Distant bounded variation and products of derivatives

by

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Abstract. Let A denote the class of real-valued functions defined on $[0, 1]$ whose product with every derivative is a derivative. In [2] it is shown that every continuous function of bounded variation belongs to A and an example of a discontinuous member of A is presented. In the present note it is established that each member of A is a bounded derivative and can have an at most finite number of points at which it is of unbounded variation. A bounded derivative f is said to be of *distant bounded variation* (BVD) at x_0 if the integral of $s(t) = |t-x_0|$ with respect to the measure induced by the total variation of f exists and satisfies a Lipschitz condition at x_0. Then $f \in A$ if and only if $f \in \mathrm{BVD}$ at x for each x in $[0, 1]$.

Throughout the following, $|E|$ will denote the Lebesgue measure of E. For a function f and an interval I, $\int_I f(x)dx$ will denote its Lebesgue integral, $(D)\int_I f(x)dx$, its wide-sense Denjoy integral, $O(f, I)$, its oscillation and $W(f, I)$, its total variation. If F is of bounded variation on I, $\int_I g(t)dF(t)$ will denote the Lebesgue–Stieltjes integral of $g(t)$ with respect to $F(t)$ on I.

Our study of the class A begins with four elementary theorems.

THEOREM 1. *If $f(x)$ and $g(x)$ belong to A, then $f(x)g(x)$ and $f(x)+g(x)$ belong to A.*

Proof. This follows immediately from the definition of A.

THEOREM 2. *If $f(x)$ is in A, then $f(x)$ is a summable derivative.*

Proof. Since $f(x) \in A$, $1\cdot f(x)$ and $f(x)f(x)$ are derivatives. Then $f^2(x)$ is D-integrable and since $f^2(x) \geqslant 0$, $f^2(x)$ is summable [4, p. 242]. Since $|f(x)| \leqslant \max(1, f^2(x))$, $f(x)$ is summable.

THEOREM 3. *If $H_1'(x) = h(x)$ on $[a, b]$ and $H_2'(x) = h(x)$ on $[b, c]$, then $H'(x) = h(x)$ on $[a, c]$ where $H(x) = H_1(x)$ on $[a, b]$ and $H(x) = H_1(b)-H_2(b)+H_2(x)$ on $(b, c]$.*

Proof. By hypothesis,

$$\lim_{x\to b^-} H_1(x)-H_1(b)/x-b = h(b) = \lim_{x\to b^+} H_2(x)-H_2(b)/x-b.$$

The theorem follows by noting that $H(x)$ is continuous at b and differs from $H_2(x)$ by a constant on $[b, c]$.

THEOREM 4. *Every member of A is approximately continuous.*

Proof. W. Wilcosz [5, p. 151] showed that if $f(x)$ and $f^2(x)$ are derivatives, then $f(x)$ and $f^2(x)$ are approximately continuous. (The condition that $f(x)$ is bounded is not used in this part of his proof).

For future reference, we state the result which is proved in [2].

THEOREM 5. *If $F(x)$ is a continuous function of bounded variation on $[0, 1]$, then $F(x)$ belongs to A.*

By relaxing the requirement that $F(x)$ be of bounded variation on the entire interval, we obtain a generalization of Theorem 5 which leads to the concept of distant bounded variation.

THEOREM 6. *Let $F(x)$ be a bounded derivative on $[0, 1]$ such that $F(x)$ is of bounded variation on $[\delta, 1]$ for every $\delta>0$. Let $V(t) = -W(F, [t, 1])$. Suppose that $\lim_{\delta\to 0}\int_\delta^1 tdV(t) = K<\infty$ and that there exists an $M>0$ such that $\frac{1}{x}\int_0^x tdV(t)<M$ (where $\int_0^x tdV(t) = K-\int_x^1 tdV(t)$) for each $x \in (0, 1]$. Then $F(x)$ belongs to A.*

Proof. We first note that since $V(t)$ is increasing on $(0, 1]$, $K \geqslant \int_{x_1}^1 tdV(t) \geqslant \int_{x_2}^1 tdV(t) \geqslant 0$ whenever $1 \geqslant x_2 > x_1 > 0$. Consequently, $\int_0^x tdV(t) \geqslant 0$ for $x \in (0, 1]$.

Let $g(x)$ be a derivative. Since $F(x)$ is a derivative, $F(x)g(x)$ is a derivative if and only if $F(x)(g(x)-g(0))$ is a derivative and we may assume that $g(0) = 0$. Let $G(x) = (D)\int_0^x g(t)dt$.

Since $F(x)$ is a derivative and is of bounded variation on $[\delta, 1]$ for each $\delta>0$, it is continuous on $[\delta, 1]$. By Theorem 5, $F(x)g(x)$ is the derivative of its D-integral on $[\delta, 1]$.

To show that $F(x)g(x)$ is D-integrable on $[0, 1]$, it suffices to show that $\lim_{\delta\to 0}(D)\int_\delta^1 F(t)g(t)dt$ exists and is finite [4, p. 258]. By [4, Theorem (2.5), p. 246],

$$(D)\int_\delta^1 F(t)g(t)dt = F(1)G(1)-F(\delta)G(\delta)-\int_\delta^1 G(t)dF(t).$$

Since $F(x)$ is bounded and $\lim_{\delta\to 0} G(\delta) = G(0) = 0$, it suffices to show that $\lim_{\delta\to 0}\int_\delta^1 G(t)dF(t)$ exists and is finite. If $0<\alpha<\beta$, then

$$\left|\int_\alpha^\beta G(t)dF(t)\right| \leqslant \int_\alpha^\beta |G(t)|dV(t) = \int_\alpha^\beta \left|\frac{G(t)}{t}\right| tdV(t).$$

Since $\lim_{t\to 0}\left|\frac{G(t)}{t}\right| = g(0) = 0$ and $\lim_{\delta\to 0}\int_0^\delta tdV(t) = 0$, for each $\varepsilon>0$, there is a $\delta = \delta(\varepsilon) > 0$ such that

$$\left|\int_\alpha^\delta G(t)\,dF(t)\right| \leqslant \int_\alpha^\delta \left|\frac{G(t)}{t}\right| t\,dV(t) < \varepsilon \quad \text{for each } \alpha$$

such that $0<\alpha<\delta$. Hence, $\lim_{\delta\to 0}\int_\delta^1 G(t)\,dF(t)$ exists and is finite, $F(x)g(x)$ is D-integrable on $[0, 1]$, and

$$(D)\int_0^1 F(t)g(t)\,dt = F(1)G(1)-F(0)G(0)-\int_0^1 G(t)\,dF(t)\,.$$

To show that $\lim_{x\to 0} x^{-1}(D)\int_0^x F(t)g(t)\,dt = F(0)g(0) = 0$, we note that

$$|x^{-1}(D)\int_0^x F(t)g(t)\,dt| = |x^{-1}(F(x)G(x)-F(0)G(0)-\int_0^x G(t)\,dF(t))|$$

$$\leqslant \left|F(x)\frac{G(x)}{x}\right| + \left|x^{-1}\int_0^x \left|\frac{G(t)}{t}\right| t\,dV(t)\right|$$

$$\leqslant \left|F(x)\frac{G(x)}{x}\right| + \left|\left(\sup_{0<t\leqslant x}\left|\frac{G(t)}{t}\right|\right) x^{-1}\int_0^x t\,dV(t)\right|$$

$$\leqslant \left|F(x)\frac{G(x)}{x}\right| + \left|\left(\sup_{0<t\leqslant x}\left|\frac{G(t)}{t}\right|\right)\cdot M\right|.$$

Since $G'(0) = g(0) = 0$ and $F(x)$ is bounded on $[0, 1]$, both terms tend to 0 as $x\to 0$ and the proof is complete.

If $f(x)$ is a continuous function of bounded variation on $[0, 1]$, then $f(x)$ satisfies the hypotheses of Theorem 6, since if $W(f, [0, 1]) = M$, then $V(0) = -M$, $V(1) = 0$ and $V(t)$ is increasing on $[0, 1]$. Hence,

$$x^{-1}\int_0^x t\,dV(t) \leqslant x^{-1}x\int_0^x dV(t) \leqslant V(x)-V(0) \leqslant M\,.$$

Moreover, if $f(x)$ is the discontinuous member of A presented in [2], then $W(f, [a_n, b_n]) = 2$ and $V(t)$ is constant on $[b_n, a_{n-1}]$. If $x\in[a_N, b_N]$, then

$$x^{-1}\int_0^x t\,dV(t) \leqslant x^{-1}\sum_{k=N}^\infty \int_{a_k}^{b_k} b_k\,dV(t) \leqslant a_N^{-1}\sum_{k=N}^\infty 2b_k \to 0$$

by condition (iv) of the example. Therefore, Theorem 6 generalizes Theorem 5.

In the following definition, the condition $\frac{1}{x}\int_0^x t\,dV(t) < M$ is translated to an arbitrary point $x_0\in[0, 1]$.

DEFINITION. A bound derivative f is said to be of *distant bounded variation from the right* at x_0 ($f\in \mathrm{BVD}^+$ at x_0) if there exist x_1, $K_{x_0}^+$ and $M_{x_0}^+$ such that $x_1 > x_0$,

f is of bounded variation on $[x_0+\delta, x_1]$ for each δ such that $0<\delta<x_1-x_0$, and letting $V_{x_0}^+(t) = -W(f, [t, x_1])$ for $t\in(x_0, x_1]$,

$$\lim_{\delta\to 0^+}\int_{x_0+\delta}^{x_1}(t-x_0)\,dV_{x_0}^+(t) = K_{x_0}^+ \quad\text{and}\quad (x-x_0)^{-1}\int_{x_0}^x (t-x_0)\,dV_{x_0}^+(t) < M_{x_0}^+$$

for each $x\in(x_0, x_1]$

(where $\int_{x_0}^x (t-x_0)\,dV_{x_0}^+(t) = K_{x_0}^+ - \int_x^{x_1}(t-x_0)\,dV_{x_0}^+(t)$). We say that f is of *distant bounded variation from the left* at x_0 ($f\in\mathrm{BVD}^-$ at x_0) if $f(1-x)\in\mathrm{BVD}^+$ at $1-x_0$. Then f is of *distant bounded variation* at x_0 if $f\in\mathrm{BVD}^+$ and BVD^- at x_0.

THEOREM 7. *If a function f satisfies* (i) *f is a bounded derivative on* $[0, 1]$, (ii) *there exist at most finitely many points $0\leqslant x_1<\ldots<x_n\leqslant 1$ at which f is of unbounded variation and* (iii) *f is of distant bounded variation at x_i, $i = 1, \ldots, n$, then f belongs to A.*

Proof. If $g(x)$ is a derivative on $[0, 1]$, then $f(x)g(x)$ is a derivative on $[0, x_1]$, $[x_n, 1]$ and $[x_{i-1}, x_i]$, $i = 2, \ldots, n$ by Theorem 6. By Theorem 3, $f(x)g(x)$ is a derivative on $[0, 1]$ and f belongs to A.

Our next objective is to show that each of the hypotheses (i), (ii) and (iii) of Theorem 7 is a necessary condition for membership in A.

We first show that every member of A is bounded.

THEOREM 8. *If $f(x)$ is a unbounded derivative, there exists a derivative $g(x)$ such that $f(x)g(x)$ is not a derivative.*

Proof. If there is no such $g(x)$, then $f(x)$ is in A. Since $f(x)$ is unbounded, $f^2(x)$ is also an unbounded member of A by Theorem 1. Thus we may assume that $f(x)\geqslant 0$. Without loss of generality, $\overline{\lim}_{x\to 0^+} f(x) = +\infty$.

Let $E_n = \{x|\ f(x)>4^n\}$. By the Denjoy-Clarkson property [1], $|E_n|>0$ and we may choose a sequence $\{x_n\}_{n=1}^\infty$ decreasing to 0 such that for each n, x_n is a point of density of E_n. We choose a sequence of sufficiently small intervals $I_n = [a_n, b_n]$ satisfying

(i) $a_n<n_n<a_{n-1}<1 = a_0$,
(ii) x_n is the midpoint of I_n,
(iii) $x_n/a_n<2$,
(iv) if J is any subinterval of I_n such that $x_n\in J$, then $|J\cap E_n|/|J|>1-1/2^n$.

Let $I_n' = [a_n', b_n']$ be the interval of length $\frac{1}{2}|I_n|$ whose midpoint is x_n.

Denote by $k(x)$ the function whose value is 2^{-n} on I_n', 0 on $[0, 1]-\bigcup_n (a_n, b_n)$, and is linear on the intervals $[a_n, a_n']$ and $[b_n', b_n]$. Then $k(x)$ is a continuous function of bounded variation on $[0, 1]$ and, therefore, belongs to A. By Theorem 1, $h(x) = f(x)k(x)$ belongs to A and to obtain the desired contradiction, it suffices to construct a derivative $g(x)$ such that $h(x)g(x)$ is not a derivative. Noting that $h(x) = 0$ on each interval $[b_n, a_{n-1}]$ and that $h(x)>2^n$ for $x\in I_n'\cap E_n$, we are ready to proceed.

Let $\alpha_n = 2^{-n}\cdot x_n$. For $x\in I_n'$, let $g(x) = \alpha_n|I_n'|^{-1}$. Let $g(a_n) = g(b_n) = g(0) = 0$. Define $g(x)$ to be linear on the intervals $[a_n, a_n']$ and $[b_n', b_n]$. Since $|I_n'| = \frac{1}{2}|I_n|$ and since above I_n, the region bounded by the x-axis and the graph of $g(x)$ is a trapezoid, it is easily seen that $\int_{I_n} g(x)dx = \frac{3}{2}\alpha_n$. On (b_n, a_{n-1}), let $g(x)$ be any continuous function such that $g(x) \leqslant 0$ on (b_n, a_{n-1}), $\lim_{x\to b_n^+} g(x) = \lim_{x\to a_{n-1}^-} g(x) = 0$, and such that $\int_{b_n}^{a_{n-1}} g(x) = -\frac{3}{2}\alpha_n$. Since $\alpha_n = 2^{-n}x_n$, $\sum_{n=1}^{\infty}\frac{3}{2}\alpha_n$ is finite and $\int_0^1 g(x)dx$ exists and equals 0. To show that $g(x)$ is a derivative, we note that it is continuous at each point x in $(0, 1]$ and, therefore, we need only show that

$$\lim_{x\to 0} x^{-1}\int_0^x g(t)dt = g(0) = 0\,.$$

Let $x \in [a_n, a_{n-1})$. Then

$$|x^{-1}\int_0^x g(t)dt| = |x^{-1}\int_{a_n}^x g(t)dt| \leqslant x^{-1}\cdot\tfrac{3}{2}\alpha_n \leqslant a_n^{-1}\cdot\tfrac{3}{2}\alpha_n\,.$$

Since $\alpha_n = x_n 2^{-n}$ and since $a_n^{-1}x_n < 2$ by condition (iii),

$$|x^{-1}\int_0^x g(t)dt| \leqslant 3\cdot 2^{-n} \quad \text{for} \quad x \in [a_n, a_{n-1})\,.$$

Hence, $g(x)$ is a derivative on $[0, 1]$.

To show that $h(x)g(x)$ is not a derivative, we note that $h(x)g(x) \geqslant 0$ for all x in $[0, 1]$ and that $h(x)g(x) > 2^n\alpha_n|I_n|^{-1}$ for x in $I_n' \cap E_n$. If $\int_0^1 h(x)g(x)dx$ does not exist, there is nothing to prove. Otherwise, let $J_n = [a_n', x_n]$. Then,

$$x_n^{-1}\int_0^{x_n} h(t)g(t)dt \geqslant x_n^{-1}\int_{J_n\cap E_n} h(t)g(t)dt \geqslant x_n^{-1}\cdot 2^n\alpha_n|I_n'|^{-1}|J_n \cap E_n|$$
$$= \tfrac{1}{2}|J_n|^{-1}|J_n \cap E_n| \geqslant \tfrac{1}{2}(1-2^{-n}) \geqslant \tfrac{1}{4}\,,$$

since $\alpha_n = 2^{-n}x_n$, $|J_n| = \frac{1}{2}|I_n'|$, and $|J_n|^{-1}|J_n \cap E_n| > 1-2^{-n}$ by condition (iv). Since $h(0)g(0) = 0$, $h(x)g(x)$ cannot be a derivative.

To show that a member of A can have only finitely many points in every neighborhood of which it is of unbounded variation, we need the following lemma.

LEMMA 1. *Let $f(x)$ be a derivative on $[0, 1]$ such that $0 \leqslant f(x) \leqslant 1$. Let $I = [a, b]$ be a subinterval of $[0, 1]$ such that $O(f, I) > \eta > 0$ (or $W(f, I) > \eta > 0$). Then for each pair of positive numbers $\omega > 0$ and $\varepsilon > 0$, there exists a piecewise linear continuous function $g(x)$ on I such that*

(i) $g(a) = g(b) = 0$,

(ii) $\int_I g(x)dx = 0$,

(iii) $O(\int_a^x g(t)dt, I) < \omega$,

(iv) $\int_I f(x)g(x)dx > \omega\eta - \varepsilon$.

Proof. We shall first prove the lemma for the case $O(f, I) > \eta$ and then extend it to the case $W(f, I) > \eta$. In the proof we shall write $g(x)$ for $g_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(x)$ and show that for sufficiently small choices of ε_1, ε_2 and ε_3, $g(x)$ satisfies the conditions of the lemma.

Since $f(x)$ has the Denjoy-Clarkson property and since $O(f, I) > \eta$, there exist positive numbers α and β such that $\beta - \alpha > \eta$ and the sets $E_1 = \{x \in I|\ f(x) > \beta\}$ and $E_2 = \{x \in I|\ f(x) < \alpha\}$ are both of positive measure. Hence, for any $\varepsilon_1 > 0$, we may choose intervals $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ in (a, b) such that $\mathrm{dist}(I_1, I_2) > 0$, $|I_1| = |I_2|$, and $|E_j \cap I_j|/|I_j| > 1-\varepsilon_1$ for $j = 1, 2$.

Given $\varepsilon_2 > 0$, let $g(x) = \omega(1-\varepsilon_2)|I_1|^{-1}$ on I_1 and $g(x) = -\omega(1-\varepsilon_2)|I_2|^{-1}$ on I_2. Then

(1) $$\int_{I_1} g(x)dx = \omega(1-\varepsilon_2) \quad \text{and} \quad \int_{I_2} g(x)dx = -\omega(1-\varepsilon_2)\,.$$

Since $f(x) \geqslant 0$ on $[a, b]$,

(2) $$\int_{I_1} f(x)g(x)dx \geqslant \int_{E_1\cap I_1} f(x)g(x)dx \geqslant \beta\omega(1-\varepsilon_2)|I_1|^{-1}|E_1 \cap I_1|$$
$$\geqslant \beta\omega(1-\varepsilon_2)(1-\varepsilon_1)\,.$$

On $E_2 \cap I_2$, $f(x)g(x) > \alpha\cdot g(x)$ and on $I_2 - E_2$, $f(x)g(x) > g(x)$ since $0 \leqslant f(x) < 1$ on I, $g(x) < 0$ on I_2, and $f(x) < \alpha$ on E_2. Therefore,

(3) $$\int_{I_2} f(x)g(x)dx = \int_{E_2\cap I_2} f(x)g(x)dx + \int_{I_2-E_2} f(x)g(x)dx$$
$$> -\alpha\omega(1-\varepsilon_2)|I_2|^{-1}|E_2 \cap I_2| - \omega(1-\varepsilon_2)|I_2|^{-1}|I_2 - E_2|$$
$$> -\alpha\omega(1-\varepsilon_2)(1-\varepsilon_1) - \omega(1-\varepsilon_2)\varepsilon_1\ (^1)\,,$$

the last inequality following from $0 < \alpha < 1$, $|I_2|^{-1}|I_2 - E_2| < \varepsilon_1$, and $|I_2|^{-1}\cdot|E_2 \cap I_2| + |I_2|^{-1}\cdot|I_2 - E_2| = 1$. Combining inequalities (2) and (3) yields

(4) $$\int_{I_1\cup I_2} f(x)g(x)dx > (\beta-\alpha)\omega(1-\varepsilon_2)(1-\varepsilon_1) - \omega(1-\varepsilon_2)\varepsilon_1$$
$$> \eta\omega(1-\varepsilon_2)(1-\varepsilon_1) - \omega(1-\varepsilon_2)\varepsilon_1\,.$$

Without loss of generality, $a < a_1 < b_1 < a_2 < b_2 < b$. Let $\varepsilon_3 > 0$ be small enough so that $a < a_1 - \varepsilon_3 < b_1 + \varepsilon_3 < a_2 - \varepsilon_3 < b_2 + \varepsilon_3 < b$. Let $K_1 = [a_1 - \varepsilon_3, a_1]$, $K_2 = [b_1, b_1 + \varepsilon_3]$, $K_3 = [a_2 - \varepsilon_3, a_2]$ and $K_4 = [b_2, b_2 + \varepsilon_3]$. Now set $g(x) = 0$ on $[a, a_1 - \varepsilon_3]$, $[b_1 + \varepsilon_3, a_2 - \varepsilon_3]$ and $[b_2 + \varepsilon_3, b]$ and define $g(x)$ to be linear on K_j,

(1) Since $|I_2|^{-1}|E_2 \cap I_2| + |I_2|^{-1}|E_2 - I_2| = 1$,

$$|I_2|^{-1}|E_2 \cap I_2| + |I_2|^{-1}|E_2 - I_2| = (1-\varepsilon_1) + \varepsilon_1.$$

Since $0 < \alpha < 1$ and $|I_2|^{-1}|E_2 \cap I_2| > 1-\varepsilon_1$,

$$\alpha|I_2|^{-1}|E_2 \cap I_2| + |I_2|^{-1}|E_2 - I_2| < \alpha(1-\varepsilon_1) + \varepsilon_1\,.$$

Note that multiplying the first term on each side of this equality by α reduces the left side more than the right.

$j = 1, 2, 3, 4$. Then $g(x)$ is a piecewise linear, continuous function on I and clearly satisfies conditions (i) and (ii) of the lemma. Moreover,

(5) $$\Big|\int_{K_j} g(x)\,dx\Big| = \tfrac{1}{2}\omega(1-\varepsilon_2)|I_1|^{-1}\varepsilon_3 \quad (j = 1, 2, 3, 4)$$

and

(6) $$\Big|\int_{K_j} f(x)g(x)\,dx\Big| \leqslant \tfrac{1}{2}\omega(1-\varepsilon_2)|I_1|^{-1}\varepsilon_3 \quad (j = 1, 2, 3, 4)$$

since $0 \leqslant f(x) \leqslant 1$. By (4) we may choose ε_1 and ε_2 small enough that

(7) $$\int_{I_1\cup I_2} f(x)g(x)\,dx > \eta\omega - \tfrac{1}{2}\varepsilon.$$

Choosing ε_3 so small that

(8) $$2\omega(1-\varepsilon_2)|I_1|^{-1}\varepsilon_3 < \min(\varepsilon_2\omega, \tfrac{1}{2}\varepsilon)$$

and noting that $O(\int_a^x g(t)\,dt, I) = \int_a^{b_1+\varepsilon_3} g(t)\,dt$, it follows from (1), (5) and (8), that $O(\int_a^x g(t)\,dt\,dt, I) < \omega$. From (6), (7) and (8), it follows that $\int_I f(x)g(x)\,dx > \eta\omega - \varepsilon$ and the proof for the case $O(f, I) > \eta$ is complete.

If $\varepsilon > 0$ and $\omega > 0$ are given and $W(f, I) > \eta$, there exist non-overlapping closed intervals $I_k \subset I$, $k = 1, \ldots, s$, such that if $O(f, I_k) = \eta_k$, then $\sum_{k=1}^{s} \eta_k > \eta$. On I_k let $g(x)$ be a piecewise linear continuous function satisfying (i), (ii) and (iii) of the lemma and $\int_{I_k} f(x)g(x)\,dx > \eta_k\omega - \varepsilon/s$. Let $g(x) = 0$ on $I - \bigcup_{k=1}^{s} I_k$. Then $g(x)$ satisfies (i), (ii) and (iii) on I and $\int_I f(x)g(x)\,dx > \sum_{k=1}^{s} \eta_k\omega - \varepsilon/s > \eta\omega - \varepsilon$. This completes the proof.

THEOREM 9. *If $f(x)$ belongs to A, the set of points in every neighborhood of which $f(x)$ is of unbounded variation is finite.*

Proof. By Theorem 8, we may assume that $f(x)$ is bounded and, without loss of generality, that $0 \leqslant f(x) < 1$. Assuming the theorem false, we may select a strictly decreasing (or increasing) sequence $\{x_n\}_1^\infty$ of points, in every neighborhood of which $f(x)$ is of unbounded variation. We assume, without loss, that $\lim_{n\to\infty} x_n = 0$. Let $I_n = [a_n, b_n]$ be a sequence of intervals such that $a_n < x_n < b_n < a_{n-1}$ for $n = 1, 2, 3, \ldots$ Let $\omega_n = 2^{-n}\cdot a_n$. Since $f(x)$ is of unbounded variation in I_n, we may choose pairwise disjoint, closed intervals $I_{n1}, \ldots, I_{nN_n}$ such that if $\eta_{nk} = O(f, I_{nk})$, then $\omega_n \cdot \sum_{k=1}^{N_n} \eta_{nk} > 2$. On $I_{nk} = [a_{nk}, b_{nk}]$, let $g(x)$ be a piecewise linear, continuous function satisfying conditions (i), (ii) and (iii) of Lemma 1 (where $I = I_{nk}$ and $\omega = \omega_n$) such that $\int_{I_{nk}} f(x)g(x)\,dx > \frac{1}{2}\omega_n\eta_{nk}$. Let $g(x) = 0$ for x in $[0, 1] - \bigcup_{n,k} I_{nk}$. Then for each n,

$$\int_{I_n} f(x)g(x)\,dx > \tfrac{1}{2}\omega_n \cdot \sum_{k=1}^{N_n} \eta_{nk} > 1\,.$$

Since $f(x)g(x) = 0$ for x not in $\bigcup_{nk} I_{nk}$, it follows that $f(x)g(x)$ is not D-integrable on $[0, 1]$ and cannot be a derivative.

Since

$$\int_{I_{nk}} g(x)\,dx = 0\,, \qquad \int_{[0,1]-\bigcup_{n,k} I_{nk}} g(x)\,dx = 0\,,$$

and

$$O\Big(\int_{a_{nk}}^{x} g(t)\,dt, I_{nk}\Big) < \omega_n = 2^{-n}a_n\,,$$

it follows from [4, Theorem (5,1), p. 257], that $g(x)$ is D-integrable on $[0, 1]$. Because $g(x)$ is continuous on $(0, 1]$, we need only show that

$$\lim_{x\to 0} x^{-1}(D)\int_0^x g(t)\,dt = g(0) = 0$$

in order to establish that $g(x)$ is a derivative. If $x \in [a_n, a_{n-1})$, then

$$\Big|x^{-1}(D)\int_0^x g(t)\,dt\Big| = \Big|x^{-1}(D)\int_{a_n}^x g(t)\,dt\Big| \leqslant x^{-1}\omega_n = 2^{-n}x^{-1}a_n \leqslant 2^{-n}.$$

This completes the proof of Theorem 9.

COROLLARY. *If $f(x)$ belongs to A, then $f(x)$ has at most finitely many discontinuities.*

Proof. Members of A, being themselves derivatives, can have discontinuities only of the second kind.

It was shown in Theorem 8, that if $f(x)$ is a non-negative unbounded derivative, then there exists a summable derivative $g(x)$ such that $f(x)g(x)$ is not a derivative. To show that the derivative constructed in Theorem 8 is not in general summable, we give the following example.

EXAMPLE. There exists a continuous function $f(x)$ such that f is of unbounded variation at $a_n = 2^{-n}$ for $n = 1, 2, 3, \ldots$, f is of distant bounded variation at a_n, $n = 1, 2, 3, \ldots$, and $f(x)g(x)$ is a derivative for each summable derivative $g(x)$.

Construction. Let $J_n = [2^{-n}, 2^{-n}+4^{-n}]$. Let M_n be an integer such that $1 \geqslant 2^{-n+1}M_n \geqslant 1/n$. Partition J_n into M_n abutting closed intervals $J_{n1}, \ldots, J_{nM_n}$. On J_{nk} let $p(x)$ denote the function whose graph is an isosceles triangle of height 2^{-n} and whose base is J_{nk}. Set $p(x) = 0$ on $[0, 1] - \bigcup_n J_n$. Then $1 \geqslant 2^{-n+1}M_n = W(p, I_n) > 1/n$ and p is of unbounded variation at 0, $0 \leqslant p(x) \leqslant x$ for each x in $[0, 1]$ and

$$2^N \sum_{k=N}^{\infty} (2^{-k}+4^{-k})\,W(p, J_n) < 4 \qquad \text{for each } N\,.$$

Since for $x \in J_N$,

$$\Big|x^{-1}\int_0^x t\,dV(t)\Big| < 2^N \sum_{k=N}^{\infty} (2^{-k}+4^{-k})\,W(p, J_n)\,, \qquad p \in \mathrm{BVD}^+ \text{ at } 0\,.$$

For $t \in I_n = [2^{-n}, 2^{-n+1}]$, let $p_n(t) = 2^{-n}p(2^{-n}x+2^{-n})$ where $t = 2^{-n}x+2^{-n}$, $x \in [0, 1]$. Then the graphs of p and p_n are similar figures and p_n is of unbounded variation at $a_n = 2^{-n}$, $0 \leqslant p_n(t) - p_n(a_n) \leqslant 1 \cdot |t - a_n|$, $t \in I_n$, and $p_n \in \mathrm{BVD}^+$ at a_n. Since $p(x) = 0$ on $[\frac{3}{4}, 1]$, $p_{n+1}(t) = 0$ on $[a_n - 3 \cdot 2^{-n-3}, a_n]$. Letting $f(x) = p_n(x)$ for $x \in I_n$ and $f(0) = 0$, we have that f is continuous on $[0, 1]$, f is of unbounded variation at a_n and $f \in \mathrm{BVD}$ at a_n for all n. Moreover, f satisfies a Lipschitz condition at each point of $[0, 1]$. (The Lipschitz constant may be chosen to be 1 at each a_n and at $x = 0$.)

Since f is continuous, f is a summable derivative and, by a result of Iosifescu [3], if g is a summable derivative, $f(x) g(x)$ is a derivative since f satisfies a Lipschitz condition at each point of $[0, 1]$. Q.E.D.

To show that if a member of A is of unbounded variation in every neighborhood of a point x_0 (without loss, $x_0 = 0$), then it must be of distant bounded variation at x_0, the following lemma is needed.

LEMMA 2. *Let $f(x)$ be a bounded derivative on $[0, 1]$ such that $f(x)$ is of bounded variation on $[\delta, 1]$ for each $\delta > 0$. Let $V(t) = -W(f, [t, 1])$ and suppose that there exist numbers $0 < a < b$ and $M > 0$ such that $\int_a^b t\,dV(t) > Mb$. Then there exists a piecewise linear continuous function $g(x)$ on $[a, b]$ such that $g(a) = g(b) = 0$,*

$$\left|\frac{1}{x}\int_a^x g(t)\,dt\right| < \frac{1}{M} \quad \textit{for each } x \in [a, b],$$

$$\int_a^b g(t)\,dt = 0 \quad \textit{and} \quad \int_a^b f(x) g(x)\,dx > \frac{b}{2}.$$

Proof. Since $\int_a^b t\,dV(t) > Mb$, there is a partition, $a = a_0 < a_1 < \ldots < a_r = b$ such that

$$\sum_{k=1}^{r} a_{k-1}(V(a_k) - V(a_{k-1})) = \sum_{k=1}^{r} a_{k-1} W(f, I_k) > Mb, \quad \text{where} \quad I_k = [a_{k-1}, a_k].$$

Let $\omega_k = a_{k-1}/M$. By Lemma 1 there exists a piecewise linear continuous function $g(x)$ on I_k such that $g(a_{k-1}) = g(a_k) = 0$

$$\int_{I_k} g(t)\,dt = 0, \qquad O\Big(\int_{a_{k-1}}^{x} g(t)\,dt, I_k\Big) < \omega_k$$

and

$$\int_{I_k} f(t) g(t)\,dt > W(f, I_k)\frac{\omega_k}{2}.$$

Then $\int_I g(t)\,dt = 0$ and if $x \in I_k$,

$$\left|\frac{1}{x}\int_a^x g(t)\,dt\right| = \frac{1}{x}\left|\int_{a_{k-1}}^{x} g(t)\,dt\right| \leqslant \frac{1}{a_{k-1}}\left|\int_{a_{k-1}}^{x} g(t)\,dt\right| < \frac{1}{a_{k-1}}\,\omega_k = \frac{1}{M}.$$

Moreover

$$\int_a^b f(t) g(t)\,dt = \sum_{k=1}^{r}\int_{I_k} f(t) g(t)\,dt > \sum_{k=1}^{r} W(f, I_k)\frac{\omega_k}{2}$$

$$= \frac{1}{2M}\sum_{k=1}^{r} a_k W(f, I_k) > \frac{1}{2M} Mb = \frac{b}{2}$$

and the proof of Lemma 2 is complete.

THEOREM 10. *If $f(x)$ belongs to A, then $f(x)$ is of distant bounded variation at each point $x_0 \in [0, 1]$.*

Proof. By Theorems 8 and 9, we may assume that $f(x)$ is a bounded derivative and that there exist at most finitely many points x_i, $i = 1, \ldots, m$, at which $f(x)$ is of unbounded variation. We have already noted that if $x_0 \neq x_i$, $i = 1, \ldots, m$, then $f \in \mathrm{BVD}$ at x_0. Suppose that $f \notin \mathrm{BVD}$ at x_k for some k, $1 \leqslant k \leqslant m$. Without loss of generality $f \notin \mathrm{BVD}^+$ at 0.

Then for each $\varepsilon > 0$ and $M > 0$, there exists a number b such that $\int_0^b t\,dV(t) > Mb$ and $0 < b < \varepsilon$. Hence there exists an a such that $0 < a < b$ and $\int_a^b t\,dV(t) > Mb$. (We note that a and b can be so chosen in case $\lim_{\delta \to 0}\int_\delta^1 t\,dV(t) = \infty$). Thus we may choose a sequence of intervals $I_n = [a_n, b_n]$ such that $a_n < b_n < a_{n-1}$ for all n and $\int_{I_n} t\,dV(t) > 2^n b_n$ for all n.

By Lemma 2 there exists a piecewise linear continuous function $g(x)$ on I_n such that $g(a_n) = g(b_n) = 0$,

$$\int_{I_n} g(t)\,dt = 0, \quad \left|\frac{1}{x}\int_{a_n}^{x} g(t)\,dt\right| < 2^{-n} \quad \text{for each } x \in I_n$$

and

$$\int_{I_n} f(t) g(t)\,dt > \frac{b_n}{2}.$$

Let $g(x) = 0$ for $x \in [0, 1] - \bigcup_n I_n$. The D-integrability of $g(x)$ follows from [4, Theorem (5.1), p. 257] and since

$$\left|\frac{1}{x}\int_0^x g(t)\,dt\right| = \left|\frac{1}{x}\int_{a_n}^x g(t)\,dt\right| < 2^{-n},$$

it follows that

$$\lim_{x\to 0}\frac{1}{x}(D)\int_0^x g(t)\,dt = g(0) = 0\,.$$

If $f(x)g(x)$ is not D-integrable, there is nothing to prove. If it is, then

$$\frac{1}{b_n}(D)\int_0^{b_n} f(x)g(x)\,dx \geqslant \frac{1}{b_n}\int_{a_n}^{b_n} f(x)g(x)\,dx \geqslant \frac{1}{2}\,.$$

Since $f(0)g(0) = 0$, $f(x)g(x)$ cannot be a derivative and the proof of Theorem 10 is complete.

THEOREM 11. *A function $f(x)$ belongs to A if and only if it is of distant bounded variation at each point x of* $[0, 1]$.

Proof. Necessity is shown in Theorem 10. To see that this condition is sufficient, we note that the definition of distant bounded variation entails that $f(x)$ be a bounded derivative. If there were infinitely many points at which $f(x)$ is of unbounded variation, they would have a limit point x_0 in $[0, 1]$. Then $f(x)$ would be of unbounded variation in every interval of the form $[x_0+\delta_1, x_0+\delta_2]$ (or $[x_0-\delta_2, x_0-\delta_1]$) for all $\delta_2>0$ and all sufficiently small $\delta_1>0$. Consequently, $f\notin$ BVD at x_0. Hence, there are at most finitely many such points and f is of distant bounded variation at each of them. Thus sufficiency follows from Theorem 7 and the proof of Theorem 11 is complete.

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Baire category from an abstract viewpoint *

by

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Abstract. The object of Point Set Theory is the investigation of methods of classifying point sets, their common properties, and their interrelationships. Included in the domain of this subject are Baire category, Lebesgue measure, Hausdorff measure, dimension, and sets of uniqueness for trigonometric series. In this paper we present a general framework for these investigations.

Introduction. In 1905 H. Lebesgue proved some basic theorems concerning sets which have the Baire property and, in particular, he proved the fundamental theorem that a linear set of the second category is everywhere of the second category in some interval (see [10], pp. 185–186). This theorem was undoubtedly known to R. Baire and he had stated earlier the analogous theorem for sets of the second category in the space of all infinite sequences of natural numbers (see [1], p. 948).

S. Banach [2] generalized the fundamental theorem to arbitrary metric spaces in 1930 and subsequently to topological spaces (see [9]). A further extension of this theorem was obtained in [13] (see Theorem 2 below). This new generalization forms the basis of an abstract theory of Baire category, an outline of which is presented in this paper. One of the main consequences of this abstract point of view is the unification of certain analogies which have been observed between properties of Lebesgue measurable sets and sets which have the Baire property (see [5], pp. 19–22, [7], [17], [20], and [22] concerning these analogies).

1. $\mathfrak{K}$-families. Let X be a (nonempty) set. Members of any family $\mathscr{A}$ of subsets of x will be called *$\mathscr{A}$-sets.*

DEFINITION 1. A family $\mathscr{C}$ of subsets of X is called a *$\mathfrak{K}$-family* if the following axioms are satisfied.

1. For each point $x\in X$ there is a $\mathscr{C}$-set containing x; i.e. $X = \bigcup\mathscr{C}$.

2. Let A be a $\mathscr{C}$-set and let $\mathscr{D}$ be a nonempty family of disjoint $\mathscr{C}$-sets which has power less than the power of $\mathscr{C}$.

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