Some set-theoretic constructions in topology

by

S. Mrówka (Buffalo, N. Y.)

Abstract. We prove a set-theoretic lemma: *If $\mathfrak{F}$ is a class of functions on a set R with* $\operatorname{card}\mathfrak{F}\leqslant\mathfrak{m}$, *then there exists a permutation π of R such that if $f\in\mathfrak{F}$ and $f\circ\pi\in\mathfrak{F}$, then, for some y,* $\operatorname{card}((R\setminus f^{-1}(y))<\mathfrak{m}$. Various topological applications of this lemma are given; e.g. we prove that there exists a class $\mathfrak{N}$ of subsets of the positive integers N such that the familiar space $N\cup\mathfrak{N}$ has only one compactification.

In $[M_2]$ we have shown a certain set-theoretic lemma (Lemma 1) which was then applied in $[M_2]$, $[M_7]$, and [FM] to construction of non-E-compact spaces. In this paper, we shall prove a stronger version of this lemma and then we will give its further applications.

§ 1. A set-theoretic lemma.

We shall prove

1.1. LEMMA. *Let $\mathfrak{m}$ be an infinite cardinal and let $\mathfrak{F}$ be a class of function on a set R where* $\operatorname{card}R=\mathfrak{m}$ *and* $\operatorname{card}\mathfrak{F}\leqslant\mathfrak{m}$. *There exists a permutation π of the set R such that, for every f, if both f and $f\circ\pi$ belongs to $\mathfrak{F}$, then, for some y*

(1) $$\operatorname{card}(R\setminus f^{-1}(y))<\mathfrak{m}\,.$$

(Of course, the above lemma is stronger than Lemma 1 of $[M_2]$ — or Lemma 1.b of $[M_7]$ — only for singular cardinals $\mathfrak{m}$.)

Proof. Let $\mathfrak{F}_0$ be the class of all f in F such that $\operatorname{card}(R\setminus f^{-1}(y))=\mathfrak{m}$ for every y. It suffices, of course, to show that there exists a permutation π of R such that $f\circ\pi\notin\mathfrak{F}$ for every $f\in\mathfrak{F}_0$.

Let $\mathfrak{F}_0=\{f_\xi\colon \xi\in\Xi\}$, $\operatorname{card}\Xi\leqslant\mathfrak{m}$; and let

$$\Xi_1=\{\xi\in\Xi\colon f_\xi \text{ has a fiber of cardinality } \mathfrak{m}\}\,,$$
$$\Xi_2=\{\xi\in\Xi\colon \text{every fiber of } f_\xi \text{ is of cardinality } <\mathfrak{m}\}$$

(a fiber of a function f is a set of the form $f^{-1}(y)$, where y is a value of f).

For every $\xi\in\Xi_1$ we denote by A_ξ one of the fibers of f_ξ with $\operatorname{card}A_\xi=\mathfrak{m}$, and we let $B_\xi=R\setminus A_\xi$. We have $\operatorname{card}B_\xi=\mathfrak{m}$. For every $\xi\in\Xi_2$ we let $C_\xi=R$ and $D_{\xi,y}=R\setminus f_\xi^{-1}(y)$ for every $y\in f_\xi(R)$. The collection $\{A_\xi\colon \xi\in\Xi_1\}\cup\{B_\xi\colon \xi\in\Xi_2\}\cup$ $\cup\{C_\xi\colon \xi\in\Xi_2\}\cup\{D_{\xi,y}\colon \xi\in\Xi_2, y\in f_\xi(R)\}$ is of cardinality $\leqslant\mathfrak{m}$ and each member of this collection is of cardinality $\mathfrak{m}$. Consequently, by a known result (quoted in

[M$_7$] as Lemma 1a), there exists a collection $\{A'_\xi\colon \xi\in\Xi_1\}\cup\{B'_\xi\colon \xi\in\Xi_1\}\cup\cup\{C'_\xi\colon \xi\in\Xi_2\}\cup\{D'_{\xi,y}\colon \xi\in\Xi_2, y\in f_\xi(R)\}$ of mutually disjoint sets of cardinality $\mathfrak{m}$ such that $A'_\xi\subset A_\xi$, $B'_\xi\subset B_\xi$, $C'_\xi\subset C_\xi$, $D'_{\xi,y}\subset D_{\xi,y}$ (for every values of the involved indices).

Functions f_ξ with $\xi\in\Xi_1$, are handled as in [M$_7$]. For a set $K\subset A'_\xi$ we denote by π_K a permutation of $A'_\xi\cup B'_\xi$ such that $\pi_K(x)=x$ for $x\in K$ and $\pi_K(x)\in B'_\xi$ for A'_ξ/K. If K and K' are distinct subsets of A'_ξ, then the functions $f_\xi\circ\pi_K$ and $f_\xi\circ\pi_{K'}$ are also distinct. Since there are 2^m subsets of A'_ξ, we can select a permutation π_ξ of $A'_\xi\cup B'_\xi$ such that $f_\xi\circ\pi_\xi\neq f|(A'_\xi\cup B'_\xi)$ for every $f\in\mathfrak{F}$.

Let $\xi\in\Xi_2$. We let $\mathfrak{m}_y=\operatorname{card}(f_\xi^{-1}(y)\cup C'_\xi)$; we have $\sum\{m_y\colon y\in f_\xi(C'_\xi)\}=\operatorname{card}C'_\xi=\mathfrak{m}$. Let $E_\xi=f_\xi(C'_\xi)$ and select a $C''_\xi\subset C'_\xi$ such that $f_\xi|C''_\xi$ is one-to-one and $f_\xi(C''_\xi)=f(C'_\xi)$; let x_y be the only element of C''_ξ with $f_\xi(x_y)=y$, $y\in E_\xi$. Consider the product $D^*=\times\{D'_{\xi,y}\colon y\in E_\xi\}$. Letting $\mathfrak{m}_y^*=\mathfrak{m}$ for every $y\in E_\xi$, we have $\mathfrak{m}_y^*>\mathfrak{m}_y$ for every $y\in E_\xi$; consequently, by Koenig's theorem

$$\operatorname{card}D^*=\prod\{\mathfrak{m}_y^*\colon y\in E_\xi\}>\sum\{\mathfrak{m}_y\colon y\in E_\xi\}=\mathfrak{m}.$$

For every element $d^*\in D^*$, $d^*=\{x_y^*\}_{y\in E_\xi}$, we denote by π_{d^*} the permutation of $C''_\xi\cup\bigcup\{D'_{\xi,y}\colon y\in E_\xi\}$ defined by $\pi_{d^*}(x_y)=x_y^*$, $\pi_{d^*}(x_y^*)=x_y$ for every $y\in E_\xi$; π_{d^*} is the identity on $\bigcup\{D'_{\xi,y}\colon y\in E_\xi\}\setminus\{x_y^*\colon y\in E_\xi\}$. We check that if d^* and d_1^* are distinct elements of D^*, then the functions $f_\xi\circ\pi_{d^*}$ and $f_\xi\circ\pi_{d_1^*}$ are also distinct. Since $\operatorname{card}D^*>\mathfrak{m}$ we can select a permutation π_ξ of $C''_\xi\cup\bigcup\{D'_{\xi,y}\colon y\in E_\xi\}$ such that $f_\xi\circ\pi_\xi\neq f|(C''_\xi\cup\bigcup\{D'_{\xi,y}\colon y\in E_\xi\})$ for every $f\in\mathfrak{F}$.

The permutations π_ξ are now defined for all $\xi\in\Xi$. Since they act in mutually disjoint sets, we can define the required permutation π to be the common extension of all the π_ξ's. The lemma is shown.

§ 2. Applications to E-compact spaces. In this section we shall extend the results of [FM]. For terminology and notations concerning E-compact spaces, we refer to [M$_5$]. First we shall eliminate the assumption of regularity of certain cardinals used in [FM].

Let $\mathfrak{m}$ be an exponential cardinal (i.e., a cardinal of the form $\mathfrak{m}=2^\mathfrak{n}$; $\mathfrak{n}$ is denoted by $\log\mathfrak{m}$); let $X^{(m)}$ be any space which contains a closed discrete subspace R of cardinality $\mathfrak{m}$ and a dense subset D of cardinality $\log\mathfrak{m}$. If π is a permutation of R, then $X_\pi^{(m)}$ will denote the space obtained from the discrete union of two disjoint copies of $X^{(m)}$ by the identification of each point $p\in R$ in the first copy with the point $\pi(p)$ in the second copy. As in [FM] we have

2.1. *$X_\pi^{(m)}$ is a perfect image of the discrete union of two copies of $X^{(m)}$.*

In addition

2.2. *$X_\pi^{(m)}$ is the union of two closed subsets each homeomorphic to $X^{(m)}$.*

Repeating the argument in [FM] but appealing to Lemma 1.1 of the present paper we can show

2.3. *Let $\mathfrak{m}$ be an exponential cardinal and let E be a space with $\operatorname{card}E\leqslant\mathfrak{m}$. There exist a permutation π of R in $X^{(m)}$ such that $X_\pi^{(m)}$ is not E-compact.*

If $\mathfrak{m}$ is Ulam non-measurable, then we can find an $\mathcal{N}$-compact space $X^{(m)}$ with the above properties (this is the content of Theorem 1, condition $S_{\log}(\mathfrak{m})$, in [M$_4$]; in fact, as $X^{(m)}$ we can take the power $\mathcal{N}^\mathfrak{m}$). Consequently, we obtain

2.4. *The class of all 0-dimensional perfect images of $\mathcal{N}$-compact space, as well as the class of all 0-dimensional spaces which are unions of two closed $\mathcal{N}$-compact subspaces, is not contained in any class of compactness $\mathfrak{K}(E)$, where E is of non-measurable cardinality.*

We have thus eliminated the assumption of regularity from the Theorem in [FM]. This has been achieved also (in a somewhat different way) by Husek [H]. Our construction, however, can be applied to an arbitrary space $X^{(m)}$ (with the stated properties).

We shall now prove the existence of a large number of 0-dimensional classes of compactness. A few introductory remarks will be in order. The smallest class of complete regularity (save for the trivial class of one-point spaces) is the class $\mathfrak{C}(\mathcal{D})$ of all 0-dimensional spaces ($\mathcal{D}$ is the two-point discrete space). We want to show that even within $\mathfrak{C}(\mathcal{D})$ there is a large number of classes of compactness. The smallest class of compactness (save for the trivial class) is the class $\mathfrak{K}(\mathcal{D})$ of all 0-dimensional compact spaces. The next one is the class $\mathfrak{K}(\mathcal{N})$; in fact, we have $\mathfrak{K}(\mathcal{D})\subseteq\mathfrak{K}(\mathcal{N})$ and, according to 4.21 in [M$_5$], *there is no E with $\mathfrak{K}(\mathcal{D})\subsetneqq\mathfrak{K}(E)\subsetneqq\mathfrak{K}(\mathcal{N})$*. On the other hand, Blefko (see [B], Ch. 3 and also [M$_5$]) has shown that if ω_δ and ω_λ are initial ordinals of different cofinality, then neither of the classes $\mathfrak{K}(S(\omega_\lambda))$ and $\mathfrak{K}(S(\omega_\sigma))$ (1) is contained in the other (if $\operatorname{cf}(\omega_\delta)=\operatorname{cf}(\omega_\lambda)$, then $\mathfrak{K}(S(\omega_\delta))=\mathfrak{K}(S(\omega_\lambda))$). He has also shown that, in contrast to the above-mentioned result on the class $K(\mathcal{N})$; there are E with $K(\mathcal{D})\subsetneqq K(E)\subsetneqq K(S(\omega_1))$. The first result of Blefko gives the existence of a large number of 0-dimensional classes of compactness, but we have to vary the size of the representative. We will show that there are a large number of classes of compactness with representatives of fixed cardinality.

We shall first observe that a space $X^{(m)}$ (with the stated properties) exists for every (not necessarily Ulam non-measurable) exponential cardinal $\mathfrak{m}$ (of course, if $\mathfrak{m}$ is measurable, then we cannot require $X^{(m)}$ to be $\mathcal{N}$-compact). One way to see this is to use the $\mathfrak{m}$th power $\mathcal{D}_{\log\mathfrak{m}}^\mathfrak{m}$ of the discrete space $\mathcal{D}_{\log\mathfrak{m}}$ of cardinality $\log\mathfrak{m}$ as $X^{(m)}$. $\mathcal{D}_{\log\mathfrak{m}}^\mathfrak{m}$ has a dense subset of cardinality $\log\mathfrak{m}$ (Hewitt–Marczewski–Pondiczery theorem) and it contains a closed discrete subspace of cardinality $\mathfrak{m}$ (2). This $X^{(m)}$ has the advantage of being $\mathcal{D}_{\log\mathfrak{m}}$-compact; if we do not care about this property, then, by taking a subspace of $\mathcal{D}_{\log\mathfrak{m}}^\mathfrak{m}$, we can obtain an $X^{(m)}$ which is of cardinality $\mathfrak{m}$.

(1) $S(\xi)$ is the space of all ordinals $<\xi$.

(2) This fact follows from considerations of [M$_3$]; however, in an explicit form, it was stated in [J] and [M$_8$]. For a more detailed comment of the whereabouts of this result see [M$_9$], Sec. 4.

Using the above remarks we shall prove

2.5. THEOREM. *Let $\mathfrak{m}$ be an exponential cardinal. There is more than $\mathfrak{m}$ 0-dimensional classes of compactness, each having a representative of cardinality $\mathfrak{m}$.*

More exactly, there exists a 0-dimensional space S with $\operatorname{card} S = \mathfrak{m}$ *and a transfinite sequence of 0-dimensional spaces*

$$E_0, E_1, \ldots, E_\xi, \ldots; \xi<\omega_\lambda,$$

where ω_λ is the first initial ordinal of cardinality $>\mathfrak{m}$, such that each E_ξ is a perfect image of S and

$$\mathfrak{K}(E_\eta)\not\supseteq\mathfrak{K}(E_\xi) \quad \textit{for every } \eta<\xi<\omega_\lambda .$$

Proof. We consider a space $X^{(\mathfrak{m})}$ of cardinality $\mathfrak{m}$ (as described above) and we let S to be the discrete union of $\mathfrak{m}$ (disjoint) copies of $X^{(\mathfrak{m})}$. We let $E_0 = S$. Assume that for a given $\xi<\omega_\lambda$ the spaces E_η are defined for every $\eta<\xi$ and they satisfy the conclusion of the theorem. We let E be the discrete union of all E_η, $E = \bigcup_d \{E_\eta : \eta<\xi\}$. E is of cardinality $\leqslant\mathfrak{m}$, hence, by 2.3, there exists a permutation π (of the set R in $X^{(\mathfrak{m})}$) such that X_π is not E-compact. We let E_ξ be the discrete union of E and X_π. Clearly, E_ξ is a perfect image of S. Since E_ξ is not E-compact, E_ξ is not E_η-compact for every $\eta<\xi$; i.e., $\mathfrak{K}(E_\xi)\not\subset\mathfrak{K}(E_\eta)$ for every $\eta<\xi$. But clearly $\mathfrak{K}(E_\eta)\subset\mathfrak{K}(E)\subset\mathfrak{K}(E_\xi)$ for every $\eta<\xi$. The theorem is shown.

If, in this theorem, we replace the condition $\mathfrak{K}(E_\eta)\not\supseteq\mathfrak{K}(E_\xi)$ by just $\mathfrak{K}(E_\eta) \neq \mathfrak{K}(E_\xi)$ (for every $\eta<\xi<\omega_\lambda$), then all the space E_ξ can be taken to be perfect images of the discrete union of two copies of $X^{(\mathfrak{m})}$.

It remains an open question whether Theorem 2.5 holds if "more than $\mathfrak{m}$" is replaced by "at least $2^{\mathfrak{m}}$".

§ 3. Application to spaces $\mathscr{N} \cup \mathfrak{R}$. Let $\mathscr{N}$ be the set of positive integers and let $\mathfrak{R}$ be a class of almost disjoint subsets of $\mathscr{N}$. By $\mathscr{N} \cup \mathfrak{R}$ we denote the space in which points of $\mathscr{N}$ are isolated and neighborhoods of a point $A\in\mathfrak{R}$ are of the form $\{A\} \cup (A\setminus S)$, where S is an arbitrary finite subset of A. This space was first considered in [M$_1$]; in [M$_1$] it was used to provide an example of a pseudo-compact non-countably compact space. In fact, it was observed in [M$_1$] that $\mathscr{N} \cup \mathfrak{R}$ is pseudo-compact if (in fact, if and only if) $\mathfrak{R}$ is maximal (with respect to almost disjointness). It is easy to construct maximal classes $\mathfrak{R}$ such that $\beta(\mathscr{N} \cup \mathfrak{R})$ is "large" (e.g., such that $\beta(\mathscr{N} \cup \mathfrak{R})\setminus(\mathscr{N} \cup \mathfrak{R})$ contains at least $2^{\aleph_0}$ points); the main result of this section is to show that there exists a maximal $\mathfrak{R}$ such that $\beta(\mathscr{N} \cup \mathfrak{R})$ is the one-point compactification.

In what follows, $\mathfrak{R}$ will always denote an almost-disjointed (but not necessarily maximal) class of infinite subsets of $\mathscr{N}$. We shall collect various observation on spaces of the form $\mathscr{N} \cup \mathfrak{R}$.

3.1. *The discrete union of two spaces of the form $\mathscr{N} \cup \mathfrak{R}$ is again of the form $\mathscr{N} \cup \mathfrak{R}$.*

3.2. *Let $\mathscr{N} \cup \mathfrak{R}$ be given and let α be a decomposition of $\mathfrak{R}$ into finite classes. The resulting quotient space is again of the form $\mathscr{N} \cup \mathfrak{R}$.*

Proof. The resulting quotient space is homeomorphic to $\mathscr{N} \cup \mathfrak{R}'$, where $\mathfrak{R}' = \bigcup \{\mathfrak{K}: \mathfrak{K}\in\alpha\}$.

3.2 will usually be applied in the following form

3.3. *Let $\mathscr{N} \cup \mathfrak{R}$ be given and let φ be a one-to-one map whose domain and counter-domain are disjoint subsets of $\mathfrak{R}$. The space obtained by the identification of each $A\in$ domain φ with $\varphi(A)$ is of the form $\mathscr{N} \cup \mathfrak{R}$.*

We have

3.4. *There are $2^{2^{\aleph_0}}$ of maximal classes $\mathfrak{R}$ (in a fixed set $\mathscr{N}$).*

Proof. Let us write $\mathscr{N} = \mathscr{N}_1 \cup \mathscr{N}_2$, where $\mathscr{N}_1$ and $\mathscr{N}_2$ are infinite disjoint. Let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be classes in $\mathscr{N}_1$ and $\mathscr{N}_2$, respectively, both maximal and of cardinality $2^{\aleph_0}$. Let φ be any one-to-one map of $\mathfrak{R}_1$ onto $\mathfrak{R}_2$ and let

$$\mathfrak{R}_\varphi = \{A \cup \varphi(A) : A\in\mathfrak{R}_1\} .$$

$\mathfrak{R}_\varphi$ is a maximal (almost-disjointed) class of subsets of $\mathscr{N}$; if $\varphi \neq \varphi'$, then $\mathfrak{R}_\varphi \neq \mathfrak{R}_{\varphi'}$.

Since a space $\mathscr{N} \cup \mathfrak{R}$ can be homeomorphic to at most $2^{\aleph_0}$ spaces of the form $\mathscr{N} \cup \mathfrak{R}'$, 3.4 yields

3.5. *There are $2^{2^{\aleph_0}}$ (non-homeomorphic) pseudocompact spaces of the form $\mathscr{N} \cup \mathfrak{R}$.*

3.6. *There is a maximal $\mathfrak{R}$ such that $\beta(\mathscr{N} \cup \mathfrak{R})\setminus(\mathscr{N} \cup \mathfrak{R})$ contains at least $2^{\aleph_0}$ points.*

Proof. Let $\mathfrak{R}'$ be an arbitrary maximal class in $\mathscr{N}$ with $\operatorname{card}\mathfrak{R}' = 2^{\aleph_0}$; for every $A\in\mathfrak{R}'$ take a maximal class $\mathfrak{R}_A$ in A and let $\mathfrak{R} = \bigcup \{\mathfrak{R}_A : A\in R'\}$. $\mathfrak{R}$ has the desired properties.

3.7. *There is an $\mathfrak{R}$ such that* $\operatorname{card}\mathfrak{R} = 2^{\aleph_0}$ *and for every continuous real function f on $\mathscr{N} \cup \mathfrak{R}$, $f|\mathfrak{R}$ is equivalent to a function of the 1st Baire class on the reals (i.e., $\mathfrak{R}$ can be mapped in a one-to-one fashion onto the reals so that the map makes $f|\mathfrak{R}$ a 1st Baire class function on the reals).*

Proof. This follows from the consideration of [M$_7$], Sec. 1. We consider the Nemytski space N discussed in the quoted paper; and, preserving the notations of that paper, we will treat $\mathscr{N}$ as the $N\setminus R$ and we let $R = \{T_p : p\in R\}$. Then $\mathscr{N} \cup \mathfrak{R}$ is homeomorphic to N, hence our conclusion follows from 1.2 in [M$_7$].

Of course, if a space $N \cup \mathfrak{R}$ has the property expressed in 3.7, then every uncountable set $E\subset\mathfrak{R}$ which is either a zero-set or a cozero-set in $N \cup \mathfrak{R}$ is, in fact, of cardinality $2^{\aleph_0}$.

3.8. *There is a maximal $\mathfrak{R}$ having the properties stated in 3.7.*

Proof. Let $\mathfrak{R}'$ be any class having the properties of 3.7; extend $\mathfrak{R}'$ to a maximal class $\mathfrak{R}''$. Since $\operatorname{card}\mathfrak{R}' = 2^{\aleph_0}$, we can find a one-to-one map φ of $\mathfrak{R}''\setminus\mathfrak{R}'$ into $\mathfrak{R}'$. Let $\mathscr{N} \cup \mathfrak{R}$ be the space obtained from $\mathscr{N} \cup \mathfrak{R}''$ by the identification of each $A\in\mathfrak{R}''\setminus\mathfrak{R}'$ with $\varphi(A)$ (see 3.3; the explicit definition of $\mathfrak{R}$ is: $\mathfrak{R} = [\mathfrak{R}'\setminus\varphi(R''\setminus\mathfrak{R}')]\cup$

$\cup \{A \cup \varphi(A)\colon A \in \mathfrak{R}''\setminus\mathfrak{R}'\})$. Pointwise, $\mathcal{N} \cup \mathfrak{R}$ is the same as $\mathcal{N} \cup \mathfrak{R}'$, and $\mathcal{N} \cup \mathfrak{R}'$ is a continuous image of $\mathcal{N} \cup \mathfrak{R}''$, hence $\mathfrak{R}$ has the desired properties.

(Remark. We have shown a slightly stronger statement:

3.8'. *For every* $\mathfrak{R}'$ *in* $\mathcal{N}$ *with* $\operatorname{card}\mathfrak{R}' = 2^{\aleph_0}$ *there exists a maximal* $\mathfrak{R}$ *in* $\mathcal{N}$ *such that* $\mathfrak{R} = \{B_A\colon A \in \mathfrak{R}'\}$ *where* $B_A \supset A$ *for every* $A \in \mathfrak{R}'$.)

3.9. *If* $\mathcal{N} \cup \mathfrak{R}$ *is pseudo-compact, then every infinite set* $E \subset R$ *which is a zero-set in* $N \cup R$ *is uncountable.*

Proof. This is an exercise that can be found in [GJ], Exercise 51, p. 79.

3.10. *Let* $X = \mathcal{N} \cup \mathfrak{R}$. *The following conditions on* $\mathfrak{R}$ *are equivalent*

(a) $\mathfrak{R}$ *is maximal;*

(b) $\mathcal{N} \cup \mathfrak{R}$ *is pseudo-compact;*

(c) *every infinite* $A \subset \mathcal{N}$ *has an accumulation point in* $\mathfrak{R}$;

(d) $\beta X\setminus\mathcal{N} = \overline{\mathfrak{R}}^{\beta X}$;

(e) *for every closed subset* A *of* X, $\overline{A}^{\beta X} \cap (\beta X\setminus X) \subset \overline{A \cap \mathfrak{R}}^{\beta X}$.

Proof. Exercise ((a)⇒(b) was given already in [M$_1$]).

We shall now prove

3.11. THEOREM. *There is a maximal class* $\mathfrak{R}$ *such that*

(a) *if* $E \subset \mathfrak{R}$ *is infinite, then* E *is a zero-set in* $\mathcal{N} \cup \mathfrak{R}$ *if and only if* $\mathfrak{R}\setminus E$ *is countable;*

and hence

(b) $\beta(\mathcal{N} \cup \mathfrak{R})$ *is the one-point compactification.*

Proof. Let $\mathfrak{R}'$ be a class in $\mathcal{N}'$ with the properties in 3.8. Let $\mathfrak{F}$ be the class of all real functions on $\mathfrak{R}'$ which can be continuously extended over $\mathcal{N}' \cup \mathfrak{R}'$. Apply Lemma 1.1 to $\mathfrak{F}$, let π be the resulting permutation of $\mathfrak{R}'$. Let X be the space obtained from the discrete union of two disjoint copies of $\mathcal{N}' \cup \mathfrak{R}'$; $\mathcal{N}' \cup \mathfrak{R}'$ and $\mathcal{N}'_1 \cup \mathfrak{R}'_1$, by the identification of each $A \in \mathfrak{R}'$ with the point $\pi(A)$ in $\mathfrak{R}'_1$. By 3.3, X can be written as $\mathcal{N} \cup \mathfrak{R}$, where $\mathcal{N} = \mathcal{N}' \cup \mathcal{N}'_1$. X, being a continuous image of a pseudo-compact space, is pseudo-compact (so $\mathfrak{R}$ is maximal). The "if" part of (a) is obvious; we shall show the "only if" part. Let $E \subset \mathfrak{R}$ be infinite and let $E = g^{-1}(0)$, where g is a continuous real function on X. Let $f = g|\mathfrak{R}$, f can be treated as a function on $\mathfrak{R}'$ and, in fact, we have that both f and $f \circ \pi$ belong to $\mathfrak{F}$. By 3.9, E is uncountable, hence, by 3.7 (see remark after the proof of 3.7), $\operatorname{card} E = 2^{\aleph_0}$. But, by Lemma 1.1, there is a y such that $\operatorname{card}(\mathfrak{R}'\setminus f^{-1}(y)) < 2^{\aleph_0}$. Clearly, $y = 0$, hence $\operatorname{card}(\mathfrak{R}\setminus E) = \operatorname{card}(\mathfrak{R}'\setminus f^{-1}(y)) < 2^{\aleph_0}$. But $\mathfrak{R}\setminus E$ is a cozero-set, hence, again by the remark after the proof of 3.7, $\operatorname{card}(\mathfrak{R}\setminus E) \leqslant \aleph_0$.

(b) follows from (a); $\mathfrak{R}$ does not contain infinite disjoint zero-sets of $N \cup \mathfrak{R}$; hence, by 3.10 (d), $\beta(\mathcal{N} \cup \mathfrak{R})\setminus(\mathcal{N} \cup \mathfrak{R})$ cannot have more than one point.

The theorem is shown.

3.11. COROLLARY. *There are* $2^{2^{\aleph_0}}$ *of (non-homeomorphic) spaces* $\mathcal{N} \cup \mathfrak{R}$ *such that* $\beta(\mathcal{N} \cup \mathfrak{R})$ *is the one-point compactification.*

Proof. Take two disjoint copies $\mathcal{N}_1 \cup \mathfrak{R}_1$ and $\mathcal{N}_2 \cup \mathfrak{R}_2$ of a space in Theorem 3.10, write $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$ and apply the proof of 3.4.

From Theorem 3.10 we shall derive

3.12. THEOREM. *There is a space* X *and a zero-set* $X_0 \subset X$ *such that*

(I) *if* $A, B \subset X_0$ *are zero-sets in* X_0 *and* $A \cup B$ *is a zero-set in* X, *then* A *or* B *is a zero-set in* X,

but it is not true that

(II) *every zero-set in* X_0 *is a zero-set in* X.

(Observe that (II) is equivalent to: every continuous real function on X_0 can be continuously extended over X.)

Proof. Take two disjoint copies $\mathcal{N}_1 \cup \mathfrak{R}_1$ and $\mathcal{N}_2 \cup \mathfrak{R}_2$ of a space with properties in Theorem 3.10. Let $\mathfrak{R}_1^* \subset \mathfrak{R}_1$ be countable; identify each point of $\mathfrak{R}_1\setminus\mathfrak{R}_1^*$ with a point from $\mathfrak{R}_2$ (by any one-to-one map of $\mathfrak{R}_1\setminus\mathfrak{R}_1^*$ onto $\mathfrak{R}_2$). Let X be the identification space, let $X_0 = \mathcal{N}_2 \cup \mathfrak{R}_1$ (X_0 can also be treated as $(\mathcal{N}_2 \cup \mathfrak{R}_2) \cup \mathfrak{R}_1^*$). (II) fails because $\mathfrak{R}_1^*$ is a zero-set in X_0 but not in X. (I) is more tedious; t can be verified by showing the following

(1) a closed set $A \subset X$ is a zero-set in X iff $A \cap \mathfrak{R}_1$ is finite or $\mathfrak{R}_1\setminus A$ is countable,

and

(2) a closed set $A \subset X_0$ is a zero-set in X_0 iff $A \cap \mathfrak{R}_2$ is finite or $\mathfrak{R}_2\setminus A$ is countable.

Theorem 3.12 yields a negative solution of a question posed by J. W. Green [G]. Green asked if the following condition (I') implies condition (II):

(I') *if* $\mathfrak{F}$ *is a* z*-ultrafilter on* X *with* $X_0 \in \mathfrak{F}$, *then* $\mathfrak{F}_0 = \{A\colon A$ *is a zero-set in* X_0 *and* $B \subset A$ *for some* $B \in \mathfrak{F}\}$ *is a* z*-ultrafilter on* X_0.

In fact, it can be shown that (I) implies (I') in pseudo-compact spaces (and the space X in Theorem 3.12 is pseudo-compact). But the relationship between (I), (I') and (II) in non-pseudo-compact spaces (say in $\mathcal{R}$-compact spaces) remains an open question.

3.13. *If* X_0 *is a zero-set in a pseudo-compact space* X, *then condition* (I) *implies* (I').

Proof. In a pseudo-compact space z-ultrafilters are closed under countable intersection and we will, in fact, show that (I) implies (I') in any space X, provided that F is closed under countable intersections. To show that F_0 is a z-ultrafilter we have to show that if C is a zero-set in X_0, then there is an $A \in \mathfrak{F}$ such that either $A \subset C$ or $A \cap C = \emptyset$. If C is a zero-set in X, then there is nothing to show. Assume that C is not a zero-set in X; let $C = f^{-1}(0)$, where f is a non-negative continuous function on X_0 and let

$$A_n = \left\{p \in X_0 \colon f(p) \geqslant \frac{1}{n}\right\}, \quad B_n = \left\{p \in X_0 \colon f(p) \leqslant \frac{1}{n}\right\}.$$

Our assumption on C implies that B_n is zero-set in X only for finitely many n (otherwise C would be zero-set in X); consequently, by (I), there is an n_0 such that A_n is a zero-set in X for every $n > n_0$. If $A_{n_1} \in \mathfrak{F}$ for some $n_1 > n_0$, then we are done; indeed, $A_{n_1} \cap C = \emptyset$. Assume therefore that $A_n \notin \mathfrak{F}$ for every $n > n_0$. Then, for every $n > n_0$, there is an $A'_n \in \mathfrak{F}$ with $A_n \cap A'_n = \emptyset$. We have $A = \bigcap_{n > n_0} A'_n \in \mathfrak{F}$ and $A \subset C$; and this ends the proof.

§ 4. Applications to rings of continuous functions. We shall consider, as usual, subrings of $C(X, \mathcal{R})$ that contain all constant functions and are closed under inversion and uniform convergence. For ease of speaking we shall call such rings *regular*. In addition, if a subring $\mathfrak{F} \subset C(X, \mathcal{R})$ separates points and closed subsets of X, we shall call $\mathfrak{F}$ an s-ring (in $C(X, \mathcal{R})$); $\mathfrak{F}$ will be called representable provided that every non-zero homomorphism $\varphi \colon \mathfrak{F} \to \mathcal{R}$ is representable by a point from X (i.e., there is a $p_0 \in X$ such that $\varphi(f) = f(p_0)$ for every $f \in \mathfrak{F}$). As a consequence of Lemma 1.1 we have the following

4.1. *The intersection of two regular representable s-subrings of $C(X, \mathcal{R})$ need not to be representable* (*even if it is a regular s-subring*); *in fact, this happens in $C(\mathcal{N}_{2^{\aleph_0}}, \mathcal{R})$.*

($\mathcal{N}_{2^{\aleph_0}}$ is the discrete space of cardinality $2^{\aleph_0}$.)

Proof. Let $\mathfrak{F}$ be a regular respresentable subring of $C(\mathcal{N}_{2^{\aleph_0}}, \mathcal{R})$ with $\operatorname{card}\mathfrak{F} = 2^{\aleph_0}$ (see remark below). Apply Lemma 1.1 to $\mathfrak{F}$, let π be the resulting permutation of $\mathcal{N}_{2^{\aleph_0}}$. Let $\mathfrak{F}_1 = \{f \circ \pi \colon f \in F\}$; $\mathfrak{F}_1$ is clearly a regular representable s-subring of $C(\mathcal{N}_{2^{\aleph_0}}, \mathcal{R})$. Both $\mathfrak{F}$ and $\mathfrak{F}_1$ contain characteristic functions of one-point sets, therefore, $\mathfrak{F} \cap \mathfrak{F}_1$ is a regular s-ring. But $\mathfrak{F} \cap \mathfrak{F}_1$ is not representable; indeed, for every $\mathfrak{F} \cap \mathfrak{F}_1$, there is a y such that $\operatorname{card}(\mathcal{N}_{2^{\aleph_0}} f^{-1}(y)) < 2^{\aleph_0}$; suffices to let $\varphi(f) = y$.

Remark. An example of a regular representable s-subring $\mathfrak{F}$ of $C(\mathcal{N}_{2^{\aleph_0}}, \mathcal{R})$ was first given by Isbell [1]. A simpler example was given in [HJ]; $N_{2^{\aleph_0}}$ is treated as the set of all points of the unit interval $[0, 1]$ and $\mathfrak{F}$ is the set of all functions of the 1st Baire class. Other examples can be provided with the aid of the following

4.2. *Let $\mathfrak{F}$ be a regular s-subring of $C(\mathcal{N}_{2^{\aleph_0}}, \mathcal{R})$. If $\mathfrak{F}$ contains a sequence $f_1, f_2, \ldots$ of functions that separate points of $\mathcal{N}_{2^{\aleph_0}}$, then $\mathfrak{F}$ is representable.*

Proof. The parametric map h corresponding to the class $\mathfrak{F}_0 = \{f_1, f_2, \ldots\}$ (see [M_5], p. 165) is a one-to-one map of $\mathcal{N}_{2^{\aleph_0}}$ into the space $\mathcal{R}^{\aleph_0}$. To put it differently, we can find a (completely regular Hausdorff) second countable topology T on $\mathcal{N}_{2^{\aleph_0}}$ such that each f_n is T-continuous. Let X^* be the $(\mathcal{N}_{2^{\aleph_0}}, T)$; the above implies that $\mathfrak{F} \cap C(X^*, \mathcal{R})$ is a regular s-subring of $C(X^*, \mathcal{R})$. Since X^* is Lindelöf we have, by Corollary 4.7 in [M_6], $\mathfrak{F} \cap C(X^*, \mathcal{R}) = C(X^*, \mathcal{R})$ thus $C(X^*, \mathcal{R}) \subset \mathfrak{F}$.

Since X^* is second countable, $C(X^*, \mathcal{R})$ is representable. Let $\mathfrak{F}^* = \{f \colon f_n \to f$ for some sequence $f_1, f_2, \ldots, f_n \in C(X^*, \mathcal{R})\}$, $f_n \to f$ stands here for pointwise convergence. Let χ_p denote the characteristic function of a point $p \in N_{2^{\aleph_0}}$; let $\mathfrak{F}_1$ be the smallest regular ring of $C(\mathcal{N}_{2^{\aleph_0}}, \mathcal{R})$ contained $C(X^*, \mathcal{R}) \cup \{\chi_p \colon p \in \mathcal{N}_{2^{\aleph_0}}\}$. Since X^* is first countable, $\chi_p \in F^*$ for every $p \in \mathcal{N}_{2^{\aleph_0}}$; thus $C(X^*, \mathcal{R}) \subset \mathfrak{F}_1 \subset \mathfrak{F}^*$. By a theorem of Mazur (quoted — with proof — in [CM], statement (A)), every homomorphism in $\varphi \colon \mathfrak{F}_1 \to \mathcal{R}$ preserves pointwise convergence; consequently, if φ restricted to $C(X^*, \mathcal{R})$ is representable by a point p_0, then the same p_0 represents φ on the entire $\mathfrak{F}_1$. It follows that $\mathfrak{F}_1$ is representable (recall that $C(X^*, \mathcal{R})$ is representable).

Finally, observe that $\mathfrak{F}$ being a regular s-subring of $C(\mathcal{N}_{2^{\aleph_0}}, \mathcal{R})$ implies that $\chi_p \in \mathfrak{F}$ for every $p \in \mathcal{N}_{2^{\aleph_0}}$; consequently $\mathfrak{F}_1 \subset \mathfrak{F} \subset C(\mathcal{N}_{2^{\aleph_0}}, \mathcal{R})$. But $\mathfrak{F}_1$ is a regular s-subring of $C(\mathcal{N}_{2^{\aleph_0}}, \mathcal{R})$; consequently, by Theorem 4 in [M_3], $\mathfrak{F}$ is also representable. This ends the proof.

4.2 implies, of course, the result of [HJ] that the ring of the 1st Baire class functions is representable. It enables us also to exhibit a still smaller representable regular s-subring of $C(\mathcal{N}_{2^{\aleph_0}}, \mathcal{R})$: once again we treat $\mathcal{N}_{2^{\aleph_0}}$ as the set of the points of the interval $[0, 1]$ and let $\mathfrak{F}$ be the smallest regular subring of $C(\mathcal{N}_{2^{\aleph_0}}, \mathcal{R})$ that contains the identity function and all the characteristic functions of points. Explicitly, this $\mathfrak{F}$ consists of all f on $[0, 1]$ such that there is a continuous g on $[0, 1]$ with $\{x \colon f(x) \neq g(x)\}$ being at most countable.

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STATE UNIVERSITY OF NEW YORK AT BUFFALO

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Pseudo-completeness in linear metrizable spaces

by

Aaron R. Todd (Brooklyn, N. Y.)

Abstract. J. C. Oxtoby has shown that the standard Baire category theorems follow from his definition of pseudo-complete spaces. Although a pseudo-complete metrizable space need not be topologically complete, pseudo-completeness implies completeness for a linear topology whose completion is stronger than a metrizable topology. Pseudo-completeness and completeness are equivalent for a linear metrizable topology. A complete linear topology stronger than a metrizable topology need not be pseudo-complete.

In a portion of his paper [2], Oxtoby nicely identifies by his pseudo-complete spaces, those common elements of several standard Baire category theorems which make them corollaries of his Proposition (5.1), *Any pseudo-complete space is a Baire space*, and his Theorem 6, *The Cartesian product of any family of pseudo-complete spaces is pseudo-complete.*

The object of this paper is to establish that, for a large class of linear topological spaces, pseudo-completeness implies completeness; indeed, for linear metrizable spaces these concepts are equivalent.

A topological space X is *quasi-regular* if and only if each non-empty open set contains the closure of a non-empty open set. A family $\mathscr{B}$ of non-empty open sets is a *pseudo-base* for X if and only if each non-empty open set contains an element of $\mathscr{B}$. A quasi-regular topological space X is *pseudo-complete* if and only if there is a sequence $(\mathscr{B}_n)$ of pseudo-bases for X such that if $U_n \in \mathscr{B}_n$ and $U_n \supset \bar{U}_{n+1}$ then $\bigcap_n U_n$ is non-empty.

It is easily seen that a pseudo-metrizable space X, which is complete in some pseudo-metric d, is pseudo-complete by considering the bases $\mathscr{B}_n$ of non-empty open sets of d-diameter less than $1/n$. That the converse is false may be seen by considering a subspace of the plane, $X = R\times(0, \infty) \cup Q\times\{0\}$, the union of the upper half plane and its set of boundary points with rational first coordinates: For each $\mathscr{B}_n$ use open disks of the plane which are contained in X and which have centers with rational first coordinates and radii less then $1/n$. If X is complete in some metric which induces its topology then X is a G_δ subset of the plane ([1], p. 96). This is not possible since Q is not a G_δ subset of R.

The following proposition characterizes a property required in the main theorem.