$P$-embedding and product spaces

by

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Abstract. Let $A$ be a subset of a topological space $X$ and $Y$ a compact Hausdorff space with weight $m$, where $m$ is an infinite cardinal number. Our main theorem asserts that $A$ is $P^m$-embedded in $X$ if and only if $A \times Y$ is $C^*$-embedded in $X \times Y$. This theorem settles all the questions posed by R. A. Ali and L. I. Senott as well as gives an analysis of the theorems obtained recently by M. E. Rudin and by M. Starbird.

§ 1. Throughout this paper by a space we shall mean a topological space and by $m$ an infinite cardinal number.

A subspace $A$ of a space $X$ is said to be $P^m$-embedded in $X$ if every locally finite cozero-set cover of $A$ with cardinality $\leq m$ has a refinement which can be extended to a locally finite cozero-set cover of $X$. In case $A$ is $P^m$-embedded in $X$ for every $m$, $A$ is said to be $P$-embedded in $X$ (H. L. Shapiro [11]). For the case $m = \aleph_0$, $P^\infty$-embedding coincides with $C$-embedding in the usual sense (T. E. Gantzer [3]).

Our main concern in this paper is to describe $P^m$ or $P$-embedding in terms of $C^*$-embedding in product spaces.

As for paracompactness and normality in products the following theorems are obtained by K. Morita [5, Theorems 2.2, 2.4 and 2.7, and 6, Theorem 1.3].

**Theorem 1.1.** For a space $X$ the following statements are equivalent.

(a) $X$ is $m$-paracompact and normal.
(b) $X \times I$ is normal for every compact Hausdorff space $I$ of weight $\leq m$.
(c) $X \times D$ is normal.
(d) $X \times D$ is normal.

Here $I$ denotes the unit interval $[0, 1]$ and $D$ the discrete space consisting of exactly two points 0 and 1.

**Theorem 1.2.** Let $X$ be a completely regular Hausdorff space and $Y$ a compact Hausdorff space containing $X$ as its subspace. Then $X$ is paracompact if $X \times Y$ is normal.

The following theorems are motivated by the above theorems.

**Theorem 1.3.** For a subspace $A$ of a space $X$ the following statements are equivalent.

(a) $A$ is $P^m$-embedded in $X$.
(b) $A \times Y$ is $C^*$-embedded in $X \times Y$ for every compact Hausdorff space $Y$ of weight $\leq m$. 
(c) $A \times I^n$ is $C^*$-embedded in $X \times I^n$.
(d) $A \times D^n$ is $C^*$-embedded in $X \times D^n$.

Theorem 1.4. Let $A$ be a subspace of a space $X$ and $Y$ a compact Hausdorff space containing $A$ as its subspace. Then $A$ is $P$-embedded in $X$ iff $A \times Y$ is $C^*$-embedded in $X \times Y$.

As for $P^*$- or $P$-embedding in product spaces, R. A. Alb and L. I. Sennott [1] have given several interesting results. The essential parts of their results follow readily from our characterizations above. In particular, all of the conjectures or questions which remained unsettled in their paper [1] are proved or solved by our Theorems 1.3 and 1.4.

Furthermore we can establish a more precise result on $P^*$-embedding.

Theorem 1.5. Let $A$ be a subspace of a space $X$ and $Y$ a compact Hausdorff space of weight $m$. Then $A$ is $P^*$-embedded in $X$ iff $A \times Y$ is $C^*$-embedded in $X \times Y$.

As was proved essentially by C. H. Dowker [2], a space $X$ is collectionwise (resp. $m$-collectionwise) normal iff every closed subset of $X$ is $P$-(resp. $P^*$)-embedded in $X$ (for a direct proof, see Theorem 3.3 below). Accordingly as a direct consequence of our Theorem 1.5 we have the following result which has recently been proved by M. E. Rudin [10].

Theorem 1.6. If the product space $X \times Y$ of a space $X$ with a compact Hausdorff space $Y$ of weight $m$ is normal, then $X$ is $m$-collectionwise normal.

§ 2. Before proving our theorems we shall need some preliminary results. The following is given in [9, Lemma 2.1].

Lemma 2.1. A subspace $A$ of a space $X$ is $C^*$-embedded in $X$ iff every finite cozero-set cover of $A$ has a refinement which can be extended to a normal open cover of $X$.

As was proved in [5, Corollary 1.3] an open cover $\{G_x \mid x \in \Omega\}$ of a normal space $X$ is normal iff there exists a normal open cover $\{U_x\}$ of $X$ such that each set $U_x$ is contained in a union of a finite number of sets of $\{G_x\}$. The following theorem may be compared with this result.

Theorem 2.2. Let $A$ be a $C^*$-embedded subspace of a space $X$. Then $A$ is $P^*$-embedded in $X$ iff for every locally finite cozero-set cover $\{H_x \mid x \in \Omega\}$ of $A$ with $\text{card} \Omega \leq m$ there exists a locally finite cozero-set cover $\{U_x\}$ of $X$ such that each set $U_x \cap A$ is contained in a union of a finite number of sets of $\{H_x\}$.

Proof. We have only to prove the "if" part. Let $\{G_x \mid x \in \Omega\}$ be a locally finite cozero-set cover of $A$ with $\text{card} \Omega \leq m$. Then there exist a cozero-set cover $\{H_x \mid x \in \Omega\}$ of $A$ and a continuous map $f_x: A \to I$ for $H_x \cap A$ such that

$$f_x(x) = \begin{cases} 0 & \text{if } x \in H_x, \\ 1 & \text{if } x \in A \setminus G_x. \end{cases}$$

Let $g_x: X \to I$ be an extension of $f_x$ and put

$$K_x = \{x \mid g_x(x) = 0\} \quad \text{and} \quad L_x = \{x \mid g_x(x) < 1\} \quad \text{for } x \in \Omega.$$
with $\text{card} \Omega \leq m$ and every collection $\{F_\alpha \mid \alpha \in \Omega\}$ of closed sets of $A$ such that $\{G_\alpha \mid A - F_\alpha\}$ is a normal open cover of $A$ for each $\alpha \in \Omega$, there exists a locally finite collection $\{H_\alpha \mid \alpha \in \Omega\}$ of cozero-sets of $X$ such that $F_\alpha \subseteq H_\alpha \cap A \subseteq G_\alpha$ for each $\alpha \in \Omega$.

Proof. Suppose that $A$ is $P^m$-embedded in $X$. Then $A$ is obviously $C$-embedded in $X$. Take $\{G_\alpha \mid \alpha \in \Omega\}$ and $\{F_\alpha \mid \alpha \in \Omega\}$ as is described in the theorem. Then, since $\{G_\alpha \mid A - F_\alpha\}$ is a normal open cover of $A$ for each $\alpha \in \Omega$, there exists a zero-set $E_\alpha$ and a cozero-set $G_\alpha$ of $A$ such that

$$F_\alpha \subseteq H_\alpha \cap A \subseteq G_\alpha$$

for each $\alpha \in \Omega$.

Let us put

$$\Psi = \bigcup_{\alpha \in \Omega} (A - F_\alpha) \cup \bigcup_{\alpha \in \Omega} \{E_\alpha\}.$$

Then $\Psi$ is a locally finite cozero-set cover of $A$ with $\text{card} \Psi \leq m$. Since $\bigcup_{\alpha \in \Omega} H_\alpha$ is a zero-set of $A$ by Lemma 2.3, and moreover we have

1. $\text{St}(K_\alpha, \Psi) \subseteq L_\alpha$ for each $\alpha \in \Omega$.

Since $A$ is $P^n$-embedded in $X$, there exists a locally finite cozero-set cover $\mathcal{V}$ of $X$ such that $\mathcal{V} \cap A$ refining $\Psi$. Then we have

2. for each set $V$ of $\mathcal{V}$ and $\alpha, \beta \in \Omega$ with $\alpha \neq \beta$; and

3. $\text{St}(K_\alpha, \mathcal{V}) \cap A \subseteq \text{St}(K_\beta, \mathcal{V})$ for $\alpha \in \Omega$.

Let $H_\alpha = \text{St}(K_\alpha, \mathcal{V})$ for $\alpha \in \Omega$. Then each $H_\alpha$ is a cozero-set of $X$. Moreover, $\{H_\alpha \mid \alpha \in \Omega\}$ is locally finite by (2) and the local finiteness of $\mathcal{V}$, and by (1) and (3) we have

$$F_\alpha \subseteq H_\alpha \subseteq G_\alpha$$

for each $\alpha \in \Omega$.

Thus, the “only if” part is proved.

Conversely, suppose that $\Psi = \{U_\alpha \mid \alpha \in \Omega\}$ is a locally finite cozero-set cover of $A$ with $\text{card} \Psi \leq m$. Then there exist a cozero-set cover $\mathcal{V} = \{U_\alpha \mid \alpha \in \Omega\}$ and a zero-set cover $\mathcal{V} = \{F_\alpha \mid \alpha \in \Omega\}$ of $A$, where $\mathcal{V}_n = \{G_\alpha \mid \alpha \in \Omega\}$ with $\text{card} \Omega \leq m$ for $n = 1, 2, \ldots$, such that

4. $\mathcal{V}$ refines $\Psi$,

5. $\mathcal{V}_n$ is discrete for $n = 1, 2, \ldots$,

6. $F_\alpha \subseteq G_\alpha$ for $\alpha \in \Omega$, $n = 1, 2, \ldots$

Then for every $n$ it is easy to see that $\{G_\alpha \mid \alpha \in \Omega\}$ and $\{F_\alpha \mid \alpha \in \Omega\}$ satisfy the assumption of the theorem. Hence there exists a locally finite collection $\{H_\alpha \mid \alpha \in \Omega\}$ of cozero-sets of $X$ such that

$$F_\alpha \subseteq H_\alpha \cap A \subseteq G_\alpha$$

for $\alpha \in \Omega$, $n = 1, 2, \ldots$. Let us put

$$D = \bigcup_{\alpha \in \Omega} \{E_\alpha\}.$$

Then $D$ is a cozero-set of $X$ and clearly contains $A$. Since $A$ is $C$-embedded in $X$, by [4, Theorem 1.18], there is a cozero-set $E$ of $X$ such that

$$E \cap A = \emptyset, \quad E \cup D = X.$$

Finally let us put

$$\mathcal{V} = \{E\} \cup \{H_\alpha\} \mid \alpha \in \Omega\}$$

Then $\mathcal{V}$ is a locally finite cozero-set cover of $X$, and hence a normal open cover of $X$ such that $\mathcal{V} \cap A$ refines $\Psi$. Thus, $A$ is $P^m$-embedded in $X$. This completes the proof of Theorem 2.4.

The following theorem which was proved in [8, Theorem 2.5] plays an essential role in the present paper.

**Theorem 2.5.** Let $X$ be a space and $Y$ a compact Hausdorff space. Let $\Psi = \{U_\alpha \mid \alpha \in \Omega\}$ be an open cover of $X \times Y$. Then there exists an open cover $\mathcal{V} = \{U_\alpha \mid A \in \mathcal{A}\}$ of $X$ satisfying conditions (a), (b) and (c) below:

(a) $\text{card} A \leq m$ or $m \geq 0$ according as $\Omega = \emptyset$ or $m < \Omega$, where $m = \text{Max}(\text{card} \Omega, \text{weight of } Y)$.

(b) For a suitable collection $\{\lambda \mid \lambda \in \mathcal{A}\}$ of finite open covers of $Y$, the collection $\{U_\alpha \times V \mid V \in \mathcal{V}, \lambda \in \mathcal{A}\}$ is an open cover of $X \times Y$ which refines $\Psi$.

(c) $\Psi$ is a normal open cover of $X$ if $\mathcal{V}$ is a normal open cover of $X \times Y$.

We need further the following two lemmas, the first of which is given in [8], and the second in [1] (for a proof see [9]).

**Lemma 2.6.** Let $X$ be a space and $Y$ a normal Hausdorff space in the sense of K. Morita [7]. If a subset $B$ of $X$ is locally compact, $\sigma$-compact and closed, then $X \times B$ is $C^*$-embedded in $X \times Y$.

**Lemma 2.7.** Let $A$ be a $P^m$-embedded subspace of a space $X$. Then $A \times X$ is $P^m$-embedded in $X \times Y$ for any compact Hausdorff space $Y$ of weight $\leq m$.

In concluding this section we shall prove one more lemma.

**Lemma 2.8.** Let $A$ be a subspace of a space $X$ and $Y$ a non-discrete compact Hausdorff space. If $A \times X$ is $C^*$-embedded in $X \times Y$, then $A$ is $C^*$-embedded in $X$.

Proof. Suppose $A \times X$ is $C^*$-embedded in $X \times Y$. First we note that $A$ is $C^*$-embedded in $X$. Let $\{U_\alpha \mid \alpha \in \Omega\}$ be a countable cozero-set cover of $A$. Since $Y$ is a non-discrete compact Hausdorff space, $Y$ contains an infinite discrete subset $B = \{y_\beta \mid \beta \in \Omega\}$. Let us put

$$H_1 = \bigcup_{\beta \in \Omega} \{y_\beta \mid \beta \in \Omega\}, \quad H_2 = \bigcup_{\beta \in \Omega} (\text{cl} B - \{y_\beta \mid \beta \in \Omega\}).$$

Then $H_1 \subseteq A$ and $H_2 \subseteq A$.
Then each of $H_1$ and $H_2$ is a cozero-set of $A \times \mathbb{C}B$ since each point $y_0$ of $B$ is isolated in $\mathbb{C}B$, and it is easy to see that $\{H_1, H_2\}$ covers $A \times \mathbb{C}B$. On the other hand, by Lemma 2.6, $A \times \mathbb{C}B$ is $C^*$-embedded in $X \times Y$ and consequently by the assumption $A \times \mathbb{C}B$ is $C^*$-embedded in $X \times \mathbb{C}B$. Then by Lemma 2.1 and Theorem 2.5, there exists a locally finite cozero-set cover $W = \{U_\lambda, \lambda \in A\}$ of $X$ such that for a suitable collection $\{Y_\lambda, \lambda \in A\}$ of open covers of $\mathbb{C}B \cup \{U_\lambda \cap A\}$, $\forall \mu \in Y_\lambda$, $\lambda \in A$ refines $\{H_1, H_2\}$. Suppose that $U_\lambda \cap A \neq \emptyset$. Since $B$ is infinite discrete and $Y$ is compact, $\mathbb{C}B - B$ is non-empty. Let $V$ be a set of $\mathbb{C}B$ with $(\mathbb{C}B - B) \cap V \neq \emptyset$. Then first we note that

$$U_\lambda \cap A = \emptyset.$$  

On the other hand, $V$ contains some $y_0$ of $B$, and so if $x$ is any point of $U_\lambda \cap A$, we have by (1)

$$(x, y_0) \in U_\lambda \times (\mathbb{C}B - \{y_0\}).$$

Consequently, $y_0 \in \mathbb{C}$, and so $x \in U_\lambda \cap A$. Therefore by Theorem 2.2, $A$ is $C^*$-embedded in $X$, and this completes the proof.

§ 3. Now we proceed to the proof of Theorems 1.3, 1.4 and 1.5.

Proof of Theorem 1.3. (a) $\Rightarrow$ (b). This follows from Lemma 2.7.

(b) $\Rightarrow$ (a). This is obvious.

(c) $\Rightarrow$ (d). Since $D^m$ is closed in $S^m$ and $S^m$ is compact Hausdorff of weight $w$, this can be seen easily by Lemma 2.6.

(d) $\Rightarrow$ (a). Our Theorem 1.5 mentioned in the introduction shows this implication. However, the following is a direct proof.

Suppose (d). Let $\{H_\lambda, \lambda \in \Omega\}$ be a locally finite cozero-set cover of $A$ with cardinality of $\Omega$. Here we may assume that $\text{card} \Omega = m$. For each $\omega \in \Omega$, let us put $Y_\omega = D^m$ and construct the product space $Y = \prod_{\omega \in \Omega} Y_\omega$, which is homeomorphic to $D^m$; we denote by $e_\omega$ the projection from $Y$ onto $Y_\omega$. Let us put

$$G_0 = \bigcup \{H_\lambda \times e_\lambda^{-1}(0) \mid \lambda \in \Omega\}, \quad G_1 = \bigcup \{H_\lambda \times e_\lambda^{-1}(1) \mid \lambda \in \Omega\}.$$  

Then $\{G_0, G_1\}$ is a binary cozero-set cover of $A \times D^m$ since

$$\{H_\lambda \times e_\lambda^{-1}(i) \mid \lambda \in \Omega, i = 1, 2\}$$

is a locally finite cozero-set cover of $A \times D^m$. Since $A \times D^m$ is $C^*$-embedded in $X \times D^m$, by Lemma 2.1 and Theorem 2.5, there exists a locally finite cozero-set cover $W = \{U_\lambda, \lambda \in A\}$ of $X$ such that for a suitable collection $\{Y_\lambda, \lambda \in A\}$ of finite open covers of $X$ the collection $\{U_\lambda \cap A \cap \mathbb{C}B, Y_\lambda, \lambda \in A\}$ refines $\{G_0, G_1\}$. Here $Y_\lambda$ can be chosen so that for some finite subset $\{a_1, ..., a_k\}$ of $A$ we have

$$Y_\lambda = \bigcap_{i=1}^k \left( \bigcup_{j \neq a_i} (U_j \cap A) \right),$$

Suppose that

$$(U_\lambda \cap A) \times \bigcap_{i \neq a} g_i^{-1}(k) = G_0.$$  

Pick a point $y$ of $\bigcap_{i \neq a} g_i^{-1}(k)$ such that $g_i(y) = 1$ for $i \neq a$. If $x \in \bigcup_{i \neq a} H_i, y \notin a \in [a_1, ..., a_k]$, then $(x, y) \notin G_0$. Hence we have

$$U_\lambda \cap A \cap \bigcap_{i \neq a} H_i = \emptyset.$$  

On the other hand, as is easily seen $A$ is $C^*$-embedded in $X$. Therefore by Theorem 2.2, $A$ is $P^m$-embedded in $X$. This completes the proof.

**Corollary 3.1.** A subspace $A$ of a space $X$ is $C^*$-embedded in $X$ if $A \times I$ is $C^*$-embedded in $X \times I$.

The equivalence of (a) and (b) in Theorem 1.3 was proved in [1] for the case $X$ is a completely regular Hausdorff space.

Let $A$ be a subset of $X$ as well as in $Y$, and let $m$ be the weight of $X$. Since the weight of $A \leq m$, $A$ is $P^m$-embedded in $X$ iff $A$ is $P^m$-embedded in $X$. Therefore Theorem 1.4 is an immediate consequence of Theorem 1.5.

Before proving Theorem 1.5 we need one more lemma. In the sequel $y$ denotes the initial ordinal number with cardinal $\gamma$.

**Lemma 3.2.** Let $Y$ be a completely regular Hausdorff space of weight $m$. Then for each $a < \gamma$ there are subsets $A_a, B_a, U_a, V_a$ of $Y$ such that

(a) $A_a$ and $B_a$ are zero-sets and $A_a \subseteq U_a$ and $B_a \subseteq V_a$,

(b) $U_a$ and $V_a$ are cozero-sets and disjoint,

(c) $Y \cap (U_a \cup V_a)$, either $A_a \not\subset U_a$ or $B_a \not\subset V_a$.

In case it is only required that each of $A_a$ and $B_a$ be closed and each of $U_a$ and $V_a$ be open, Lemma 3.2 is proved by M. Stone (cf. [12]), and his proof can be modified easily so as to yield our lemma.

Now we shall prove Theorem 1.5.

**Proof of Theorem 1.5.** Since the "only if" part follows readily from Theorem 1.3, we have only to prove the "if" part.

Suppose that $Y$ is a compact Hausdorff space of weight $m$ and $A \times Y$ is $C^*$-embedded in $X \times Y$.

First let us note that $A$ is $C^*$-embedded in $X$ by Lemma 2.8. Let $\{G_\alpha, \alpha < \gamma\}$ be a discrete collection of open sets of $A$ and $\{F_\alpha < \gamma\}$ a collection of closed sets of $A$ such that $\{G_\alpha, A - F_\alpha\}$ is a normal open cover of $A$ for each $\alpha < \gamma$. Take a zero-set $K_\alpha$ and a cozero-set $L_\alpha$ of $A$ so that

$$F_\alpha \subseteq K_\alpha \subseteq L_\alpha \subseteq G_\alpha$$

for $\alpha < \gamma$.

Let $A_\alpha, B_\alpha, U_\alpha$ and $V_\alpha$ be the subsets of $Y$ with the properties described in Lemma 3.2. Then, each $K_\alpha \times A_\alpha$ is a zero-set of $A \times Y$, $\{L_\alpha \times U_\alpha\}$ is a locally finite collection...
of cozero-sets of $A \times Y$ and $K_x \times A \subseteq L_x \times U_x$ for $\alpha < \gamma$. Hence the set
$$Z_1 = \bigcup (K_x \times A, \alpha \leq \gamma)$$
is a zero-set of $A \times Y$ by Lemma 2.3. Similarly the set
$$Z_2 = \bigcup (K_x \times (Y - U_y), \alpha < \gamma)$$
is a zero-set of $A \times Y$, and it is easy to see that $Z_1$ and $Z_2$ are disjoint. By the same argument as above we also have that the sets
$$Z_3 = \bigcup (K_x \times B_y, \alpha < \gamma)$$
and
$$Z_4 = \bigcup (K_x \times (Y - V_y), \alpha < \gamma)$$
are mutually disjoint zero-sets of $A \times Y$. Since $A \times Y$ is $C^*$-embedded in $X \times Y$ (by Lemma 2.1 and Theorem 2.5) there exists a locally finite cozero-set cover $\mathcal{M} = (M_x, \lambda \in A)$ of $X$ with the property that for a suitable collection $\{M_x, \lambda \in A\}$ of open covers of $Y$ $\{M_x \times N, N \in \mathcal{M}_x, \lambda \in A\}$ covers $X \times Y$ and that
$$\{M x \cap N | N \in \mathcal{M}_x, \lambda \in A\}$$
refines
$$\{A \times Y - Z_1, A \times Y - Z_2\}$$
and
$$\{A \times Y - Z_3, A \times Y - Z_4\}$$
For a set $M_x$ of $\mathcal{M}$ we have
$$(1) \quad M_x \cap K_y \neq \emptyset \quad \text{and} \quad M_x \cap K_y \neq \emptyset \implies \alpha = \beta$$
To prove this, suppose $\beta < \alpha$. Then by Lemma 3.2.
either $A_x - U_y \neq \emptyset$ or $B_y - Y_y \neq \emptyset$.
For example, let $A_x - U_y \neq \emptyset$. Then there is a set $N$ of the cover $\mathcal{M}_N$ of $Y$ with $(A_x - U_y) \cap N \neq \emptyset$. We have then
$$\{M_x \cap N, (M_x \cap N) \cap (K_y \times A_y) \neq \emptyset, (M_x \cap N) \cap (K_y \times A_y) \neq \emptyset,$$On the other hand, we have
$$\{M_x \cap N, (M_x \cap N) \cap (K_y \times (Y - U_y)) \neq \emptyset, (M_x \cap N) \cap (K_y \times (Y - U_y)) \neq \emptyset.$$But this is a contradiction since $\{M_x \cap N, (M_x \cap N) \cap (K_y \times A_y) \neq \emptyset,$refines
$$\{A \times Y - Z_1, A \times Y - Z_2\}$$. Thus (1) is proved.
Let us put
$$H_x = \mathrm{St}(K_x, \mathcal{M}) \quad \text{for} \quad \alpha < \gamma;$$
then by (1) $\{H_x, \alpha < \gamma\}$ is locally finite, and each $H_x$ is a cozero-set of $X$ containing $K_x$. Since $A$ is clearly $C^*$-embedded in $X$, there is a cozero-set $L_x$ of $X$ such that $L_x \cap A = L_x$. Then $\{H_x \cap L_x, \alpha < \gamma\}$ is a locally finite collection of cozero-sets of $X$, and we have
$$F_x \cap H_x \cap L_x \cap A \subseteq L_x \subseteq G_x \quad \text{for} \quad \alpha < \gamma.$$Therefore by Theorem 2.4, $F_x$ is $P^\alpha$-embedded in $X$. This completes the proof of Theorem 3.5.

As is known, a space $X$ is said to be $m$-collectionwise normal if for every discrete collection $\{F_x, x \in \Omega\}$ of closed sets of $X$ with $\operatorname{card} \Omega \leq m$ there exists a discrete collection $\{G_x, x \in \Omega\}$ of open sets of $X$ such that $F_x \subseteq G_x$ for each $x \in \Omega$. The following theorem can be obtained by putting together the several results in C. H. Dowker [2]. Here we shall give a direct proof.

**Theorem 3.5.** A space $X$ is $m$-collectionwise normal iff every closed subset of $X$ is $P^\alpha$-embedded in $X$.

Proof. The "only if" part follows from Theorem 2.4. To prove the "if" part, let $\{F_x, x \in \Omega\}$ be a discrete collection of closed sets of $X$ with $\operatorname{card} \Omega \leq m$. Then $\{F_x, x \in \Omega\}$ is a locally finite cozero-set cover with $\operatorname{card} \Omega$ of the closed set $A = \bigcup (F_x, x \in \Omega)$. By assumption there is a locally finite cozero-set cover $\mathcal{U}$ of $X$ such that $\mathcal{U} \cap A$ refines $\{F_x, x \in \Omega\}$. Since $\mathcal{U}$ is normal, there is a locally finite open cover $\mathcal{V}$ of $X$ which is a star-refinement of $\mathcal{U}$. Let us put
$$G_x = \operatorname{St}(F_x, \mathcal{V}) \quad \text{for} \quad x \in \Omega.$$Then it is easy to see that $\{G_x, x \in \Omega\}$ is a discrete collection of open sets such that $F_x \subseteq G_x$ for each $x \in \Omega$. Hence $X$ is $m$-collectionwise normal, and this completes the proof.

Combining Theorem 1.5 with Theorem 3.3, we have readily the following theorem which was proved by M. Starbird [12] with a different method.

**Theorem 3.6.** Let $Y$ be a compact Hausdorff space with weight $m$. Then a space $X$ is $m$-collectionwise normal iff for every closed set $A$ of $X \times Y$ with $A \times Y \subseteq K^\alpha$ is $C^*$-embedded in $X \times Y$.

Added in proof. In a letter (dated Mar. 29, 1975) to K. Morita, T. Przymusinski communicated, without proof, the equivalence of conditions (a) and (c) in our Theorem 1.5 and the validity of Theorem 1.4 in the case of $Y$ being $\beta X$.

**References**
K. Morita and T. Hoshina


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