References

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Characterizations of real functions by continua

by

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Abstract. For a real-valued function f with domain an open interval, definitions for the functional concepts of Darboux at a point and connected at a point are examined and two-sided conditions in these definitions are reduced to one-sided conditions. The existence and characterizations of two new subclasses of Darboux functions are obtained and several examples are given to indicate that none of the four classes mentioned above are equivalent.

1. Introduction. This paper deals with properties of real functions which can be characterized by the types of continua which they intersect. A function f is said to be a *Darboux function* if $f(C)$ is connected whenever C is a connected subset of the domain of f. Equivalently, a real function f is a *Darboux function* if every horizontal interval which meets $f(+)$ and $f(-)$ meets f. A function which has a connected graph is called a *connected function*. In a paper published in 1965, [3], Bruckner and Ceder define what it means for a function to be Darboux at a point and in a paper published in 1971, [4], Garrett, Nelms, and Kellum define what it means for a function to be connected at a point. The main theorems in this paper reduce these definitions and still retain the results of [3] and [4]. Also, we exhibit two new classes of real functions each of which are subclasses of the class of Darboux functions and each of which contain the class of connected functions. The author wishes to express his appreciation to Harvey Rosen for many helpful ideas.

2. Notation. If M is a subset of the plane E then $(M)_X$ denotes the X-projection of M and M_K denotes those points of M which have X-projection in K where K is a subset of the X-axis. We denote the vertical line through the point $(x, 0)$ by l_x. If f is a real function with domain a subset of the real line R then $f(+)$ denotes the subset of E consisting of all ordered pairs (x, y) where x is in $(f)_X$ and $y > f(x)$. We define $f(-)$ similarly. A continuum is a closed connected subset of E. A horizontal segment is a bounded open connected subset of a horizontal line and a horizontal interval is the closure of a horizontal segment. Unless otherwise stated, all functions considered are real functions with domain an open connected subset of R. No distinction will be made between a function and its graph.

3. Preliminaries.

DEFINITION 1. The function f is said to be *Darboux from the left* (*right*) *at the point* z *of its domain* if whenever (z, a) and (z, b) are two limit points of f from the left (right) then the horizontal interval H contains a point of f whenever $(H)_X$ has right (left) end z and the point H_z is between (z, a) and (z, b) [4], [3].

PROPERTY 1. Let f be a function which has the property that each point $(z, f(z))$ belonging to f is a limit point of f from the left and the right. If A and B are mutually separated sets, $A \cup B = f$, and K denotes the boundary of $(A)_X$ relative to the domain of f, then relative to $(f)_X$, K is the boundary of $(B)_X$, K is a perfect set, and each of $(A)_X \cap K$ and $(B)_X \cap K$ is dense in K [7].

PROPERTY 2. No compact Hausdorff space and no complete metric space is both perfect and countable [5], p. 88.

PROPERTY 3. Let P be a perfect set in R and let 0 be a horizontal segment of R such at 0 meets P. Then 0 contains a point of P which is a limit point of P from both the left and right.

PROPERTY 4. Let f be a connected function whose domain is a connected subset of R. If M is a continuum such that $(M)_X$ is a subset of $(f)_X$, M intersects $f(+)$, and M intersects $f(-)$, then M intersects f [4].

4. Darboux at a point.

LEMMA 1. *Let f be a function which has the property that each point of f is a limit point of f from the left and right. If f is not Darboux, then there exists a horizontal interval H meeting $f(+)$ and $f(-)$ but missing f and furthermore there is a point p in the interior of $(H)_X$ such that (p, a) and (p, b) are two limit points of f from the left with (p, a) above H and (p, b) below H.*

Proof. If f is not Darboux, then there exists a horizontal interval H meeting $f(+)$ and $f(-)$ which misses f. Now f restricted to $(H)_X$ is disconnected. Hence by Property 1 we have boundary of $(A)_X$ in $(H)_X$ = boundary of $(B)_X$ in $(H)_X = K$ where $A = H(+) \cap f$ and $B = H(-) \cap f$, K is a perfect set in $(H)_X$, and each of $K \cap (A)_X$ and $K \cap (B)_X$ is dense in K.

For each positive integer i, denote by D_i the set of all ordered pairs (x, y) with x in K and $(H)_Y + i \leqslant y < (H)_Y + i + 1$ and denote by D_{-i} the set of all ordered pairs (x, y) with x in K and $(H)_Y - i \geqslant y > (H)_Y - i - 1$. Also for each positive integer j, $j \geqslant 2$, denote by F_j the set of all ordered pairs (x, y) with x in K and $(H)_Y + 1/(j-1) > y \geqslant (H)_Y + 1/j$ and denote by F_{-j} the set of all ordered pairs (x, y) with x in K and $(H)_Y - 1/j \geqslant y > (H)_Y - 1/(j-1)$. Now since K which is a subset of $(H)_X$, is a complete metric space and

$$K = \Big[\bigcup_{m=\pm i} (D_m)_X\Big] \cup \Big[\bigcup_{n=\pm j} (F_n)_X\Big],$$

one of the $(D_m)_X$ or $(F_n)_X$ is dense in some open subset of K.

For sake of argument, suppose for some $i>0$ we have $(D_i)_X$ is dense in some subinterval O_1 of $(H)_X \cap K$, that is, $O_1 = C \cap K$ where C is a horizontal subinterval of $(H)_X$ such that the interior of C meets K. Let k be an element of O_1 which is a limit point of O_1 from both the left and the right. If k is in $(B)_X$ we may use k as the desired p since D_i is a subset of A. If k is not in $(B)_X$ then k is in $(A)_X$ and since $K \cap (B)_X$ is dense in K, there is a point k_1 to the right of k such that k_1 is in the interior of $O_1 \cap (B)_X$. If k_1 is a limit point of K from the left, k_1 is the desired p to satisfy the lemma. Otherwise, if k_1 is not a limit point of K from the left, then there exists an interval O_2 (an interval in $(H)_X$) with right end k_1 which lies entirely in $(B)_X$ and hence inside C. We consider the component containing O_2 in $(B)_X$. This component is of the form $[k_2, k_1]$, where k_2 is in $(B)_X$ (since each point of f is a limit point of f from the right and left). Now $[k_2, k_1]$ is a subset of C and k_2 is the desired p of the lemma.

THEOREM 1. *Let f be a function having the property that each point of f is a limit point of f from the left and right. Then the following statements are equivalent:*

(1) *f is Darboux from the left at each point of $(f)_X$,*

(2) *f is Darboux from the right at each point of $(f)_X$,*

(3) *f is a Darboux function.*

Proof. We first show (1) implies (2) using a proof by contradiction. Suppose (x, b) and (x, a) are two limit points of f from the right such that (x, c) is between (x, b) and (x, a) and (x, c) is a left endpoint of a horizontal interval H containing no point of f. Now H meets $f(+)$ and $f(-)$ but misses f and hence by applying Lemma 1 there is a point p in the interior of $(H)_X$ such that (p, r) and (p, s) are two limit points of f from the left with $r > c > s$. Since f satisfies (1), f must meet H, and this is a contradiction. Hence our assumption that f does not satisfy condition (2) is false and therefore (1) $\Rightarrow$ (2).

By using a symmetric argument we see (2) $\Rightarrow$ (1) and by similar arguments we see (2) and (1) hold if and only if (3) holds. This reduces the definition of Darboux at a point given in [3] and [4].

5. Connected at a point.

The next theorem reduces the definition of connected at a point given in [4].

DEFINITION 2. The function f has *property* CL (*property* CR) at the point z of its domain when the following condition holds: if (z, a) and (z, b) are two limit points of f from the left (right), then the continuum M contains a point of f whenever $(M)_X$ is a nondegenerate set with right (left) end z and M_z is a subset of the vertical segment with ends (z, a) and (z, b).

THEOREM 2. *Let f be a function with the property that each point $(z, f(z))$ in f is a limit point of f from the left and right. Then the following statements are equivalent:*

(1) *f has property* CL *at each point of $(f)_X$,*

(2) *f has property* CR *at each point of $(f)_X$,*

(3) *f is connected.*

Proof. We first show (1) $\Rightarrow$ (2) using a proof by contradiction. Let f have property CL at each point of $(f)_X$ and assume there exist two points (r, a) and

(r, b), each limit points of f from the right, and a continuum M such that $(M)_X$ is a nondegenerate subset of $(f)_X$ with left end r, M_r lies between (r, a) and (r, b), and M misses f. Now $(M)_X$ is a connected set and from Property 4 we know f restricted to $(M)_X$ is disconnected. Hence there exist two open sets (open in f restricted to $(M)_X$) A and B, $A \cup B = f$ restricted to $(M)_X$, such that boundary of $(A)_X$ in $(M)_X =$ boundary of $(B)_X$ in $(M)_X = K$, K is a perfect set, each of $(A)_X \cap K$ and $(B)_X \cap K$ is dense in K, and A and B are mutually separated.

Also there exist mutually separated sets O_A and O_B (open in $[(M)_X \times R] - M$) containing A and B respectively such that $O_A \cup O_B = [(M)_X \times R] - M$. Now since f is Darboux by the preceding theorem, we find a point z, as in Theorem 1 of [4], such that z is in the interior of $(M)_X$ and such that there is a limit point (z, a) of f in O_A and a limit point (z, b) of f in O_B. If each of (z, a) and (z, b) are limit points from the left, we use the following argument referenced as Case I and obtain a contradiction.

Case 1. Suppose (z, a) and (z, b) are limit points of f from the left lying in O_A and O_B respectively with $a>b$. We choose two circles C_1 and C_2 with interiors as well as boundaries contained in O_A and O_B respectively such that (z, a) is the center of C_1, (z, b) is the center of C_2, and both C_1 and C_2 have radius r. Denote by S the subspace of E such that (x, y) is in S if and only if $z-r \leqslant x \leqslant z$ and there are two points (x, r_1) and (x, r_2), belonging to C_1 and C_2 respectively, such that $r_2 \leqslant y \leqslant r_1$. Now $M \cap S$ is closed and $S-M$ is disconnected since O_A and O_B are separated. By Lemma 1 of Roberts [9], we know there is a continuum M_1 separating (z, a) and (z, b) from each other in S such that M_1 is a subset of M. Now M_1 must meet l_z between (z, a) and (z, b). Since f satisfies (1), we know f meets M_1 and hence f and this is a contradiction.

Case 2. Hence each of (z, a) and (z, b) are limit points of f from the right and only one is a limit point of f from the left. We will assume without loss of generality that $(z, a) = (z, f(z))$ is the point which is a limit point of f from both the left and right. Repeating the argument used in Case I from the right instead of the left (i.e. S consists of all ordered pairs (x, y) where $z \leqslant x \leqslant z+r$ and there are two points (x, r_1) and (x, r_2), belonging to C_1 and C_2 respectively, such that $r_2 \leqslant y \leqslant r_1$) we obtain a bounded continuum M_2 such that M_2 is a subset of M, M_2 intersects l_z strictly between the points (z, a) and (z, b), and M_2 has nondegenerate X-projection with left end z.

Now f restricted to $(M_2)_X$ is disconnected. Hence there exist two mutually separated open sets (open in f restricted to $(M_2)_X$) A' and B', $A' \cup B' = f$ restricted to $(M_2)_X$, such that boundary of $(A')_X$ in $(M_2)_X =$ boundary of $(B')_X$ in $(M_2)_X = K'$, K' is perfect, and each of $(A')_X \cap K'$ and $(B')_X \cap K'$ is dense in K'. Also there exist mutually separated open sets (open in $(M_2)_X \times R$) $O_{A'}$ and $O_{B'}$ containing A' and B' respectively such that $O_{A'} \cup O_{B'} = [(M_2)_X \times R] - M_2$. M_2 is a closed, compact set. Hence there exists a sequence of domains $D_1, D_2, \ldots$ such that (a) for each n, D_n contains M_2 and the closure of D_{n+1} is a subset of D_n and (b) M_2 is the common part of the domains of this sequence ([8], p. 172).

Now let k_0 be a point in K' such that k_0 is a limit point of K' from both the left and the right, in particular from the left. Now $(k_0, f(k_0))$ is either in $O_{A'}$ or $O_{B'}$, say for arguments sake in $O_{A'}$. Then there exists a sequence of points $\{(b_i, f(b_i))\}_{i=1}^{\infty}$ in $O_{B'}$ such that the sequence $\{b_i\}_{i=1}^{\infty}$ converges to k_0 from the left and since M_2 is bounded and f is Darboux, the sequence $\{(b_i, f(b_i))\}_{i=1}^{\infty}$ has a convergent subsequence which converges to some finite point (k_0, p) on l_{k_0}. Now (k_0, p) cannot be in $O_{A'}$ since $O_{A'}$ is open. Hence (k_0, p) is in $O_{B'}$ or M_2. If (k_0, p) is in $O_{B'}$, we repeat Case I and reach a contradiction. Therefore assume (k_0, p) is in M_2 and without loss of generality we may assume for each such sequence $\{b_j\}_{j=1}^{\infty}$ converging to k_0 from the left the convergent subsequences of $\{b_j, f(b_j)\}_{j=1}^{\infty}$ converge to a point in M_2.

Hence there exists an interval $s_0 = [h_0, k_0]$, $h_0 \neq k_0$, such that if q is in $(B')_X \cap K' \cap [h_0, k_0]$, $(q, f(q))$ is in D_1. Let a_1 be a point of $(A')_X \cap K' \cap [h_0, k_0]$, $a_1 \neq k_0$, $a_1 \neq h_0$. Let y_1 be a point of $(B')_X \cap K' \cap s_0$ such that y_1 is to the right of a_1. Now y_1 is either a limit point of K' from the left or it is not. If y_1 is a limit point from the left, we either have Case 1 or an interval $s_1 = [h_1, y_1]$, $h_1 \neq y_1$, which is a subset of s_0 such that if q is in $(A')_X \cap K' \cap s_1$, $(q, f(q))$ is in D_1. If y_1 is not a limit point of K' from the left, there is an interval w_0 of the form $[m_0, y_1]$ such that if q is in $[m_0, y_1]$, then $(q, f(q))$ is in B'. Let J_0 be the component containing w_0 in $(B')_X$. J_0 is of the form $[n_0, y_1]$ and n_0 is in $(B')_X$ since each point of f is a limit point of f from the left and right. Now there is either an interval s_1' of the form $[t_0, n_0]$ such that if q is in $[t_0, n_0] \cap (A')_X \cap K'$, $(q, f(q))$ is in D_1 or we have Case I as applied at the point n_0 for z. Hence we either use Case I and obtain a contradiction or obtain an interval j_0 equal to s_1 or s_1' such that if q is in $j_0 \cap (A')_X \cap K'$, $(q, f(q))$ is in D_1. Now since j_0 is a subset of s_0, we have f restricted to $j_0 \cap K'$ is in D_1. By a symmetric argument replacing A' by B' and B' by A' and starting with y_1 instead of k_0 if we used s_1 for j_0 or starting with n_0 instead of k_0 if we used s_1' for j_0 we obtain an interval j_1 such that j_1 is a subset of j_0 and f restricted to $j_1 \cap K'$ is in D_2.

Repetition of this process yields a sequence of closed sets such that $K' \cap j_m$ is a subset of $K' \cap j_{m-1}$ for each positive integer m. Thus there is a point t in $T = \bigcap_{m=0}^{\infty} (K' \cap j_m)$ such that $(t, f(t))$ is in $M_2 = \bigcap_{n=1}^{\infty} D_n$ and this contradicts the assumption that f misses M. Since Case I or Case II leads to a contradiction, we see that (1) $\Rightarrow$ (2). In [4] it is shown that (1) and (2) together are equivalent to (3). Using a similar argument as above we see (2) $\Rightarrow$ (1) and the equivalence of the three statements is completed.

4. Examples. The next property was proven in [4] and motivates the following definitions.

Property 5. A necessary and sufficient condition that a function f be connected is that every continuum M in E intersect f whenever (1) $(M)_X$ is a subset of $(f)_X$, (2) M intersects $f(+)$, and (3) M intersects $f(-)$.

DEFINITION 3. A real function f is said to have *property* A if and only if each continuous function g (with a closed connected nondegenerate domain) which meets $f(+)$ and $f(-)$ meets f, where $(g)_X$ is contained in $(f)_X$.

DEFINITION 4. A real function f is said to have *property* B if and only if each arc A intersecting $f(+)$ and $f(-)$ meets f, whenever $(A)_X$ is contained in $(f)_X$.

We have the following implications holding for real functions with connected domain. Almost continuous [10] $\Rightarrow$ connected $\Rightarrow$ property B $\Rightarrow$ property A $\Rightarrow$ Darboux $\Rightarrow$ each point of f is a limit point of f from the left and right. However none of these is reversible unless further hypothesis is stated. For instance, if a real function f is of Baire Class 1, all of the above are equivalent ([2] and [7]).

In [6] as well as many other places, an example is given showing connectivity does not imply almost continuous. Example 4.1 in [3] is Darboux and fails to have property A. The function g defined to be zero if x is rational and 1 if x is irrational is an example of a function in which each point of g is a limit point of g from the left and right but which is not Darboux. In 1969, Brown [1] gave an example of a function having property A but not property B. We now construct other examples concerning property A and property B.

EXAMPLE 1. There exists a function f with property B which is not connected.

Proof. We first construct a continuum K with the following properties:

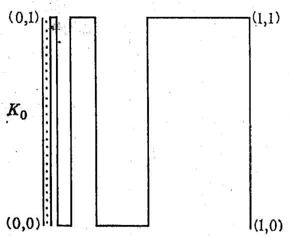
(a) $(K)_X$ is nondegenerate,

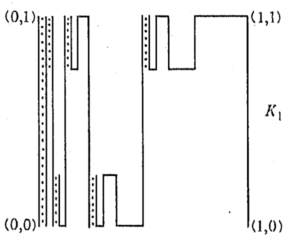
(b) the set of all z in $(K)_X$ such that K_z is nondegenerate is a countable dense subset B of $(K)_X$,

(c) for each z in $(K)_X$, K_z is degenerate or a vertical interval, and

(d) the only arcs in K are vertical intervals.

The continuum K will be the limit of a sequence of sets $\{K_i\}_{i=0}^{\infty}$ each of which is a countable union of "square $\sin(1/x)$ curves" plus a countable union of lines.





We picture K_0 and K_1 above. K_{i+1} is obtained from K_i by replacing each horizontal segment in K_i with a "square $\sin(1/x)$ curve" whose vertical intervals are $\frac{1}{4}$ the length of those vertical intervals placed in K_i.

We now construct a function f with domain $(-1, 2)$ and range $(-1, 2)$ with the desired properties. Let g be a connected function dense in $(-1, 2)\times(-1, 2)$.

We define f as follows: if $(x, g(x))$ is in K, $f(x) = \frac{3}{2}$ and if $(x, g(x))$ is not in K, $f(x) = g(x)$. We first show f has property B.

Let A be an arc in $(-1, 2)\times(-1, 2)$. If A misses K or $(A \cap K)_X$ is degenerate we know A meets f provided $(A)_X$ is nondegenerate. Hence we assume that $(A)_X$ is nondegenerate and that a and b are two points of $A \cap K$ with $(a)_X \neq (b)_X$. Then since $A \cap K$ is closed and K has condition (d) there exists a subarc H of A such that H misses K and H separates a and b in A. Now either for some two points s and t in $A \cap K$ with $(s)_X \neq (t)_X$ there exists an arc H such that $(H)_X$ is nondegenerate, H misses K, and H separates s and t in A or there exists no such arc. If such an arc H exists, we know H meets $g(+)$ and $g(-)$ and hence H meets f since H misses K.

On the other hand assume no arc H exists such that $(H)_X$ is nondegenerate and H misses K. Then every subarc k of $A-K$ separating two points of $A \cap K$ has X-projection a point. We see each endpoint p of a maximal vertical subarc D of A is in K provided p is a cutpoint of A. To see this, suppose p is a cutpoint of A, p is not in K, and p is the endpoint of a maximal vertical subarc D of A. Then there is a circle C with center p and radius r such that K misses C and its interior and $(C)_X$ is a subset of $(A)_X$. Since A is locally connected at p, there is a subarc F of A such that F lies in C and its interior and $(F)_X$ is nondegenerate and this is a contradiction. Therefore we may assume each maximal vertical subarc M of A which misses the noncutpoints of A meets K in the endpoints of M. Now since K_x is connected for each x in $(K)_X$, M is a subset of K. Let $O = \{z\colon z \text{ is in } M \text{ and } M \text{ is a vertical subarc of } A\}$. Now O is not dense in A for if it were K would contain a nonvertical arc. Hence there exists a point N in A and a disk S containing N in its interior such that S misses O. Now there exists an arc U in $A \cap S$ such that U misses O, that is, U contains no vertical subarcs. Hence there exists an arc U' such that U' is in $A \cap S$ and $(U')_X$ is nondegenerate and this is a contradiction. Hence we may conclude f has property B. Since f misses K and K meets $f(+)$ and $f(-)$, by Property 5 we see f is not connected.

Remark. The question whether such an example as above existed was asked by B. D. Garrett. The next example is due to B. D. Garrett and is similar to Example 1 of [1].

EXAMPLE 2. There exists a function f which has property A but not property B,

Proof. We construct a continuum M in the following manner. Form the middle-thirds Cantor set on the interval $[0, 1]$ of the Y-axis. Take the interval $[\frac{1}{3}, \frac{2}{3}]$ of the Y-axis and move it horizontally to the vertical line $l_{1/2}$. Now construct a pseudo-arc M_1 from the point $(\frac{1}{2}, \frac{2}{3})$ to the point $(\frac{1}{2}, \frac{1}{3})$ such that M_1 lies in the rectangle with vertices $(\frac{1}{2}, \frac{2}{3})$, $(\frac{1}{2}, \frac{1}{3})$, $(\frac{3}{4}, \frac{2}{3})$, $(\frac{3}{4}, \frac{1}{3})$. Next take $[\frac{1}{9}, \frac{2}{9}]$ horizontally to $l_{3/4}$ and $[\frac{7}{9}, \frac{8}{9}]$ horizontally to $l_{1/4}$. Construct pseudo-arcs M_2 and M_3 such that M_2 is constructed from the point $(\frac{3}{4}, \frac{1}{9})$ to the point $(\frac{3}{4}, \frac{2}{9})$ lying in the rectangle with vertices $(\frac{3}{4}, \frac{1}{9})$, $(\frac{3}{4}, \frac{2}{9})$, $(\frac{3}{4}+(\frac{1}{3})^2, \frac{1}{9})$, $(\frac{3}{4}+(\frac{1}{3})^2, \frac{2}{9})$ and M_3 is constructed from the point $(\frac{1}{4}, \frac{7}{9})$ to the point $(\frac{1}{4}, \frac{8}{9})$ lying in the rectangle with vertices $(\frac{1}{4}, \frac{7}{9})$, $(\frac{1}{4}, \frac{8}{9})$, $(\frac{1}{4}+(\frac{1}{3})^3, \frac{7}{9})$, $(\frac{1}{4}+(1)^3, \frac{8}{9})$. Continuing this process, construct an appropriate pseudo-arc M_i for each segment removed to form the Cantor set. M = closure of $(\bigcup_{i=1}^{\infty} M_i)$ is a compact continuum in I^2, $I = [0, 1]$, which contains no arc. Now let f be a connected function dense in $I \times [0, 2]$ such that $f(x) = 0$ whenever x is a dyadic rational. Define a function g from I to $[0, 2]$ such that if x is in $(f \cap M)_X$, $g(x) = 2$ and if x is not in $(f \cap M)_X$, $g(x) = f(x)$. Then g has property A since any continuous function which has a closed connected nondegenerate domain and which meets M intersects M in at most a set whose X-projection is nowhere dense in $(M)_X$. However, g will not have property B since D misses g where D is an arc such that for each dyadic rational x in $(D)_X$, $D \cap l_x$ is an interval with its endpoints in M.

The above method of construction may not be used for an arbitrary continuum and a dense function to construct functions which have property B but which are not connected, for if M above had been used for K in Example 1, we will not get a function with property B.

COROLLARY. *Let f be a connected function from I to I, and furthermore suppose f is dense in the unit square. Then there exists a function g which has property* A *but not property* B *such that if x is in $I-D$, $f(x) = g(x)$, where D is a dense subset of I such that $I-D$ is dense in I.*

QUESTION 1 (1). Let f be a function with property B. Characterize the continua which must meet f whenever they meet $f(+)$ and $f(-)$. For example, does the pseudo-arc have this property?

Remark. It is not hard to construct examples of functions f with property A which miss a pseudo-arc meeting $f(+)$ and $f(-)$ or to construct functions g which meet every pseudo-arc meeting $f(+)$ and $f(-)$ but do not have property A. This is because the intersection of a continuous function and a pseudo-arc has a nowhere dense X-projection.

7. Property A **and** B **at a point.** We now state a few natural definitions and results pertaining to property A and property B.

DEFINITION 5. The function f has *property* BL (BR) at the point z of its domain when the following condition holds: if (z, a) and (z, b) are two limit points of f from the left (right), then the arc M contains a point of f whenever $(M)_X$ is a nondegenerate set with right (left) end z and M_z is a subset of the vertical segment with ends (z, a) and (z, b). We define property AL (AR) at a point z similarly replacing arc M by continuous function g with domain a closed connected set.

THEOREM 3. *Let f be a function with the property that if $(z, f(z))$ is a point of f, then $(z, f(z))$ is a limit point of f from the left and right. Then the following statements are equivalent using either property* B *or property* A *throughout.*

(1) In a paper by Bruckner and Ceder (see *On jumping functions by connected sets*, Czech. Math. Journal 22 (1972), p. 443), they seem to answer Question 1 negatively concerning the pseudo-arc with their function f_4. However, they made the false assumption that if M is an indecomposable continuum and A is an arc, then the X-projection of $A-M$ is uncountable. Hence Question 1 concerning the pseudo-arc is still open.

(1) *f has property* BL *(property* AL*) at each point of* $(f)_X$,
(2) *f has property* BR *(property* AR*) at each point of* $(f)_X$,
(3) *f has property* B *(property* A*).*

Proof. The proof that (1) ⇒ (2) and (2) ⇒ (1) follows as in Theorem 2. Now suppose f satisfies (1) and not (3). Since f does not have property B, there exists an arc K intersecting $f(+)$ and $f(-)$ which misses f. Consider $(K)_X \times R$. Then $[(K)_X \times R] - K$ is disconnected and hence by Property 1 as applied to f restricted to $(K)_X$, there are mutually separated open sets O_A and O_B containing f and as in Theorem 2, there is a point z in the interior of $(K)_X$ such that there is a limit point (z, a) of f in O_A and a limit point (z, b) of f in O_B. We may take $b<a$, $(z, b) = (z, f(z))$ and (z, a) to be limit point from one side or the other. Since (1) ⇒ (2) we reach a contradiction. Hence f has property B.

It is clear that property B ⇒ (1), (2). Hence we have shown equivalence of the statements for property B. The proofs for property A are the same with the only change being to replace arc by continuous function with closed connected domain.

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