued map whose domain is a subfamily of $\exp X$ which contains the singletons. Recall that

$$X_\theta = \{x \in X \colon \theta\{x\} \neq \emptyset\} .$$

3.22. Remark. Given a T_1 space $(X, \mathscr{X})$ and a subset A, there is always an intersection epimorphism θ on X for which $A = X_\theta$ viz. the map $\theta \colon \mathscr{X} \to \mathscr{X} \cap A$ defined by $\theta(F) = F \cap A$.

3.23. Theorem. *Let $\mathscr{X}$ and $\mathscr{Y}$ be the families of closed sets of T_1 spaces X and Y respectively, and let θ be an intersection morphism of $\mathscr{X}$ onto $\mathscr{Y}$. Then*

$$\overline{\tilde{\theta}(X_\theta)} = \theta(\overline{X}_\theta) ,$$

where $\tilde{\theta}$ is defined by $\{\tilde{\theta}(x)\} = \theta(\{x\})$.

Proof. Since $\{x\} \subseteq \overline{X}_\theta$ for every x in X_θ, we have

$$\tilde{\theta}(X_\theta) \subseteq \theta(\overline{X}_\theta) ,$$

and since $\theta(\overline{X}_\theta)$ is closed, it follows that

$$\overline{\tilde{\theta}(X_\theta)} \subseteq \theta(\overline{X}_\theta) .$$

On the other hand,

$$\overline{\tilde{\theta}(X_\theta)} = \theta(L)$$

for some $L \in \mathscr{X}$.

We show that $L \supseteq X_\theta$.

If $L \not\supseteq X_\theta$, then $L \cap X_\theta \neq X_\theta$, and since $\tilde{\theta}$ is one to one on X_θ, this implies that $\tilde{\theta}(L \cap X_\theta) \neq \tilde{\theta}(X_\theta)$. But then

$$\begin{aligned}\tilde{\theta}(X_\theta) \neq \tilde{\theta}(L \cap X_\theta) &= \theta(L) \cap \tilde{\theta}(X_\theta) \quad \text{(by part (3) of Corollary 3.10)} \\ &= \overline{\tilde{\theta}(X_\theta)} \cap \tilde{\theta}(X_\theta) \\ &= \tilde{\theta}(X_\theta) .\end{aligned}$$

Since $L \supseteq X_\theta$, and L is closed, we have $L \supseteq \overline{X}_\theta$, and so

$$\theta(L) \supseteq \theta(\overline{X}_\theta) .$$

Thus, $\overline{\tilde{\theta}(X_\theta)} \supseteq \theta(\overline{X}_\theta)$, and so

$$\overline{\tilde{\theta}(X_\theta)} = \theta(\overline{X}_\theta) .$$

§ 4. Intersection morphisms II: compact spaces.

4.1. Lemma. *Let X be a compact T_1 space for which $\mathscr{B}$ is a closed subbasis which is closed under finite intersections. Then, for every distinct pair, (x_1, x_2) of elements of X, disjoint elements B_1 and B_2 of $\mathscr{B}$ exist for which $x_1 \in B_1$ and $x_2 \in B_2$.*

Proof. For $i = 1, 2$, let

$$\mathscr{B}(x_i) = \{B \in \mathscr{B} \colon x_i \in B\} .$$

If the conclusion of the lemma were false, then, for all finite families

$$\{B_1^1, B_1^2, \ldots, B_1^n\} \subseteq \mathscr{B}(x_1) \quad \text{and} \quad \{B_2^1, B_2^2, \ldots, B_2^m\} \subseteq \mathscr{B}(x_2) ,$$

we would have

$$B_1^1 \cap B_1^2 \cap \ldots \cap B_1^n \cap B_2^1 \cap B_2^2 \cap \ldots \cap B_2^m \neq \emptyset .$$

Thus, $\mathscr{B}(x_1) \cup \mathscr{B}(x_2)$ would have the finite intersection property. Since X is compact, it would follow that

$$(*) \qquad \emptyset \neq \bigcap [\mathscr{B}(x_1) \cup \mathscr{B}(x_2)] = [\bigcap \mathscr{B}(x_1)] \cap [\bigcap \mathscr{B}(x_2)] .$$

Since X is T_1, $\{x_i\}$ $(i = 1, 2)$ is an intersection of finite unions of elements of $\mathscr{B}$, and therefore $\{x_i\} = \bigcap \mathscr{B}(x_i)$. Thus, equation (*) implies $\emptyset \neq \{x_1\} \cap \cap \{x_2\} = \emptyset$, and this contradiction establishes the lemma.

4.2. The following example shows that sets X and Y exist with respective closed bases $\mathscr{X}$ and $\mathscr{Y}$ which are closed under finite unions and which are union isomorphic, but which generate nonhomeomorphic compact T_1 topologies. Theorem 4.5 establishes the validity of the corresponding statement for intersections.

4.3. Example. Let

$$X = \{z \in C\colon \operatorname{Im}(z) = 0 \text{ and } -1 \leqslant \operatorname{Re}(z) \leqslant 1\},$$
$$Y = \{z \in C\colon \operatorname{Im}(z) = 1 \text{ and } 0 \leqslant \operatorname{Re}(z) \leqslant 1\} \cup \{z \in C\colon \operatorname{Im}(z) = -1 \text{ and } -1 \leqslant \operatorname{Re}(z) \leqslant 0\},$$

and let θ_1 be the projection of Y onto X.

Let $\mathscr{X}(\mathscr{Y})$ be the family of finite unions of closed intervals of $X(Y)$, where each of the intervals contains either no elements or infinitely many elements.

The point map θ_1 generates a union isomorphism of $\mathscr{X}$ onto $\mathscr{Y}$; however, $(X, \mathscr{X})$ and $(Y, \mathscr{Y})$ are nonhomeomorphic compact T_2 spaces.

4.4. Example 3.14 shows that if $\mathscr{X}$ and $\mathscr{Y}$ are closed subbases (even bases) for T_1 topological spaces X_1 and X_2, which are closed under finite intersections (and even unions as well) and if X_1 is compact, then the existence of an intersection morphism θ of $\mathscr{X}$ onto $\mathscr{Y}$ with $\theta^{-1}(\emptyset) = \{\emptyset\}$ (even the existence of a lattice isomorphism) does not necessarily imply that X_1 and X_2 are homeomorphic (or even that a continuous map of X_1 onto X_2 exists).

The following theorem shows, however, that the compactness of X_2 ensures such a homeomorphism.

4.5. Theorem. *Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be compact T_1 spaces with closed subbases $\mathscr{X}$ and $\mathscr{Y}$ respectively which are closed under finite intersections. If an intersection morphism θ of $\mathscr{X}$ onto $\mathscr{Y}$ exists with $\theta^{-1}(\emptyset) = \{\emptyset\}$, then $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ are homeomorphic.*

4.6. Remark. This generalizes Theorem 9 of [1, p. 225].

4.7. Proof of Theorem 4.4. For each x in X, let

$$\mathscr{X}_x = \{B \in \mathscr{X}\colon x \in B\}.$$

For every finite family $\{B_1, B_2, \ldots, B_n\} \subseteq \mathscr{X}_x$, we have $\bigcap_{m=1}^{n} B_m \neq \emptyset$. Since $\theta^{-1}(\emptyset) = \{\emptyset\}$, $\bigcap_{m=1}^{n} \theta(B_m) = \theta[\bigcap_{m=1}^{n} B_m] \neq \emptyset$.

Thus, $\theta[\mathscr{X}_x]$ has the finite intersection property, and since $(Y, \mathscr{V})$ is compact, $\bigcap \theta[\mathscr{X}_x] \neq \emptyset$.

We show now that $\bigcap \theta[\mathscr{X}_x]$ is a singleton.

Let $\{a, b\} \subseteq \bigcap \theta[\mathscr{X}_x]$ with $a \neq b$. Let L be an element of $\mathscr{X}$ for which $b \in \theta(L)$ and $a \notin \theta(L)$.

For every element P of $\mathscr{X}_x$, we have

$$\theta(P \cap L) = \theta(P) \cap \theta(L) \neq \emptyset,$$

and so $P \cap L \neq \emptyset$. The family $\mathscr{X}_x \cup \{L\}$ therefore has the finite intersection property ($\mathscr{X}_x$ being closed under finite intersections), and so

$$\emptyset \neq L \cap \bigcap \mathscr{X}_x = L \cap \{x\}.$$

From this it follows that $x \in L$, and so $L \in \mathscr{X}_x$. Thus,

$$\theta(L) \supseteq \bigcap \theta(\mathscr{X}_x) \supseteq \{a, b\},$$

which contradicts $a \notin \theta(L)$, and so $\bigcap \theta(\mathscr{X}_x)$ is a singleton.

Define $\tilde{\theta}\colon X \to Y$ by

$$\{\tilde{\theta}(x)\} = \bigcap \theta(\mathscr{X}_x).$$

The map $\tilde{\theta}$ is one-to-one since if $x \neq y$, then by Lemma 4.1 disjoint elements B_1 and B_2 of $\mathscr{X}$ exist with $x \in B_1$ and $y \in B_2$.

Thus,

$$\{\tilde{\theta}(x)\} \cap \{\tilde{\theta}(y)\} = \left(\bigcap \theta(\mathscr{X}_x)\right) \cap \left(\bigcap \theta(\mathscr{X}_y)\right) \subseteq \theta(B_1) \cap \theta(B_2) = \theta(B_1 \cap B_2) = \theta(\emptyset) = \emptyset,$$

and so $\tilde{\theta}(x) \neq \tilde{\theta}(y)$.

We show that $\tilde{\theta}$ is onto. Let $y \in Y$. Now, $\{y\} = \bigcap \mathscr{Y}_y$ where $\mathscr{Y}_y = \{D \in \mathscr{Y}\colon y \in D\}$. Let

$$\mathscr{K} = \theta^{-1}[\mathscr{Y}_y].$$

For every finite subfamily $\{K_1, K_2, \ldots, K_n\}$ of $\mathscr{K}$,

$$\theta[\bigcap_{i=1}^{n} K_i] = \bigcap_{i=1}^{n} \theta(K_i) \supseteq \bigcap \mathscr{Y}_y = \{y\}.$$

Thus, $\bigcap_{i=1}^{n} K_i \neq \emptyset$, and $\mathscr{K}$ has the finite intersection property. Since X is compact, $\bigcap \mathscr{K} \neq \emptyset$.

Let $x \in \bigcap \mathscr{K}$. Thus,

$$\{\tilde{\theta}(x)\} = \bigcap \{\theta(B)\colon x \in B \in \mathscr{X}\} \subseteq \bigcap \{\theta(K)\colon K \in \mathscr{K}\} = \bigcap \mathscr{Y}_y = \{y\},$$

and so $\tilde{\theta}(x) = y$.

The map $\tilde{\theta}$ has the property that $\tilde{\theta}(F) = \theta(F)$ for every F in $\mathscr{X}$.

It is clear from the definition of $\tilde{\theta}$ that $\tilde{\theta}(F) \subseteq \theta(F)$.

Let $y \in \theta(F)$. Defining $\mathscr{K} \subseteq \mathscr{X}$ as before by $\mathscr{K} = \theta^{-1}[\mathscr{Y}_y]$, we have $F \in \mathscr{K}$. As shown above, if $x \in \bigcap \mathscr{K}$, then $\tilde{\theta}(x) = y$. Since $x \in F$, this proves that $\theta(F) \subseteq \tilde{\theta}(F)$.

The proof that $\tilde{\theta}$ is closed and continuous follows in the same way as in Corollary 3.15 since $\tilde{\theta}$ is a one-to-one map of X onto Y.

§ 5. Union morphisms.

5.1. Example. The existence of a *union* morphism θ from even the totality of closed sets of a T_1 space onto that of another T_1 space for which $\theta^{-1}(\emptyset) = \{\emptyset\}$ does not generally imply that the two spaces are homeomorphic. Let X and Y be nonhomeomorphic T_1 spaces for which a closed continuous map f of X onto Y exists. (E.g. $X=$ unit square, $Y=$ unit interval and $f=$ projection.) A union morphism θ of 2^X into 2^Y is thereby induced, and since $f(f^{-1}(K)) = K$ for every $K \in 2^Y$, $\theta(2^X) = 2^Y$.

5.2. Results concerning union morphisms which are *dual* to those for intersection morphisms, however, are easily arrived at. The following corollary, for example, is the dual of Corollary 3.10.

5.3. Corollary. *Let α be a union morphism of $\mathscr{A} \subseteq \exp X$ onto $\mathscr{B} \subseteq \exp Y$ where X and Y contain at least two points and $\mathscr{A}$ and $\mathscr{B}$ are closed under finite unions and contain the cosingletons of X and Y respectively.*

Define X^α by

$$X^\alpha = \{x \in X\colon \alpha(\{x\}') \neq Y\}.$$

Then

(1) *the map $\tilde{\alpha}\colon X^\alpha \to Y$, defined by $\{\tilde{\alpha}(x)\} = \alpha(\{x\}')'$, is one-to-one;*

(2) *if $\alpha^{-1}(Y) = \{X\}$, then $\tilde{\alpha}$ maps X onto Y; and*

(3) *for every L in $\mathscr{A}$, $\tilde{\alpha}[L \cap X^\alpha] = \alpha(L) \cap \tilde{\alpha}[X^\alpha]$.*

Proof. Let $\mathscr{X}$ and $\mathscr{Y}$ denote the families of complements of $\mathscr{A}$ and $\mathscr{B}$ respectively, and define $\theta\colon \mathscr{X} \to \mathscr{Y}$ by

$$\theta(A) = \alpha(A')'.$$

The families $\mathscr{X}$ and $\mathscr{Y}$ and the map θ satisfy the conditions of Corollary 3.10. We note too that for every x in X,

$$\{\tilde{\theta}(x)\} = \alpha(\{x\}')' = \{\tilde{\alpha}(x)\}.$$

(Also, $X_\theta = X^\alpha$.)

Parts (1) and (2) therefore follow easily from parts (1) and (2) of Corollary 3.15.

Part (3) is proved as follows.

For every L in $\mathscr{A}$,

$$\begin{aligned} \tilde{\alpha}[L \cap X^\alpha] &= \tilde{\theta}[L \cap X_\theta] \\ &= \tilde{\theta}(X_\theta) \sim \tilde{\theta}[L' \cap X_\theta] \quad (\text{since } \tilde{\theta} \text{ is one-to-one on } X_\theta) \\ &= \tilde{\theta}(X_\theta) \sim [\theta(L') \cap \tilde{\theta}(X_\theta)] \quad (\text{by part (3) of Corollary 3.15}) \\ &= (\theta(L'))' \cap \tilde{\theta}(X_\theta) = \alpha(L) \cap \tilde{\alpha}(X^\alpha). \end{aligned}$$

5.4. Lemma 3.3 shows that meet epimorphisms between meet-semilattices map atoms to atoms or to 0. (If $(S, \circ)$ is a semilattice, $(S, \leqslant_\wedge)$ is called the corresponding meet semilattice.)

Join epimorphisms between join semilattices with zeros do not necessarily map atoms to elements which are either zero or atoms, as the following example shows.

5.5. Example. Let f be any one-to-one map of $(0,1)$ onto $\boldsymbol{R}$, and define $G\colon \boldsymbol{R} \to \exp \boldsymbol{R}$ by

$$G(x) = \begin{cases} \{f(x)\} & \text{if} \quad x \in (0,1), \\ \boldsymbol{R} & \text{if} \quad x \notin (0,1). \end{cases}$$

The map G extends in a natural way to a union morphism θ of $\exp \boldsymbol{R}$ onto $\exp \boldsymbol{R}$, defined by $\theta(S) = \{G(s)\colon s \in S\}$. $\operatorname{Exp} \boldsymbol{R}$ is a semilattice under $\cup$, $\leqslant_\vee$ is equivalent to $\subseteq$, $A \vee B = A \cup B$, and θ a join epimorphism. However, the image of the atom $\{1\}$ is $\boldsymbol{R}$, which is neither zero nor an atom.

5.6. Note. The set of $\leqslant_\vee$-atoms of a semilattice L will be denoted L^a.

5.7. Lemma. *Every join-isomorphism between atomic semilattices maps atoms to atoms in the $\leqslant_\vee$ ordering.*

Proof. Let S and T be atomic semilattices and $\theta\colon S \to T$ a join isomorphism.

Let $j \in S^a$ and let z be a $\leqslant_\vee$-atom of T with $z \leqslant \theta(j)$. Suppose that $z = \theta(b)$ for $b \in S$.

Now,

$$\theta(j \vee b) = \theta(j) \vee \theta(b) = \theta(j) \vee z = \theta(j),$$

which implies that $j \vee b = j$, and so $b \leqslant j$.

Since $j \in S^a$, either $b = j$ or S has a zero and $b = 0$. If $b = 0$, then

$$0 \neq z = \theta(b) = \theta(0) = 0$$

by Lemma 3.2, and so $b = j$. Thus,

$$\theta(j) = \theta(b) = z,$$

which is an atom.

5.8. Theorem. *Let S and T be semilattices with $(T, \leqslant_\vee)$ atomic. Let $\theta\colon S \to T$ be a join isomorphism. For every $k \in S$,*

$$\theta[\{l \in S^a\colon l \leqslant k\}] = \{z \in T^a\colon z \leqslant \theta(k)\}.$$

Proof. Let $l \in S^a$. By Lemma 5.7, $\theta(l) \in T^a$. If $l \leqslant k$, then $\theta(l) \leqslant \theta(k)$. Thus,

$$\theta[\{l \in S^a\colon l \leqslant k\}] \subseteq \{z \in T^a\colon z \leqslant \theta(k)\}.$$

The converse inclusion follows similarly from the fact that θ^{-1} is a join isomorphism.

5.9. Let $\mathscr{X}$ and $\mathscr{Y}$ be families of subsets of X and Y respectively and suppose that they are closed under finite unions and C-generate T_1 topologies. Example 3.14 shows that, without the additional condition that $\mathscr{X}$ contains the singletons, the existence of a union morphism of $\mathscr{X}$ onto $\mathscr{Y}$ is not generally sufficient to ensure that the topologies which they C-generate are homeomorphic. (In this example, $\mathscr{Y}$ also contains the singletons.)

Example 4.3 illustrates the same point when $\mathscr{X}$ and $\mathscr{Y}$ both generate compact T_2 topologies. (In this example, neither space contains singletons.)

5.10. Corollary. *If* $(X, \mathscr{U})$ *and* $(Y, \mathscr{V})$ *are spaces having closed subbases which* (i) *contain the singletons,* (ii) *are closed under finite unions, and* (iii) *are union isomorphic, then* $(X, \mathscr{U})$ *and* $(Y, \mathscr{V})$ *are homeomorphic.*

Proof. Let $\mathscr{X}$ and $\mathscr{Y}$ be the respective closed subbases in $(X, \mathscr{U})$ and $(Y, \mathscr{V})$, and let θ be an isomorphism of $\mathscr{X}$ onto $\mathscr{Y}$. By Lemma 5.6, the equation $\tilde{\theta}(x) = \{\theta(\{x\})\}$ defines a map of $\mathscr{X}$ to $\mathscr{Y}$, and this map is clearly a bijection.

By Theorem 5.8, we have $\tilde{\theta}(K) = \theta(K)$ for every K in $\mathscr{X}$, and the reasoning of Corollary 3.15 is applicable to showing that $\tilde{\theta}$ is a homeomorphism.

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On $\aleph_0$-categoricity and the theory of trees

by

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Abstract. The principal result is: every $\aleph_0$-categorical tree is decidable. This follows from a general theorem which asserts that every nuclear, $\aleph_0$-categorical structure is finitely axiomatizable. Other facts about trees are also proved. For example, the finitely axiomatizable, $\aleph_0$-categorical trees are characterized.

In this paper we investigate $\aleph_0$-categoricity, in general, and $\aleph_0$-categorical trees, in particular. The concept of a nuclear structure is introduced in § 1, where it is shown that each nuclear, $\aleph_0$-categorical structure is finitely axiomatizable. Any $\aleph_0$-categorical, linearly ordered set is easily seen to be nuclear, so we get the result of Rosenstein [4] that every $\aleph_0$-categorical linearly ordered set is finitely axiomatizable. This result is extended in § 2, using the notion of nuclearity, to show that every $\aleph_0$-categorical tree is decidable. In addition, those $\aleph_0$-categorical trees which are also finitely axiomatizable are characterized.

Our method of proof is quite different from Rosenstein's. His is based on an analysis of linear orderings similar to ones given by Erdös and Hajnal [2], or by Läuchli and Leonard [3] in their proof of the decidability of the theory of linearly ordered sets. It is hoped that our method can more easily be applied to other theories. More generally, it is hoped that the notion of a nuclear structure will lead to a classification of those $\aleph_0$-categorical theories which are finitely axiomatizable.

For a first-order theory T we denote by $\varrho(T)$ the similarity type of T. We will consider only T for which $\varrho(T)$ is finite. For convenience we assume that $\varrho(T)$ contains only relation symbols (although none of our results depends on this restriction). A theory is $\aleph_0$-*categorical* iff all of its countable (and here we include the possibility of finite) models are isomorphic, so that an $\aleph_0$-categorical theory is automatically complete. If T is a theory then an n-*type* (of T) is a maximal set of n-ary formulas (i.e. those formulas all of whose free variables are among $v_0, \ldots, v_{n-1}$) which is consistent with T. We denote the set of n-types of T by $S_n(T)$. We will often attribute to a structure $\mathfrak{A}$ a property that $\mathrm{Th}(\mathfrak{A})$ has (e.g. $\aleph_0$-categoricity, decidability).

* Many of the results contained herein were annouced in [6]. The intended proof of Theorem 1 in [6] was erroneous; however, this has no effect on the contents of this paper.