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STATE UNIVERSITY OF NEW YORK AT BUFFALO

Accepté par la Rédaction le 11. 4. 1974

Pseudo-completeness in linear metrizable spaces

by

Aaron R. Todd (Brooklyn, N. Y.)

Abstract. J. C. Oxtoby has shown that the standard Baire category theorems follow from his definition of pseudo-complete spaces. Although a pseudo-complete metrizable space need not be topologically complete, pseudo-completeness implies completeness for a linear topology whose completion is stronger than a metrizable topology. Pseudo-completeness and completeness are equivalent for a linear metrizable topology. A complete linear topology stronger than a metrizable topology need not be pseudo-complete.

In a portion of his paper [2], Oxtoby nicely identifies by his pseudo-complete spaces, those common elements of several standard Baire category theorems which make them corollaries of his Proposition (5.1), *Any pseudo-complete space is a Baire space*, and his Theorem 6, *The Cartesian product of any family of pseudo-complete spaces is pseudo-complete.*

The object of this paper is to establish that, for a large class of linear topological spaces, pseudo-completeness implies completeness; indeed, for linear metrizable spaces these concepts are equivalent.

A topological space X is *quasi-regular* if and only if each non-empty open set contains the closure of a non-empty open set. A family $\mathscr{B}$ of non-empty open sets is a *pseudo-base* for X if and only if each non-empty open set contains an element of $\mathscr{B}$. A quasi-regular topological space X is *pseudo-complete* if and only if there is a sequence $(\mathscr{B}_n)$ of pseudo-bases for X such that if $U_n \in \mathscr{B}_n$ and $U_n \supset \bar{U}_{n+1}$ then $\bigcap_n U_n$ is non-empty.

It is easily seen that a pseudo-metrizable space X, which is complete in some pseudo-metric d, is pseudo-complete by considering the bases $\mathscr{B}_n$ of non-empty open sets of d-diameter less than $1/n$. That the converse is false may be seen by considering a subspace of the plane, $X = R\times(0, \infty) \cup Q\times\{0\}$, the union of the upper half plane and its set of boundary points with rational first coordinates: For each $\mathscr{B}_n$ use open disks of the plane which are contained in X and which have centers with rational first coordinates and radii less then $1/n$. If X is complete in some metric which induces its topology then X is a G_δ subset of the plane ([1], p. 96). This is not possible since Q is not a G_δ subset of R.

The following proposition characterizes a property required in the main theorem.

Proposition. *If $\mathscr{I}$ is a linear topology, then the following are equivalent:*
(a) *$\mathscr{I}$ is finer than some linear metrizable topology.*
(b) *$\mathscr{I}$ is finer than some metrizable topology.*
(c) *$\mathscr{I}$ contains a countable subfamily which intersects in a singleton.*

Proof. Clearly (a) implies (b), and (b) implies (c). Suppose (U_n) is a sequence in $\mathscr{I}$ such that $\bigcap_n U_n$ is a singleton. We may suppose the singleton is $\{0\}$. Using the neigborhoods $\{U_n\}$ of 0, we may easily obtain a countable family of balanced neighborhoods $\{V_n\}$ of 0 with $V_{n+1}+V_{n+1}\subset V_n \cap U_n$ which satisfies the metrization theorem ([1], p. 48) and so forms a local base for a linear metrizable topology weaker than $\mathscr{I}$. That is, the linear topology with local base $\{V_n\}$ is induced by some translation invariant metric defined on E. ■

The main theorem now follows.

Theorem. *If the topology of the completion of a linear topological space E contains a countable subfamily which intersects in a singleton, then E is complete if it is pseudo-complete.*

Proof. From the above proposition, there is a translation invariant metric d on the completion F of E which induces on F a topology coarser than the topology of F. Suppose that $(\mathscr{B}_n)$ is a sequence of pseudo-bases for E as in the definition of pseudo-completeness. We shall show that E equals F, and so E is complete.

By the earlier cited result of Oxtoby, the pseudo-complete space E is a Baire space. The Baire space E is a dense subset of F, and so F also is a Baire space. Therefore the intersection of a countable family of dense open subsets of F is a non-meager Borel subset of F. With the aid of this fact and the following claim we shall obtain a non-meager Borel subset A of F (1). By the difference theorem ([1], p. 92), $A-A$ is a neighborhood of 0 in F. Finally, we show that E contains A, thus E contains $\bigcup_n n(A-A) = F$, and so $E = F$, which will complete the proof of the theorem.

Claim. *There is a sequence $(\mathscr{C}_n)$ of families of disjoint open subsets of F such that, for $k\geqslant 0$,*
(0) *$U_k = \cup\mathscr{C}_k$ is dense in F,*
for each $C\in\mathscr{C}_{k+1}$,
(1) *d-diam$(C)\leqslant 1/(k+1)$,*
(2) *$C\cap E\in\mathscr{B}_{k+1}$, and*
(3) *there is $D\in\mathscr{C}_k$ with $C^-\subset D$.*

Proof. Start with $\mathscr{C}_0 = \{F\}$, whose union is certainly dense in F. For $m\geqslant -1$, suppose that $\mathscr{C}_0, \mathscr{C}_1, \ldots, \mathscr{C}_{m+1}$ satisfy condition (0) for all k such that $0\leqslant k\leqslant m+1$ and the remaining conditions for all k such that $0\leqslant k\leqslant m$. Let $n = m+1$. (We shall say that a family of sets is a disjoint family if no two different elements meet.) Let $\mathscr{C}$ be the collection of all open subsets C of F which satisfy the following: (a) d-diam(C) $\leqslant 1/(n+1)$, (b) $C\cap E\in\mathscr{B}_{n+1}$, and (c) for some $D\in\mathscr{C}_n$, $C^-\subset D$. Let $\mathscr{C}_{n+1}$ be maximal among the subfamilies of $\mathscr{C}$ each of which is a disjoint family.

Now $U_{n+1} = \bigcup\mathscr{C}_{n+1}$ is dense in F. For suppose not, then let U be a non-empty open subset of $F\setminus U_{n+1}$ with d-diameter less than $1/(n+1)$. Since U_n is dense, some $D\in\mathscr{C}_n$ meets U. Let W be a non-empty open subset of F such that $W^-\subset D\cap U$. Because $\mathscr{B}_{n+1}$ is a pseudo-base of E, and E is dense in its completion F and inherits its topology from F, there is an open subset C of F such that $C\cap E\in\mathscr{B}_{n+1}$ and $C\cap E$ is contained in $W\cap E$. But C is open and E is dense, so

$$C^- = (C\cap E)^-\subset(W\cap E)^-\subset W^-\subset D\cap U\subset F\setminus\bigcup\mathscr{C}_{n+1}.$$

Thus d-diam$(C)\leqslant 1/(n+1)$ and $C^-\subset D\in\mathscr{C}_n$, so that $\mathscr{C}_{n+1}\cup\{C\}$ is a disjoint subfamily of $\mathscr{C}$. But $\mathscr{C}_{n+1}\cup\{C\}$ properly contains $\mathscr{C}_{n+1}$, which is a contradiction.

Condition (0) is now satisfied for $0\leqslant k\leqslant n+1 = m+2$, and the remaining conditions are satisfied for $0\leqslant k\leqslant n = m+1$. The claim is now established by induction.

Now we let $A = \bigcap_n U_n$. Each U_n is a dense open subset of the Baire space F. Thus A is a non-meager Borel subset of F, and so $A-A$ is a neighborhood of 0 in F. We now show that A is a subset of E, which completes the proof of the main theorem. For $x\in A$, there are $C_n\in\mathscr{C}_n$ with $x\in C_n$. Fix $n\geqslant 1$. From the above, there is $D\in\mathscr{C}_n$ with C_{n+1}^- contained in D. However x is in both C_n and C_{n+1}, and so C_n and D meet. But $\mathscr{C}_n$ is a disjoint family, whence C_n and D are the same set. Thus for all $n\geqslant 1$, $C_{n+1}^-\subseteq C_n$.

Since $C_n\cap E$ is in $\mathscr{B}_n$ and C_{n+1} is open in F and E is dense in F, we have $(C_{n+1}\cap E)^-\cap E = C_{n+1}^-\cap E$. Thus $C_n\cap E\supset C_{n+1}^-\cap E = (C_{n+1}\cap E)^-\cap E$. Now $\bigcap_n (C_n\cap E)$ is a non-empty subset of E by the choice of $(\mathscr{B}_n)$. Also the d-diameter of C_n is less than $1/n$, so that $\bigcap_n (C_n\cap E)$ is a singleton and necessarily $\{x\}$. ■

In the corollary we specialize this result to the case of linear metrizable spaces.

Corollary. *If E is a linear metrizable space, then the following are equivalent:*
(a) *E is pseudo-complete.*
(b) *E is complete.*
(c) *E is complete in some metric which induces its topology.*

Proof. Clearly (b) implies (c); the converse is well known ([1], p. 96). The theorem and proposition shows (a) implies (b), and any complete metric space is pseudo-complete, so (c) implies (a). ■

Of course completeness does not imply that a linear topological space is a Baire space, and so completeness does not imply pseudo-completeness. The example of Saxon [3] used here has a topology finer than a metric topology by the first proposition. Let E be the union of the increasing sequence of the Banach spaces (l_n) and provide E with the strongest locally convex topology $\mathscr{I}$ for which each injection $i_n\colon (l_n, \|\ \|_n)\hookrightarrow(E,\mathscr{I})$ is continuous. Let S_n be the closed unit ball of $(l_n, \|\ \|_n)$

(1) The choice of A uses suggestions for which the author is deeply indebted to Professor Jack Brown of Auburn University, Auburn, Alabama.

and $S = \bigcup_n S_n$. Now S is a neighborhood of 0 in E, and it is easily shown that each S_n is complete in the Hausdorff space E, and so each S_n is closed in E. But each S_n is balanced and convex, yet not absorbing, and so is rare in E. Therefore E is not a Baire space. Also E is a barrelled space and the union of an increasing sequence (nS_n) of balanced convex complete sets, thus, by a theorem of Valdivia ([3], Th. 1), E is complete. Moreover, S contains no ray from 0, and so the countable family $\{(1/n)S\}$ of neighborhoods of 0 intersects in the singleton $\{0\}$.

There are incomplete normed spaces which are Baire spaces ([1], p. 95), and so a normed Baire space need not be pseudo-complete. The question of the existence of a pseudo-complete linear topological space which is not complete seems to be open.

The author gratefully acknowledges the encouragement and helpful suggestions of his teacher S. A. Saxon.

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DEPARTMENT OF MATHEMATICS
BROOKLYN COLLEGE, C.U.N.Y.
Brooklyn, New York

Accepté par la Rédaction le 29. 7. 1974

Totally-disconnected compact metric groups

by

Joseph M. Rosenblatt (Vancouver)

Abstract. Any totally-disconnected compact group has a basis at the identity $\{N_i\}$ consisting of closed and open normal subgroups of finite index. If the group contains a fintely-generated dense subgroup then the topology is a metric topology and the basis at the identity can be taken to be countable. We say that a group is r-separable if there is a dense subgroup with r generators. Let F be a class of finite groups. For certain F, there is a largest r-separable totally-disconnected compact group G_0 such that all the factor groups G_0/N_i are in F. Examples include for the class F the class of all finite groups, the class of all finite p-groups for a prime number p, and the class of all finite nilpotent groups. The largest r-separable totally-disconnected compact group with factors finite nilpotent is the Cartesian product over all primes p of the largest r-separable totally-disconnected compact groups with finite p-group factors. Totally-disconnected compact groups in some ways have a more complex algebraic structure than connected compact groups. There are r-separable totally-disconnected compact solvable and nilpotent groups with derived and central series of any given length. The question of which r-separable totally-disconnected compact groups satisfy non-trivial algebraic laws is a difficult problem concerning the residual properties of free groups. It is shown that if a compact group contains a non-abelian free group then it contains a free group on a continuum of free generators.

Introduction. The class of totally-disconnected compact metric groups which contain a finitely-generated dense subgroup can be classified by the residual properties of finite rank free groups. Some of these groups satisfy non-trivial algebraic laws and others contain subgroups which are free groups with a continuum of generators. If a compact group contains a subgroup with two free generators, then it contains a subgroup on a continuum of free generators.

Section 1. A *Cantor group* is any topological group which has the Cantor discontinuum as its underlying topological space. Montgomery–Zippin [7] show that any totally-disconnected compact topological group G has a basis $\{N_i\}$ at the identity e consisting of open and closed normal subgroups. It follows that G/N_i is a finite group for all i. If G has a metric topology then the basis at e can be assumed to be a sequence with $N_i \supset N_{i+1}$ for all $i \geqslant 1$ and G is a Cantor group. Let P be the Cartesian product $\prod_{i=1}^{\infty} G/N_i$ with the product topology and let $\varphi\colon G \to P$ be defined by $\varphi(g)(i) = gN_i$. Then φ is an isomorphism of G onto a closed subgroup of $\prod_{i=1}^{\infty} G/N_i$. Let S_n be the symmetric group on n symbols and let P_0 be the Cartesian