Higher Tall axioms

by

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Abstract. Tall has considered a weakened version of Martin's axiom which applies only to sets of forcing conditions which are constructible from some real number. We show the consistency (relative to the existence of a weakly compact cardinal) of a natural modification of Tall's axiom which applies to $\kappa$ closed sets of forcing conditions with the $\kappa^n$ chain condition.

Solovay and Tennenbaum [6] proved the relative consistency with the negation of the continuum hypothesis of an axiom which has since come to be known as Martin's axiom. A modification of this axiom was later introduced by Tall [9]. Both of these axioms are strengthenings of the Rasiowa-Sikorski lemma for countable chain condition Boolean algebras. It is natural to look for modification of these axioms which will apply to other algebras. In this paper, we show the consistency (relative to a weakly compact cardinal) of such a generalization of Tall's axiom.

§ 1. Preliminaries. We will generally follow the notation of Jech [2]. In particular, if $\mathfrak{P} = (P, \leq_\mathfrak{P})$ is a poset then the initial segments of $\mathfrak{P}$ generate a topology on $P$. The regular open sets of this topological space form a complete Boolean algebra (cBa), $\mathfrak{B} = \langle B, \leq_\mathfrak{B} \rangle = \text{RO}(\mathfrak{P})$. There is a canonical embedding of $\mathfrak{P}$ as a dense subset of $\langle B - \{0\}, \leq_\mathfrak{B} \rangle$.

If $M$ is a standard transitive model of set theory and $\mathfrak{B} \in M$ is a cBa in $M$ then $M[\mathfrak{B}]$, is the Scott-Solovay Boolean valued model of set theory. $[\varphi] = [\varphi]_\mathfrak{B} \in \mathfrak{B}$ is the truth value of the statement $\varphi$ in $M[\mathfrak{B}]$.

There is a canonical embedding of $M$ into $M[\mathfrak{B}]$ where the image in $M[\mathfrak{B}]$ of $m \in M$ is $m$. If $G$ is an $M$-generic ultrafilter on $\mathfrak{B}$ then we may regard the elements of $M[\mathfrak{B}]$ as names for the elements of $M[G]$; let $\iota_\mathfrak{B}$ be the denotation function so that $m \in M[\mathfrak{B}]$ is a name for $\iota_\mathfrak{B}(m) \in M[G]$. For $m \in M$, $\iota_\mathfrak{B}(\check{m}) = m$.

Two elements of a poset are called incompatible if they have no common predecessor. Two elements of a Boolean algebra are called incompatible if their product is $0$. (The ambiguity which arises from the fact that a Boolean algebra is a particular kind of poset should cause no serious confusion.)

A poset or a Boolean algebra is said to satisfy the $x$-chain condition (soc) provided that every set of pairwise incompatible elements has cardinality less than $x$. It is easy to see that a poset $\mathfrak{P}$ is soc if $\mathfrak{B} = \text{RO}(\mathfrak{P})$ is soc.
If every decreasing sequence in $\Psi$ of length less than $x$ has a lower bound in $\Psi$ then $\Psi$ is said to be $x$-closed. If $\mathbf{B}$ is a cBa and there is a $x$-closed poset $\mathbf{Q}$ dense in $\langle B - \{0\}, \leq \rangle$ then we say that $\mathbf{B}$ is $x$-closed.

The following facts are well-known:

1. 1. PROPOSITION. If $\mathbf{B} = \langle B, \leq, \alpha \rangle$ is a cBa, $\Psi = \langle P, \leq \rangle$ is a substructure of $\mathbf{B}$ such that $P$ is dense in $\langle B - \{0\}, \leq \rangle$ and $G = \{ b \in B \mid (\exists p \in P \cap G) (p \leq b) \}$, then the following are equivalent:
   a. $G$ is an $M$-generic ultrafilter on $\mathfrak{B}$,
   b. $G \cap P$ is $\Psi$-generic over $M$.

   Proof. [7, pp. 30-34].

1.1. PROPOSITION. Suppose $\mathbf{B} \in M$ is nec and $G$ is $\Psi$-generic over $M$. If $x < \lambda$ is a cardinal in $M$ then $\lambda$ is a cardinal in $M[G]$. If $x$ is a regular cardinal in $M$ then $x$ is a regular cardinal in $M[G]$.

   Proof. [2, p. 63].

1.2. PROPOSITION. Suppose $\mathbf{B} \in M$ is $x$-closed and $G$ is $\Psi$-generic over $M$ and $\lambda < x$ then $(\Psi(\lambda) \in M) = (\Psi(\lambda) \in M[G])$. 

   Proof. [5, p. 372].

When $X$ is a set we write $[X]^{2}$ for the set of all unordered pairs of distinct elements of $X$. A subset $X$ of $x$ is said to be homogeneous for a function $f : [x]^{2} \rightarrow \lambda$. An uncountable cardinal is said to be weakly compact if whenever $\lambda \in X$ and $f : [\lambda]^{2} \rightarrow \lambda$, then there is an $\subseteq \lambda$ of power $x$ which is homogeneous for $f$. A weakly compact cardinal is strongly inaccessible [5, Theorem 4.5].

2. Some generalizations of the Rasiowa–Sikorski lemma.

2.0. THEOREM (Rasiowa–Sikorski). If $\mathbf{B}$ is a $x$-closed poset and $F \leq x$ then there is a $G$ which is $\Psi$-generic over $F$.

   Proof. [7, p. 29].

The most natural way to try to strengthen the Rasiowa–Sikorski theorem is to allow $F$ to be larger. However, to avoid collapsing cardinals it is necessary to restrict $\Psi$ in some way.

DEFINITION. The $x$-Martin axiom ($x$-MA) is the statement that if $\mathbf{B}$ is a $x$-closed and $\Psi$-cc poset and $\mathfrak{B} \leq x$ then there is a $G$ which is $\Psi$-generic over $F$.

The $x$-Martin axiom is easily seen to be a consequence of $2^x = x^+$. Solovay and Tennenbaum constructed a model of ZFC$+$CH$\vdash x$-MA. Such a model satisfies the Souslin hypothesis and has many other interesting properties [3], [8]. Whether ZFC$+$x-MA$\vdash 2^x = x^+$ is consistent for an uncountable $x$ remains an open question.

Tall has suggested a weakened form of $x$-MA which we will call the $x$-Tall axiom.
Notice that \( \text{dom}(p_\beta) \cap \text{dom}(p_\gamma) = \{ \gamma \} \neq \emptyset \). Thus when \( \alpha < \beta < \lambda \), there is a \( \gamma \neq \emptyset \) such that \( p_\beta(D^\gamma) \) and \( p_\gamma(D^\gamma) \) are incompatible. Let \( \mathcal{F}(\alpha, \beta) \) be the least such \( \gamma \).

Since \( \lambda \) is weakly compact we may assume that \( \mathcal{F} \) assumes a constant value \( \alpha \). But then \( p_\beta(D^\gamma) \not\prec \lambda \) is a pairwise incompatible subset of \( \mathcal{U}_\nu \), which is impossible.

Let us say that a topological space satisfies the \( \kappa \)-chain condition (ccc) if every set of pairwise disjoint open sets has power less than \( \kappa \). The proof of Theorem 3.2 may be used to show the following.

**3.3. Theorem.** Let \( \kappa < \nu \) let \( X_\alpha \) be a \( \kappa \)ccc topological space. Then \( X = \prod X_\alpha \) is a \( \kappa \)ccc topological space.

**4. A model of \( \kappa \)-TA.** Let \( M \) be a countable standard transitive model of ZFC in which \( \kappa \) is a weakly compact cardinal and \( \kappa, \nu \) are regular cardinals with \( 2^\kappa \leq \nu \) and \( \kappa < \lambda < \nu \). Let \( \langle \Psi_\alpha : \alpha < \nu \rangle \) be an enumeration in \( M \) of all \( \kappa \)-closed \( \lambda \)-closed posets of power less than \( \nu \) which are constructible from a bounded subset of \( \kappa \). Let \( \langle \Psi_{\alpha+1} : \alpha < \kappa \rangle \) be the set of \( \kappa \)-closed \( \lambda \)-closed posets for collapsing \( \kappa \) to \( \nu \) where \( \omega \leq \kappa \). We may assume that each \( \Psi_{\alpha+1} \) appears unboundedly often in the enumeration \( \langle \Psi_{\alpha} : \alpha < \kappa \rangle \).

For \( \gamma \in \kappa \), define \( \Psi^\gamma = \prod_{\alpha < \gamma} \Psi_{\alpha} \). Let \( G^\gamma \) be \( \Psi^\gamma \)-generic over \( M \) and define (as in Theorem 3.1),

\[
G^\gamma = G^\gamma \cap P^\gamma, \quad G^\gamma = \{ p(\gamma) : p \in G \}.
\]

**4.0. Theorem.** \( M[G^\gamma] \) is a model of ZFC+\( \kappa \)-TA and \( 2^{\kappa} = \nu \).

**Proof.** By Theorem 3.2, \( \Psi^\gamma \) is a \( \kappa \)-closed poset and clearly \( \Psi^\gamma \) is \( \kappa \)-closed. Thus \( \kappa \) is a cardinal in \( M[G^\gamma] \) and \( \kappa^\gamma = \lambda \). Since each \( G_{\alpha+1} \) introduces a new subset of \( \kappa \), the remaining cardinal equalities are easily seen.

Suppose that in \( M[G^\gamma] \), \( \Psi \) is a \( \kappa \)-closed \( \lambda \)-closed poset which is constructible from some \( \alpha \in \kappa \). Since \( \alpha \in M, \Psi \in M \). Any decreasing sequence from \( \Psi \) of length less than \( \nu \) which is in \( M[G^\gamma] \) is also in \( M \), so \( \Psi \) is \( \kappa \)-closed in \( M \). Any subset of \( P \) of power \( \lambda \) in \( M[G^\gamma] \) must also be of power \( \lambda \) in \( M[G^\gamma] \), so \( \Psi \) is \( \kappa \)-closed in \( M \).

In addition to the assumptions of the preceding paragraph, \( F \) is a set of subsets of \( P \) of power less than \( \nu \) in \( M[G^\gamma] \), then for some \( \gamma < \nu, \Phi \in M[G^\gamma] \). For some \( \delta \geq \gamma \), \( \Psi \) and \( \Psi^\gamma \) are isomorphic. By Theorem 3.1, \( G_{\delta} \in M[G^\gamma] \) is (within isomorphism) \( \Psi^\gamma \)-generic over \( F \).

Remark. We really have a somewhat stronger result. Suppose \( \Gamma \models \mathcal{M} \) is such that if \( \alpha \in \mathcal{M} \) is a bounded subset of \( \kappa \) then \( \langle \mathcal{N}(\beta) \in \mathcal{M} \rangle = \nu \). Then \( \langle \Psi_{\alpha} \rangle : \alpha < \nu \rangle \) could be taken as an enumeration of all posets relatively constructible from \( \mathcal{M} \) and a bounded subset of \( \kappa \). We would then have an \( \mathcal{N} \)-realization of the \( \kappa \)-TA.

The author conjectures that this construction for \( \mathcal{M} = \mathcal{N} \) will, possibly with some additional large cardinal assumptions, give a model of \( \kappa \)-MA. We recall the following lemma of Solovay and Tennenbaum [6] which says that two stage forcing is no better than one stage forcing.

**References**


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