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Compactly generated shape theories

by

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Abstract. For locally compact metric spaces, Borsuk's weak extension of shape to metric spaces and compactly generated shape are equivalent.

1. Introduction. Among the extensions of K. Borsuk's shape theory [1] to non-compact spaces are the ones given by Borsuk for metric spaces [2] and L. Rubin and the author for Hausdorff spaces [9]. Relationships that exist between these two extensions are discussed in [10].

The approach to shape in [9] is through the compact subsets of the Hausdorff space, hence the name "compactly generated shape". A weakened version [3] [4] of Borsuk's approach in [2] is also through the compact subsets of the metric space. In private communication, B. J. Ball posed the question as to whether or not these two approaches are equivalent. We are able to answer affirmative for locally compact metric spaces. The reader is referred to [9], [12] for the development of compactly generated shape. We denote the compactly generated shape category of [12] by $\mathscr{H}$. The full subcategory of $\mathscr{H}$ consisting of locally compact metric spaces is denoted by $\mathscr{H}_0$. We use AR and ANR to denote, respectively, absolute retract and absolute neighborhood retract for general (i.e. possibly not compact) metric spaces.

2. Weak shape. Suppose M and N are AR's and X and Y are closed subsets of M and N, respectively. A *weak sequence* from X to Y in (M, N), $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$, is a sequence of maps $\varphi_k\colon M \to N$ that satisfy the following condition:

(2.1) For every compactum $A \subset X$ there is a compactum $B \subset Y$ such that for every neighborhood V of B (in N) there is a neighborhood U of X (in M) and an integer K such that if $k \geqslant K$ then

$$\varphi_k|_U \simeq \varphi_{k+1}|_U \quad \text{in } V.$$

Note that a fundamental sequence $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ as defined in [2] is a weak sequence. Intuitively, we have dropped the "external" conditions imposed on a fundamental sequence in [2] and have retained only the "internal" conditions. Compositions and identities may be defined as in [2]. A weak sequence $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ is an *extension* of $\underline{\psi} = \{\psi_k, X', Y\}_{M,N}$ if $X' \subset X$ and $\varphi_k(x) = \psi_k(x)$ for all $x \in X'$

and $k = 1, 2, \ldots$ If X and Y are compact then a weak sequence is a fundamental sequence [2].

Two weak sequences $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ and $\underline{\psi} = \{\psi_k, X, Y\}_{M,N}$ are said to be *homotopic* (notation $\underline{\varphi} \cong \underline{\psi}$) if they satisfy the following condition:

(2.2) For every compactum $A \subset X$ there is a compactum $B \subset Y$ such that for every neighborhood V of B (in N) there is a neighborhood U of A (in M) and an integer K such that if $k \geqslant K$ then

$$\varphi_k|_U \cong \psi_k|_U \quad \text{in } V.$$

Note that if $\underline{\varphi}$ and $\underline{\psi}$ are homotopic fundamental sequences [2] then they are homotopic as weak sequences.

If M', N' are also AR's containing X, Y, respectively, then there are extensions $\alpha\colon M \to M'$ and $\beta\colon N' \to N$ of the imbeddings $i_X\colon X \subset M'$ and $i_Y\colon Y \subset N$. Let $\underline{\alpha} = \{\alpha, X, X\}_{M,M'}$ and $\underline{\beta} = \{\beta, Y, Y\}_{N',N}$ denote the fundamental sequences, considered as weak sequences, determined by α and β, respectively. Two weak sequences $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ and $\underline{\psi} = \{\psi_k, X, Y\}_{M',N'}$ are *homotopic* if $\underline{\beta}\,\underline{\varphi}$ and $\underline{\psi}\,\underline{\alpha}$ are homotopic in the sense of (2.2). It can be shown that this is an equivalence relation that preserves composition, that it does not depend on the extensions α and β, and that this definition agrees with (2.2) in case $M = M'$ and $N = N'$. Furthermore, there is a category $\mathscr{B}$ whose objects are metric spaces and whose morphisms are equivalence classes of weak sequences. If X and Y are metric spaces that are equivalent objects in $\mathscr{B}$, then they have the *same weak shape*. We use $\mathscr{B}_0$ to denote the full subcategory of $\mathscr{B}$ whose objects are locally compact metric spaces.

3. Equivalence. Let $\mathscr{M}$ denote the full subcategory of the compact shape category, defined by Mardešić in [6], whose objects are compact metric spaces. Let $\mathscr{S}$ denote the full subcategory of $\mathscr{B}$, whose objects are compact metric spaces. There are functors $K\colon \mathscr{S} \to \mathscr{M}$ and $L\colon \mathscr{M} \to \mathscr{S}$ that are the identity on objects and such that both compositions are the identity functors (see [10] Proposition (2.6)). An inspection of [10] shows that there is a functor $K^*\colon \mathscr{B}_0 \to \mathscr{H}_0$ that is the identity on objects and such that, for $[\underline{\varphi}] = [\{\varphi_k, X, Y\}_{M,N}]$, $K^*[\underline{\varphi}] = [f, \underline{f_A}]$ where $f\colon c(X) \to c(Y)$ is any φ-compadibility function and $[\underline{f_A}] = K[\underline{\varphi}|_A]$. Here, $[\underline{\varphi}]$ and $[f, \underline{f_A}]$ denote equivalence classes of weak sequences and CS-morphisms, respectively, and $\underline{\varphi}|_A = \{\varphi_k, A, f(A)\}_{M,N}$. We now construct a functor $L^*\colon \mathscr{H}_0 \to \mathscr{B}_0$.

(3.1) THEOREM. *Suppose $\{X_n|\ n = 1, 2, \ldots\}$ is a sequence of compact metric spaces embedded in M, an* AR, *$X = \bigcup_{n=1}^{\infty} X_n$, and $X_n \subset \operatorname{int}_X X_{n+1}$ for $n = 1, 2, \ldots$ If $[\underline{\varphi_n}] = [\{\varphi_k^n, X_n, Y\}_{M,N}]$ is a sequence of homotopy classes of fundamental sequences such that $[\underline{\varphi_n}] = [\underline{\varphi_{n+1}}][\underline{i_n}]$, where $\underline{i_n} = \{1_M, X_n, X_{n+1}\}_{M,M}$ for $n = 1, 2, \ldots$ is the inclusion weak sequence, then there is a unique homotopy class $[\underline{\varphi}] = [\{\varphi_k, X, Y\}_{M,N}]$ such that $[\underline{\varphi_n}] = [\underline{\varphi}][\underline{j_n}]$, where $\underline{j_n} = \{1_M, X_n, X\}_{M,N}$ for $n = 1, 2, \ldots$ is the inclusion weak sequence.*

Proof. By Theorem 3.1, p. 9 of [11] there is a sequence of representatives $\underline{\varphi_n} = \{\varphi_k^n, X_n, Y\}_{M,N}$ of the homotopy class $[\underline{\varphi_n}]$ such that $\underline{\varphi_{n+1}}$ is an extension of $\underline{\varphi_n}$ for $n = 1, 2, \ldots$ Let $\varphi_k = \varphi_k^{k+1}\colon M \to N$. Then $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ is representative of the required homotopy class. The uniqueness also follows in a straightforward manner.

The following indicates that topological sums are sums in the category $\mathscr{B}$.

(3.2) THEOREM. *Suppose $\{X_a|\ a \in A\}$ is a collection of metric spaces and M is an* AR *such that $X = \sum_{a \in A} X_a$ is a closed subset of M. Let Y be a closed subset of N, an* AR. *If $[\underline{\varphi_a}] = [\{\varphi_k^a, X_a, Y\}_{M,n}]$ is a family of homotopy classes of weak sequences, then there is a unique homotopy class of weak sequences $[\underline{\varphi}] = [\{\varphi_k, X, Y\}_{M,N}]$ such that $[\underline{\varphi_a}] = [\underline{\varphi}][\underline{i_a}]$, where $\underline{i_a} = \{1_M, X_a, X\}_{M,M}$ is the inclusion weak sequence.*

Proof. For each n, define $\varphi_n(x) = \varphi_n^a(x)$ if $x \in X_a$. Then $\varphi_n\colon X \to Y \subset N$ and N is an AR, so there is an extension $\varphi_n\colon M \to N$. Let $A \subset X$ be compact. Then $A = \sum_{a \in A'} A_a$ where A' is a finite subset of A and $A_a \subset X_a$ is compact. Let B_a, $a \in A'$, be the family of compact subsets of Y guaranteed by $\underline{\varphi_a}$, $a \in A'$, being a family of weak sequences. Let $B = \sum_{a \in A'} B_a$. Then $B \subset Y$ is compact. Suppose V is an open neighborhood of B in N. Let K be an integer such that if $k \geqslant K$ and $a \in A'$ then there is a homotopy $H_a\colon X_a \times I \to V$ between $\varphi_k^a|_{X_a}$ and $\varphi_{k+1}^a|_{X_a}$. Since V is an ANR, there is a neighborhood U of X in M and a homotopy $H\colon U \times I \to V$ between $\varphi_k|_U$ and $\varphi_{k+1}|_U$.

For uniqueness, suppose $\underline{\psi} = \{\psi_k, X, Y\}_{M,N}$ is any weak sequence such that $\underline{\varphi_a} \cong \underline{\psi}\,\underline{i_a}$. By an argument analogous to the above proof that $\underline{\varphi}$ is a weak sequence it follows that $\underline{\psi} \cong \underline{\varphi}$.

We are now able to define a functor $L^*\colon \mathscr{H}_0 \to \mathscr{B}_0$ that is the identity on objects. Let Y and $X = \sum_{a \in A} (\bigcup_{n=1}^{\infty} X_{a,n})$, where ([5], Theorem XI. 7.3) $X_{a,n}$ is compact and $X_{a,n} \subset \operatorname{int} X_{a,n+1}$ for all $a \in A$ and $n = 1, 2, \ldots$, be locally compact metric spaces. If $[f, \underline{f_A}]\colon X \to Y$ is a homotopy class of CS-morphisms then by Theorems 3.2 and 3.3 there is a unique homotopy class of weak sequences $[\underline{\varphi}] = [\{\varphi_k, X, Y\}]$ such that

$$[\{\varphi_k|_A, A, f(A)\}] = L(\underline{f_A}) \quad \text{for} \quad A = X_{a,n},\ a \in A \text{ and } n = 1, 2, \ldots$$

Define $L^*([f, \underline{f_A}]) = [\underline{\varphi}]$.

One then checks that the functors $L^*\colon \mathscr{H}_0 \to \mathscr{B}_0$ and $K^*\colon \mathscr{B}_0 \to \mathscr{H}_0$ are inverse functors, i.e., the compositions are the appropriate identity functors. Thus the concepts of Borsuk's weak shape and compactly generated shape agree on locally compact metric spaces.

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The automorphism group of a p-group of maximal class with an abelian maximal subgroup *

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Abstract. In this paper we give a detailed description of the automorphism group of a p-group of maximal class with a maximal subgroup which is abelian.

§ 1. In this paper we will always let G denote a p-group of maximal class of order p^n, $n \geqslant 4$, p an odd prime, and we will let $\mathscr{A}$ be the group of all automorphisms of G. First we note that G has a characteristic cyclic series, that is, there are characteristic subgroups, G_i, $0 \leqslant i \leqslant n$, of G with G_i/G_{i+1} cyclic such that

$$G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_n = E. \tag{1.1.1}$$

This follows from Lemmas 14.2 and 14.4 in [7]. From Durbin and McDonald's result in [3] or [1], $\mathscr{A}$ is supersolvable and its exponent divides $p^t(p-1)$ for some $t>0$. Thus the Sylow p-subgroup P of $\mathscr{A}$ is normal in $\mathscr{A}$, and so it has a p'-complement H.

The characteristic series (1.1.1) may be taken as a composition series, in that case the factors G_i/G_{i+1} have prime order p. Thus any automorphism α of G acts on G_i/G_{i+1} as a power map, i.e. if α is an automorphism of G restricted to G_i/G_{i+1} then $(aG_{i+1})\alpha = \alpha^n G_{i+1}$ for all $aG_{i+1} \in G_i/G_{i+1}$. Consider H' the commutator subgroup of H; clearly H' stabilizes (1.1.1), that is, if $h \in H'$ then h acts trivially on G_i/G_{i+1}, $i = 0, \ldots, n-1$. By Theorem 1 of P. Hall's paper [6] H' is nilpotent, and by Corollary 3.3 of [4], p. 179, it is a p-group. Therefore H' is trivial giving us that H is an abelian p'-group.

LEMMA 1.1. *The automorphism group $\mathscr{A}$ of a p-group G of maximal class is the semidirect product of P by H where P is the normal Sylow p-subgroup of $\mathscr{A}$ and where H is the p'-complement of P. Furthermore H is an abelian p'-subgroup of $\mathscr{A}$ with exponent dividing $p-1$.*

A p'-group H of automorphisms of G may be represented faithfully on the H-module $G/\Phi(G)$ over the field Z_p (integers modulo p). Here $\Phi(G)$ denotes the Frat-

* A p-group of order p^n is of maximal class if it has class $n-1$. N. Blackburn studies these groups in detail in his paper [2]; most of his results are presented in Huppert [7], pp. 361–377.