and $S = \bigcup_n S_n$. Now S is a neighborhood of 0 in E, and it is easily shown that each S_n is complete in the Hausdorff space E, and so each S_n is closed in E. But each S_n is balanced and convex, yet not absorbing, and so is rare in E. Therefore E is not a Baire space. Also E is a barrelled space and the union of an increasing sequence (nS_n) of balanced convex complete sets, thus, by a theorem of Valdivia ([3], Th. 1), E is complete. Moreover, S contains no ray from 0, and so the countable family $\{(1/n)S\}$ of neighborhoods of 0 intersects in the singleton $\{0\}$.

There are incomplete normed spaces which are Baire spaces ([1], p. 95), and so a normed Baire space need not be pseudo-complete. The question of the existence of a pseudo-complete linear topological space which is not complete seems to be open.

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Totally-disconnected compact metric groups

by

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Abstract. Any totally-disconnected compact group has a basis at the identity $\{N_i\}$ consisting of closed and open normal subgroups of finite index. If the group contains a fintely-generated dense subgroup then the topology is a metric topology and the basis at the identity can be taken to be countable. We say that a group is r-separable if there is a dense subgroup with r generators. Let F be a class of finite groups. For certain F, there is a largest r-separable totally-disconnected compact group G_0 such that all the factor groups G_0/N_i are in F. Examples include for the class F the class of all finite groups, the class of all finite p-groups for a prime number p, and the class of all finite nilpotent groups. The largest r-separable totally-disconnected compact group with factors finite nilpotent is the Cartesian product over all primes p of the largest r-separable totally-disconnected compact groups with finite p-group factors. Totally-disconnected compact groups in some ways have a more complex algebraic structure than connected compact groups. There are r-separable totally-disconnected compact solvable and nilpotent groups with derived and central series of any given length. The question of which r-separable totally-disconnected compact groups satisfy non-trivial algebraic laws is a difficult problem concerning the residual properties of free groups. It is shown that if a compact group contains a non-abelian free group then it contains a free group on a continuum of free generators.

Introduction. The class of totally-disconnected compact metric groups which contain a finitely-generated dense subgroup can be classified by the residual properties of finite rank free groups. Some of these groups satisfy non-trivial algebraic laws and others contain subgroups which are free groups with a continuum of generators. If a compact group contains a subgroup with two free generators, then it contains a subgroup on a continuum of free generators.

Section 1. A *Cantor group* is any topological group which has the Cantor discontinuum as its underlying topological space. Montgomery–Zippin [7] show that any totally-disconnected compact topological group G has a basis $\{N_i\}$ at the identity e consisting of open and closed normal subgroups. It follows that G/N_i is a finite group for all i. If G has a metric topology then the basis at e can be assumed to be a sequence with $N_i \supset N_{i+1}$ for all $i \geq 1$ and G is a Cantor group. Let P be the Cartesian product $\prod_{i=1}^{\infty} G/N_i$ with the product topology and let $\varphi\colon G \to P$ be defined by $\varphi(g)(i) = gN_i$. Then φ is an isomorphism of G onto a closed subgroup of $\prod_{i=1}^{\infty} G/N_i$. Let S_n be the symmetric group on n symbols and let P_0 be the Cartesian product $\prod_{n=1}^{\infty} S_n$ with the product topology. Then P_0 is a Cantor group and P is a closed subgroup of P_0 because each group G/N_i is a subgroup of some S_n where n depends on i. Thus, the Cantor groups are exactly the closed subgroups of P_0.

A group G is *residually finite* if given $g \neq e$ in G there is a subgroup H of G with the index $[G:H]$ of H in G finite such that $g \notin H$. M. Hall [3] describes a topology in certain residually finite groups via normal subgroups of finite index. Suppose G is residually finite and $\{G_i\}$ is a decreasing sequence of normal subgroups of finite index in G with $\bigcap_{i=1}^{\infty} G_i = (e)$. One defines a topology denoted $\tau(G_i)$ in G by letting $\{G_i\}$ be a basis at e. The open sets then have $\{gG_i\}$ as a basis. This is the same topology in G as the relative topology in G when it is embedded in the product $\prod_{i=1}^{\infty} G/G_i$ with the product topology by mapping $g \to \langle gG_i \rangle$. The *completion of* $(G, \tau(G_i))$ *by Cauchy sequences* is denoted $C(G; G_i)$ and can be identified with the closure of G when embedded in $\prod_{i=1}^{\infty} G/G_i$ as above. This identifies $C(G; G_i)$ with the inverse limit $\prod_{\overleftarrow{i}} G/G_i$ consisting of all $\langle g_i G_i \rangle$ where $g_{i+1}G_i = g_i G_i$ for all $i \geqslant 1$. Also, $C(G; G_i)$ is uniquely determined by its containing $(G, \tau(G_i))$ as a dense subtopological group.

Any Cantor group can be represented as above. Let H be a dense subgroup of a Cantor group G and let $\{N_i\}$ be a decreasing sequence of closed normal subgroups of finite index in G which is a basis at e in G. Let $H_i = H \cap N_i$. With $\{H_i\}$ as a basis at e in H, H has the relative topology in G. Hence, $C(H; H_i)$ is isomorphic to G. This representation is of interest in the case that H is finitely-generated or at least countable. Since any Cantor group G is separable, G will always have a representation as $C(H; H_i)$ for some countable subgroup H.

DEFINITION. We say G is an *r-separable Cantor group* if G is a Cantor group with a dense subgroup generated by r elements.

Remark. The definition is slightly redundant. If G is a compact totally-disconnected topological group and contains a dense subgroup H generated by a finite number of elements then G has a metric topology. The proof of this is based on the fact that if H is finitely-generated then H contains only finitely many subgroups of a given index; see M. Hall [3] for a proof. In fact, if $\{N_i\}$ is any collection of closed and open normal subgroups in G then $\{N_i \cap H\}$ is countable. Since the closure of $N_i \cap H$ in G is N_i, $\{N_i\}$ is also countable.

We see that any r-separable Cantor group is of the form $C(H; H_i)$ where H is generated by r elements and $\{H_i\}$ is a decreasing sequence of normal subgroups of finite index in H with $\bigcap_{i=1}^{\infty} H_i = (e)$. If we represent H as a factor group of the free group F_r on r generators by a normal subgroup N then N must be the intersection of a sequence of finite index normal subgroups. The classification of all r-separable Cantor groups in this way depends on the residual properties of the free group F_r. In M. Hall [3], the main concern is the question for which decreasing sequences $\{N_i\}$ of finite index normal subgroups of F_r does $\bigcap_{i=1}^{\infty} N_i = (e)$? In any such case one has a Cantor group $G = C(F_r; N_i)$ with a basis $\{\overline{N}_i\}$ at e where $\overline{N}_i$ is the closure of N_i in G with respect to the topology determined by $\tau(N_i)$.

1.1. PROPOSITION (Takahasi [9]). *If $\{N_i\}$ is a properly decreasing sequence of subgroups of F_r such that each N_{i+1} is characteristic in N_i for $i \geqslant 1$, then $\bigcap_{i=1}^{\infty} N_i = (e)$.*

Proof. See [6] for the details. ■

This proposition gives some important examples of r-separable Cantor groups. For instance, let p be a prime number. Let $N_0 = F_r$ and let N_{i+1} be the intersection of all normal subgroups N in N_i with $[N_i:N] = p$. Then $N_{i+1} \subsetneqq N_i$ and N_{i+1} is characteristic in N_i. By 1.1, $\bigcap_{i=1}^{\infty} N_i = (e)$. Also, each N_i is index a power of p in F_r and we have a Cantor group $C(F_r; N_i)$ which we will denote henceforth as $C(F_r; p)$. Another example is to take M_i to be the intersection of all normal subgroups N in F_r with $[F_r:N] \leqslant i$. Again, $\bigcap_{i=1}^{\infty} M_i = (e)$ and we denote $C(F_r; M_i)$ by $C(F_r;\text{ tot})$.

Any r-separable Cantor group G is a continuous homomorphic image of $C(F_r;\text{ tot})$. To prove this we use the following lemma.

DEFINITION. A *resolution in a group* G is a decreasing sequence $\{N_i\}$ of finite index normal subgroups of G with $\bigcap_{i=1}^{\infty} N_i = (e)$.

1.2. PROPOSITION. *Let $p\colon G_1 \to G_2$ be an onto homomorphism and let $\{G_1(j)\}$ and $\{G_2(j)\}$ be resolutions of G_1 and G_2 respectively. Assume that for all $j \geqslant 1$, $p^{-1}(G_2(j)) \supset G_1(k)$ for k sufficiently large depending on j. Then there is a continuous onto homomorphism $\overline{p}\colon C(G_1; G_1(j)) \to C(G_2; G_2(j))$ such that $\overline{p}$ restricted to G_1 is p.*

Proof. Choose a subsequence $\{G_1(k_j)\}$ with $\bigcap_{j=1}^{\infty} G_1(k_j) = (e)$ and $G_1(k_j) \subset p^{-1}(G_2(j))$ for all $j \geqslant 1$. We can identify $C(G_1; G_1(j))$ as $\prod_{\overleftarrow{j}} G_1/G_1(k_j)$ and $C(G_2; G_2(j))$ as $\prod_{\overleftarrow{j}} G_2/G_2(j)$. Define $\overline{p}$ by $\overline{p}\langle g_j G_1(k_j)\rangle = \langle p(g_j) G_2(j)\rangle$. It is easy to verify the properties of $\overline{p}$. ■

Remark. Another description of $\overline{p}$ is to take any $g \in C(G_1; G_1(j))$ and some sequence $g_j \to g$ with $g_j \in G_1$ for all j. Then $\{p(g_j)\}$ is $\tau(G_2(j))$ — Cauchy and has a unique limit point $\overline{p}(g)$ in $C(G_2; G_2(j))$.

1.3. PROPOSITION. *If $C(F_r; N_i)$ and $C(F_r; M_j)$ are given with each N_i containing some M_j where j depends on i then there is a continuous homomorphism p of $C(F_r; M_j)$ onto $C(F_r; N_i)$ which is the identity when restricted to F_r.*

1.4. Proposition. *$C(F_r, \text{tot})$ has every r-separable Cantor group as a continuous homomorphic image. If G_0 is an r-separable Cantor group with this property then G_0 is isomorphic to $C(F_r; \text{tot})$.*

Proof. The first part comes from Proposition 1.2 and the definition of $C(F_r; \text{tot})$. Suppose G_0 has the property above. Then, in particular, there is a continuous onto homomorphism $p\colon G_0 \to C(F_r; \text{tot})$. Let H be an r-generator subgroup which is dense in G_0. In F_r we have the resolution $\{N_i\}$ where N_i is an intersection of a family $\{N\colon N \text{ is normal in } F_r \text{ and } [F_r\colon N] \leqslant i\}$. By construction, if $\{\overline{N}_i\}$ is the sequence of closures in $C(F_r; \text{tot})$, then $\{\overline{N}_i\}$ is a basis at e of closed normal subgroups and $C(F_r; \text{tot})/\overline{N}_i$ is isomorphic to F_r/N_i for all $i \geqslant 1$. This gives us a sequence $\{H_i\}$ defined by $H_i = H \cap p^{-1}(\overline{N}_i)$. Since $\overline{N}_i$ is closed and open, $p^{-1}(\overline{N}_i)$ is closed and open and $G_0/p^{-1}(\overline{N}_i)$ is H/H_i by the density of H.

We claim that H is isomorphic to F_r and each $H_i \subset N_i$ up to this isomorphism. To show this let π be an onto homomorphism from F_r to H. Then $\{\pi^{-1}(H_i)\}$ is a decreasing sequence of normal subgroups of finite index. But the lattice of subgroups $\{N\colon N \supset \pi^{-1}(H_i) \text{ and } N \text{ is normal in } F_r\}$ is in 1-1 correspondence with the $\{N\colon N \supset H_i \text{ and } N \text{ is normal in } H\}$. Also, this correspondence preserves the index. Since H/H_i is F_r/N_i through the isomorphism of $G_0/p^{-1}(\overline{N}_i)$ with $C(F_r, \text{tot})/\overline{N}_i$, we also have $\{N\colon N \supset H_i \text{ and } N \text{ is normal in } H\}$ in 1-1 index-preserving correspondence with $\{N\colon N \supset N_i \text{ and } N \text{ is normal in } F_r\}$. The definition of N_i must then imply $\{N\colon N \supset \pi^{-1}(H_i) \text{ and } N \text{ is normal in } F_r\}$ contains as many distinct normal subgroups N with $[F_r\colon N] \leqslant i$ as there are in $\{N\colon N \supset N_i \text{ and } N \text{ is normal in } F_r\}$. Since there are only a finite number of such subgroups and the same number in both cases, $\pi^{-1}(H_i) \subset N_i$ for all $i \geqslant 1$. Thus, π is an isomorphism because the kernel of π is contained in $\bigcap_{i=1}^{\infty} \pi^{-1}(H_i) \subset \bigcap_{i=1}^{\infty} N_i = (e)$.

The conclusion of the argument is that if τ is the topology of G_0 then τ restricted to H is at least as fine as $\tau(H_i) \supset \tau(N_i)$. Since G_0 is a Cantor group, the topology τ when restricted to H is no finer than (H, tot). Hence, $(H, \tau|H)$ is isomorphic to (F_r, tot) and G_0 is isomorphic to $C(F_r, \text{tot})$. ■

By Schreier's index formula, if F is a subgroup of F_s with index r then F is a free group on $r(s-1)+1$ generators. From this one can see that $C(F_s; \text{tot})$ contains $C(F_t; \text{tot})$ as a normal closed subgroup of index r if $t = r(s-1)+1$. Also, any closed subgroup H of $C(F_s; \text{tot})$ with index r is just $C(F_t; \text{tot})$. Another property of $C(F_r; \text{tot})$ is that it does not contain a dense subgroup with fewer then r generators.

In the sense of Proposition 1.4, $C(F_r; \text{tot})$ is the largest r-separable Cantor group. The group $C(F_r; p)$ for a fixed prime p is in the same sense the largest of the Cantor groups $C(H; H_i)$ where H is generated by r elements and H/H_i is a p-group for all $i \geqslant 1$. Similarly, there is an r-separable Cantor group $C(F_r; N_i)$ with F_r/N_i nilpotent for all $i \geqslant 1$ which is the largest r-separable Cantor group among all Cantor groups of the form $C(G; G_i)$ where G has r generators and G/G_i is nilpotent for $i \geqslant 1$. This group is denoted $C(F_r; \text{nil})$.

1.5. Proposition. *The group $C(F_r; \text{nil})$ is isomorphic to the Cartesian product of the groups $C(F_r; p)$ where p runs through all the primes.*

Proof. Let $\{p_j\}$ be an enumeration of the prime numbers. Let $\{N_i(p_j)\}$ be a basis at e as in the definition of $C(F_r; p_j)$. Let $\mathcal{N}_i = N_i(p_1) \cap \ldots \cap N_i(p_i)$. Then $C(F_r; \mathcal{N}_i)$ is well-defined and is isomorphic to $C(F_r; \text{nil})$. Also, $F_r/\mathcal{N}_i$ is isomorphic to the direct sum $F_r/N_i(p_i) \oplus \ldots \oplus F_r/N_i(p_i)$ because the primes $\{p_1, \ldots, p_i\}$ are distinct and $[F_r\colon N_i(p_s)]$ is a power of p_s for all $i, s \geqslant 1$. Hence, $C(F_r; \text{nil})$ is just $\prod_i [F_r/N_i(p_i) \oplus \ldots \oplus F_r/N_i(p_i)]$. This in turn is $\prod_i (\prod_{j \geqslant i} F_r/N_j(p_i))$. Since, $\prod_{j \geqslant i} F_r/N_j(p_i)$ is $C(F_r; p_i)$, we have $C(F_r; \text{nil})$ isomorphic to $\prod_i C(F_r, p_i)$. ■

Remark. Let G be a topological group. Let C_1 be the closure of the subgroup generated by $\{ghg^{-1}h^{-1}\colon g, h \in G\}$. Let C_2 be the closure of the subgroup generated by $\{ghg^{-1}h^{-1}\colon g \in G \text{ and } h \in C_1\}$. Continue this inductively to get a decreasing sequence of closed normal subgroups $\{C_i\}$ with G/C_i nilpotent for all i. If G is any totally-disconnected compact group with a basis $\{N_j\}$ of closed and open normal subgroups having nilpotent factors G/N_j then $\bigcap_{i=1}^{\infty} C_i = (e)$. In particular, G is an inverse limit of a sequence of totally-disconnected compact nilpotent groups. Similarly, if each G/N_i is solvable then G is an inverse limit of a sequence of compact solvable groups.

Section 2. Let $W(X_1, \ldots, X_n)$ be a word in n free variables. A group G satisfies a law $W(X_1, \ldots, X_n)$ when the equation $W(g_1, \ldots, g_n) = e$ holds for all $g_1, \ldots, g_n \in G$. Balcerzyk and Mycielski [1] have shown that a connected compact group is either abelian or contains a free group on a continuum of generators. It follows that if a connected compact group satisfies a non-trivial law $W(X_1, \ldots, X_n)$ then the group is abelian.

In general, if a compact group G satisfies a law $W(X_1, \ldots, X_n)$ which is non-trivial then we also have the law $W(e, \ldots, X, \ldots, e)$ in the group. Hence, either the group satisfies a law X^n where $n \geqslant 1$ or the sum of the exponents on each X_i in $W(X_1, \ldots, X_n)$ is 0. The latter occurs if and only if $W(X_1, \ldots, X_n)$ is a product of commutators. Also, if a compact group satisfies a law X^n or if the compact group has only elements of finite order then the group is totally disconnected. To sum this up, if a compact group satisfies a law $W(X_1, \ldots, X_n)$ then the group is totally-disconnected and satisfies a law X^n or $W(X_1, \ldots, X_n)$ is a product of commutators and the identity component of the group is abelian.

The case of a totally-disconnected compact group which satisfies a law X^n has special algebraic significance. The Restricted Burnside Conjecture is that an r-separable Cantor group satisfying a law X^n is finite. Kostrikin [5] has shown this is true for primes n. P. Hall and Higman [4] have shown that it follows from this for other special integers like products of two distinct primes.

Also, unlike in the connected case, there are non-abelian Cantor groups satisfying commutator identities. Let $W(X_1, \ldots, X_n)$ be either $\big((\ldots((X_1, X_2), X_3), \ldots), X_n\big)$ or $\big((\ldots((X_1, X_2), (X_3, X_4)), \ldots), (X_{n-1}, X_n)\big)$ where $(Z, Y) = ZYZ^{-1}Y^{-1}$. Let $N(W)$ be the normal subgroup of F_r generated by $\{W(g_1, \ldots, g_n): g_1, \ldots, g_n \in G\}$. Then in either case, $F_r/N(W)$ is residually finite and, in fact, is residually a finite p-group for any prime p. A reference for this is Gruenberg [2]. If $\{N_i\}$ is a resolution of $F_r/N(W)$ then $C(F_r/N(W); N_i)$ is an r-separable Cantor group satisfying W. In general, these groups are solvable in the sense that there is a sequence $(e) = S_1 \subset \ldots \subset S_n = G$ of closed normal subgroups with each S_{i+1}/S_i an abelian Cantor group. Depending on which of the two examples given for W above that one uses, the group $C(F_r/N(W); N_i)$ will have derived or central series of any given length.

We would like to have simple conditions under which a compact group satisfies no non-trivial laws and to show that in this case the group contains large free groups. At least, any group of the form $C(F_r; N_i)$ with $r \geq 2$ will satisfy no law. Also, any such group contains a free group on a countable number of free generators because F_2 does. We will show in the remainder of this section that any group of this form contains a free group on a continuum of generators. It will follow that any compact group which contains a free group on two generators must contain a free group on a continuum of generators.

Let $\{N_i\}$ be a resolution of F_2. Assume x and y are free generators of F_2. Let $O_x = \{y^j x y^{-j}: j \geq 1\}$. For each $i \geq 1$, $O_x \bmod N_i$ is a finite set containing M_i elements. Each M_i divides M_{i+1} and M_i increases to ∞ as $i \to \infty$. If $\{M_i\}$ were bounded then there would be a $K \geq 1$ with $y^K x y^{-K} N_i = x N_i$ for all $i \geq 1$. Since $\bigcap_{i=1}^{\infty} N_i = (e)$, this would mean $y^K x y^{-K} = x$ which is impossible. We assume without loss of generality for what follows that $M_{i+1} > M_i$ for all $i \geq 1$ and $M_1 \geq 2$. Let $M_0 = 1$.

We want to define a sequence $\{r_i\}$ of homomorphisms of a free group on a continuum of generators into F_2. It is enough to define $\{r_i\}$ on the generators of such a group. Let F_∞ be a free group with free generators $\{x_\lambda: \lambda \in [0, 1)\}$. Partition $[0, 1)$ into M_1 intervals $X_1^1, \ldots, X_{M_1}^1$ of the same length and of the form $[\cdot, \cdot)$. Do this so that $\lambda_i < \lambda_j$ if $\lambda_i \in X_i^1$ and $\lambda_j \in X_j^1$ with $i < j$. Define r_1 by the formula $r_1(x_\lambda) = y^{j-1} x y^{-j+1}$ if $\lambda \in X_j^1$.

Suppose now $i \geq 1$ and r_i has been defined in the following manner. There is a partition $\{X_1^i, \ldots, X_{M_i}^i\}$ of $[0, 1)$ by intervals of the same length and of the form $[\cdot, \cdot)$. If $\lambda \in X_j^i$ then $r_i(x_\lambda) = x^{j-1} x y^{-j+1}$. Let $d = M_{i+1}/M_i$. Divide each X_j^i into d intervals $X_j^i(1), \ldots, X_j^i(d)$ of the same length and of the form $[\cdot, \cdot)$. We assume $\lambda_s < \lambda_t$ if $\lambda_s \in X_j^i(s)$ and $\lambda_t \in X_j^i(t)$ with $s < t$. We enumerate the $(i+1)$-partition $\{X_j^i(k): j = 1, \ldots, M_i \text{ and } k = 1, \ldots, d\}$ lexicographically. That is, the $(i+1)$-partition is $\{X_1^{i+1}, \ldots, X_{M_{i+1}}^{i+1}\}$ where $X_{(k-1)M_i+j}^{i+1} = X_j^i(k)$ for $j = 1, \ldots, M_i$ and $k = 1, \ldots, d$. Define r_{i+1} by $r_{i+1}(x_\lambda) = y^{j-1} x y^{-j+1}$ if $\lambda \in X_j^{i+1}$.

The fundamental property of $\{r_i\}$ is that for all $w \in F_\infty$ $r_i(w) N_i = r_{i+1}(w) N_i$ for all $i \geq 1$. Hence, $r_{i+1}^{-1}(N_{i+1}) \subset r_i^{-1}(N_i)$ because $r_{i+1}(w) \in N_{i+1}$ implies $r_{i+1}(w) N_i = N_i$ and, therefore, $r_i(w) \in N_i$. It will not be necessarily true that $\bigcap_{n=1}^{\infty} r_n^{-1}(N_n) = (e)$ but if we suitably restrict the generators of F_∞ then this will be true.

Any word $w(x, y)$ in the free variables x and y has a unique expression in the form $x^{i_1} y^{j_1} \ldots x^{i_m} y^{j_m}$ where $i_1, j_1, \ldots, i_m, j_m$ are integers which are all non-zero except possibly i_1 and j_m. The length of w, denoted $\|w\|$, is $\sum_{s=1}^{m} |i_s| + |j_s|$. Let D_n be the largest whole number such that all words $w(x, y) \neq e$ such that $\|w\| < D_n$ are not in N_n. The sequence $\{D_n\}$ increases to ∞ as $n \to \infty$ because $\bigcap_{n=1}^{\infty} N_n = (e)$.

Choose a sequence $\{R_i\}$ of positive integers with the following properties: $R_0 = 1$ and $R_i + 1 < R_{i+1}$ for all $i \geq 0$; also, $M_{R_i+1}/D_{R_{i+1}} \leq 1/2^{i+1}$ for all $i \geq 0$. This is possible because $D_n \to \infty$ as $n \to \infty$. Any $\lambda \in [0, 1)$ has a unique expression of the form $\sum_{j=1}^{\infty} c_j/M_j$ if $0 \leq c_j < M_j/M_{j-1}$ and $c_j \neq M_j/M_{j-1} - 1$ frequently. Let $\lambda(n)$ be defined by the equation $r_n(x_\lambda) = y^{\lambda(n)} x y^{-\lambda(n)}$. It is easy to see that $\lambda(n) = \sum_{j=1}^{n} c_j M_{j-1}$. We define Λ_0 to be all $\lambda \in [0, 1)$ having $c_i(\lambda) = 0$ unless $i = R_j + 1$ for some $j \geq 0$ and $c_j(\lambda) \in \{0, 1\}$ for all $i \geq 1$. The property of $\lambda \in \Lambda_0$ we will use is that for all $i \geq 0$,

$$\lambda(R_{i+1}) = \sum_{j=1}^{R_{i+1}} c_j M_{j-1} \leq \sum_{j=1}^{R_i+1} (M_j/M_{j-1} - 1) M_{j-1} \leq M_{R_i+1} \leq D_{R_{i+1}}/2^{i+1}.$$

Hence, $\lambda(R_{i+1})/D_{R_i+1} \to 0$ as $i \to \infty$. Let F_0 be the free group with generators $\{x_\lambda: \lambda \in \Lambda_0\}$.

2.1. Proposition. *F_0 is a free group on a continuum of generators and there is a sequence of homomorphisms $p_i: F_0 \to F_2$ such that if $w \in F_0$ and $w \neq e$ then $p_i(w) \notin N_i$ for i sufficiently large. Also, if $w \in F_0$, then $p_i(w) N_i = p_{i+1}(w) N_i$ for all $i \geq 1$.*

Proof. Take p_i to be r_i restricted to F_0. If $w \in F_0$ and $w \neq e$ then $w = V(x_{\lambda_1}, \ldots, x_{\lambda_m})$ where V is freely reduced and not the identity and $\lambda_1, \ldots, \lambda_m$ are distinct elements of Λ_0. For any $n \geq 1$, we have

$$r_n(w) = V(y^{\lambda_1(n)} x y^{-\lambda_1(n)}, \ldots, y^{\lambda_m(n)} x y^{-\lambda_m(n)}).$$

Since $\lambda_1, \ldots, \lambda_m$ are distinct, eventually $r_n(x_{\lambda_1}), \ldots, r_n(x_{\lambda_m})$ are distinct and $r_n(w) \neq e$. The length

$$\|r_n(w)\| \leq \|V\| \big(1 + 2\,\mathrm{MAX}(\lambda_1(n), \ldots, \lambda_m(n))\big).$$

Since each $\lambda_i \in \Lambda_0$, we have

$$\|r_{R_K}(w)\| \leq \|V\| (1 + 2D_{R_K}/2^K).$$

Hence, $\|r_{R_K}(w)\|/D_{R_K}\to 0$ as $K\to\infty$. This says $r_{R_K}(w)\notin N_{R_K}$ for K sufficiently large. That is, $p_i(w)\notin N_i$ for i sufficiently large.

To finish the proof we need only prove that Λ_0 is of the same cardinality as $\prod_{i=1}^{\infty}\{0,1\}$. But Λ_0 is in 1-1 correspondence with $\prod_{i=1}^{\infty}\{0,1\}\setminus D$ where D is countable. Hence, Λ_0 has the same cardinality as the continuum. ■

2.2. Corollary. *$C(F_2; N_i)$ contains a subgroup isomorphic to a free group on a continuum of generators.*

Proof. Let $\varphi\colon F_0\to\prod_i F_2/N_i$ be defined by $\varphi(w)(i) = p_i(w)N_i$. Then φ is an isomorphism of the free group F_0 into $C(F_2; N_i)$. ■

We should remark here that with a little more effort we can get a free group on a continuum of generators as a dense subgroup of $C(F_2; N_i)$. Let F be the free group with free generators $\{x_\lambda\colon \lambda\in[0,1]\}$. Define a sequence of homomorphisms $\{s_n\}$ by letting $s_n(x_\lambda) = r_n(x_\lambda)$ if $\lambda<1$ and $s_n(x_1) = y$ for all $n\geqslant 1$. Define an equivalence relation on Λ_0 by $\lambda_1\sim\lambda_2$ if and only if $\{\lambda_1(n)-\lambda_2(n)\}$ is bounded in absolute value. Let Λ^* be a set of representatives of the equivalence classes of this equivalence relation. Let F^* be the free group generated by $\{x_\lambda\colon \lambda = 1$ or $\lambda\in\Lambda^*\}$.

2.3. Proposition. *F^* is a free group on a continuum of generators and there is an isomorphism of F^* with a dense subgroup of $C(F_2; N_i)$.*

Proof. We use the fact that if $j = R_{i+1}+1$ and $c_j(\lambda_1)\neq c_j(\lambda_2)$, then we have

$$\begin{aligned}|\lambda_1(j)-\lambda_2(j)| &\geqslant M_j-\sum_{s=1}^{j-1}|c_s(\lambda_1)-c_s(\lambda_2)|\,M_{s-1}\\ &\geqslant M_j-\sum_{s=1}^{R_i+1}(M_s/M_{s-1}-1)M_{s-1}\\ &\geqslant M_j-M_{R_i+1}+1\\ &\geqslant 3M_{R_i+1}.\end{aligned}$$

It follows that $\lambda_1\sim\lambda_2$ if and only if $c_i(\lambda_1) = c_i(\lambda_2)$ for i sufficiently large. Also, if λ_1 is not equivalent to λ_2 then $|\lambda_1(n)-\lambda_2(n)|\to\infty$.

One immediate consequence of these facts is that Λ^* has the cardinality of the continuum. This is because Λ_0 has the same cardinality as $\prod_{i=1}^{\infty}\{0,1\}$ while each $\lambda\in\Lambda_0$ has a countable equivalence class. Hence, the representatives Λ^* have cardinality the same as Λ_0.

We let $q_n\colon F^*\to F_2$ be the map s_n restricted to F^*. An argument similar to the one in the proof of 2.1 will show that $\bigcap_{n=1}^{\infty} q_n^{-1}(N_n) = (e)$. One needs to use the fact that if $\lambda_1,\lambda_2\in\Lambda^*$ and $\lambda_1\neq\lambda_2$ then $|\lambda_1(n)-\lambda_2(n)|\to\infty$ as $n\to\infty$. This gives us an isomorphism $\varphi\colon F^*\to\prod_i F_2/N_i$ by $\varphi(w)(i) = q_i(w)N_i$. The image $\varphi(F^*)$ is dense in $C(F_2; N_i)$ because $q_i(F^*) = F_2$ for all $i\geqslant 1$. We know this because $q_i(x_1) = y$ for all $i\geqslant 1$ and if $\lambda\sim 0\in\Lambda_0$ then $q_i(x_\lambda) = y^Kxy^{-K}$ for some constant K when i is sufficiently large. Hence, $q_i(F^*) = F_2$ for large i and, therefore, for all i. ■

2.4. Theorem. *If a compact group contains a free group on two free generators then it contains a free group on a continuum of generators.*

Proof. Suppose G is compact and F_2 is a subgroup. Let H be the closure of F_2. Then H is compact and F_2 is dense in H. Let H_0 be the identity component of H. If $H_0\cap F_2\neq(e)$ then it is a non-trivial normal subgroup of F_2 and, hence, is a free group with at least two generators. Balcerzyk and Mycielski [1] showed that in this case H_0 contains a free group on a continuum of generators. If $H_0\cap F_2 = (e)$ then H/H_0 contains a dense subgroup on two free generators. Since H/H_0 is totally-disconnected H/H_0 is isomorphic to $C(F_2; N_i)$ for some resolution $\{N_i\}$. Now Corollary 2.2 implies H/H_0 contains a free group on a continuum of generators. Therefore, in any case G contains a free group on a continuum of generators. ■

The above theorem can be extended to locally compact groups G with identity component G_0 such that G/G_0 is compact using [1] and 2.4. It is not true for all locally compact groups. If a compact group G does contain a non-abelian free group F then the index of F in G is necessarily the cardinality of the continuum. Also, the technique we use here is easily adapted for showing that a locally compact group with an element x for which the subgroup generated by x is not discrete in the relative topology must contain a free abelian group on a continuum of generators.

It was suggested by Jan Mycielski that the main theorem of [8] would give a categorical proof of Theorem 2.4. One wants to show that if a compact group contains a free subgroup on two generators, then it contains a free group on a continuum of generators. As in the proof of Theorem 2.4, we need only show this for compact totally-disconnected groups. By taking a subgroup, we may assume we have a totally-disconnected compact group G and a dense subgroup on a countable infinity of free generators $\{x_i\colon i = 1,2,3\ldots\}$. Using the theorem in [8], we can conclude that G contains a continuum of free generators if we can show that for each non-trivial word $W(z_1,\ldots,z_m)$ in free variables $z_1,\ldots,z_m$ the set $R_W = \{(g_1,\ldots,g_m)\colon W(g_1,\ldots,g_m) = e\}$ is nowhere dense in $G\times\ldots\times G$. Since the set is closed, we need only show that it has no interior. If it had interior then there would be open-closed normal subgroups $N_1,\ldots,N_m$ in G and elements $h_1,\ldots,h_m$ in the subgroup generated by $\{x_i\colon i = 1,2,3,\ldots\}$ such that $W(h_1n_1,\ldots,h_mn_m) = e$ for all $(n_1,\ldots,n_m)$ in $N_1\times\ldots\times N_m$. Choose distinct free generators $x_{i_1},\ldots,x_{i_m}$ in $\{x_i\}$ which are not involved in the free reduced forms of any of $h_1,\ldots,h_m$. For some large integer K, $x_{i_j}^K$ is in N_j for all $j = 1,\ldots,m$. Hence, $W(h_1x_{i_1}^K,\ldots,h_mx_{i_m}^K) = e$. But this is impossible by the choice of $\{x_{i_j}\colon j = 1,\ldots,m\}$. Hence, R_W is nowhere dense. This shows that the theorem of Mycielski [8] is really more general than Theorem 2.4. However, the proof of Corollary 2.2 is quite explicit in the case of a totally-disconnected compact group; moreover, it gives us Proposition 2.3 which is not obvious from the categorical method.

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Semilattice theory with applications to point-set topology

by

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Abstract. By means of semilattice theory, it is shown that if an intersection-preserving epimorphism θ exists between a pair of closed subbases containing the singletons, and $\theta^{-1}(\emptyset) = \{\emptyset\}$, then the generated topologies are homeomorphic. The necessity of including the singletons in this context is studied, and a similar theorem is proved for subbases which do not necessarily contain the singletons, but which generate compact T_1 topologies. These results generalize theorems of Birkhoff concerning bases or entire families of closed sets and set maps which are lattice isomorphisms.

1. Introduction. Theorems expressing conditions under which the isomorphism of the lattices of closed sets of a pair of topological spaces implies the topological equivalence of the spaces are an established part of the lattice theory literature. (See, for example, [1], [3], and [4].)

By means of general semilattice theorems, it is shown here that, for T_1 spaces, consideration of these questions can be profitably extended to *semilattices* of closed sets which are *subbases* for the topologies involved — instead of totalities of closed sets or of bases, — and to *intersection-* or *union-epimorphisms* — instead of lattice isomorphisms (e.g. for an intersection morphism, $f(A \cap B) = f(A) \cap f(B)$ for all A, B). Several counterexamples are presented to delineate the extent to which some of the hypotheses can be weakened.

In Section 3 we study meet epimorphisms between semilattices (Theorem 3.6). The results obtained are applied to show that if an intersection morphism exists from a closed subbasis containing singletons onto a second, then the generated topologies are homeomorphic (Corollary 3.15). This generalizes a theorem of Birkhoff. A counterexample shows the necessity of including the singletons, set theoretic investigations yield Corollary 3.12 which essentially concerns cardinalities, and further topological considerations are discussed which stem naturally from the study of intersection morphisms (Corollaries 3.19 and 3.32).

Section 4 has as its main result (Theorem 4.5) a generalization of another theorem of Birkhoff. It is proved that if an intersection morphism exists from one closed subbasis onto another, and if the topologies which they generate are com-