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A note on tangentially equivalent manifolds

by

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Abstract. Let M_1^m, M_2^m be two closed 1-connected smooth manifolds which are tangentially equivalent. B. Mazur proved that, for large values of k, $k \geqslant m+2$, there exists a diffeomorphism

$$F\colon M_1^m \times D^k \to M_2^m \times D^k.$$

In this note we define an obstruction theory for the existence of such a diffeomorphism in the metastable range, $k \geqslant \frac{1}{2}(m+4)$ and for $m \geqslant 5$.

Recall that two closed oriented smooth manifolds M_1^m and M_2^m are called *tangentially equivalent* iff there exists a smooth homotopy equivalence

$$f\colon M_1^m \to M_2^m,$$

such that $f^*\bar{\tau}(M_2) = \bar{\tau}(M_1)$, where $\bar{\tau}(M_i)$ is the stable tangent bundle of $M_i (i = 1, 2)$. B. Mazur in [6] proved that if M_1^m and M_2^m are two closed simply connected tangentially equivalent manifolds, then for large k, $k \geqslant m+2$, there exists a diffeomorphism

$$F\colon M_1^m \times D^k \to M_2^m \times D^k$$

such that the following diagram is commutative up to homotopy:

$$\begin{array}{ccc} M_1^m \times D^k & \xrightarrow{F} & M_2^m \times D^k \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

where the vertical maps are projections on the first factor, and f the tangential equivalence. In this note we define an obstruction theory for the existence of the diffeomorphism for values of k in the metastable range, i.e. for $k \geqslant \frac{1}{2}(m+4)$ and for 1-connected manifolds, $m \geqslant 5$.

Let $f\colon M_1^m \to M_2^m$ be a tangential equivalence between the 1-connected closed manifolds M_1 and M_2. Consider the composition map $i \circ f = f'$

$$M_1^m \xrightarrow{f} M_2^m \xrightarrow{i} M_2^m \times D^k,$$

i is the inclusion map, f' induces an isomorphism between the homotopy groups in all dimensions, hence by Haefliger theorem, for $k \geqslant \frac{1}{2}(m+4)$, [2], f' can be approximated, within its homotopy class, by an imbedding

$$g\colon M_1^m \to M_2^m \times D^k.$$

PROPOSITION 1. *The normal bundle ν_1 of $g(M_1^m)$ in $M_2^m \times D^k$ is stably trivial.*

Proof. The following relation is obvious

$$\nu_1 \oplus \tau(M_1^m) \simeq g^*(P_1^* \tau(M_2^m)) \oplus P_2^* \tau(D^k)$$

where

$$P_1\colon M_2^m \times D^k \to M_2^m,$$
$$P_2\colon M_2^m \times D^k \to D^k$$

are projections on the first and second factors respectively and $\oplus$ stands for the whitney sum of bundles. Therefore,

$$\nu_1 \oplus \tau(M_1^m) \oplus \varepsilon' \simeq f^*(\tau(M_2)) \oplus \varepsilon^{k+1}.$$

Here ε^m stands for an m-dimensional trivial bundle. Since

$$\tau(M_1^m) \oplus \varepsilon' \simeq f^*(\tau(M_2^m) \oplus \varepsilon^1),$$

it follows immediately that the stable class of ν_1 is zero, i.e. ν_1 is stably trivial.

The following is a special case of Theorem A in [3]:

PROPOSITION 2. *The normal bundle ν_1, defined above, is fibre homotopically trivial.*

Proof. Put $U_2 = M_2 \times D^k$. Let U_1 be a closed tubular neighborhood of $g(M_1^m) \subseteq U_2$. Let N_1, N_2 be the boundaries of U_1 and U_2, respectively. Then N_1 and N_2 are, respectively, the sphere bundles of ν_1 and the trivial k-bundle over M_2. One can show that $U_2 - \operatorname{int} U_1$ is an h-cobordism between N_1 and N_2 (see [7]). Then by a theorem of Dold [3], we deduce that ν_1 is fibre homotopically trivial. This finishes the proof.

Let H_n be the H-space of maps of S^{n-1} onto itself with degree 1. It is known (theorem of Dold and Lashof [1]) that the set of fibre homotopy equivalence classes of H_n-bundles over a finite complex X is in 1-1 correspondence with $[X, BH_n]$ = set of homotopy classes of maps of $X \to BH_n$, BH_n is the classifying space of H_n. Now if our SO_k-bundle ν_1 is trivial over the $(i-1)$-skeleton of M_1^m, then the obstruction to the triviality of this bundle over the ith skeleton is an element $\sigma_i \in H^i(M_1, \pi_{i-1}(SO_k))$. Notice that σ_i does not depend on the approximation of f' by Haefliger theorem [2]. (Any two approximations g_1 and g_2 of f' are isotopic for $k \geqslant \frac{1}{2}(m+4)$).)

There are natural homomorphisms

$$S_{ik}\colon \pi_{i-1}(SO_k) \to \pi_{i-1}(SO),$$
$$j_{ik}\colon \pi_{i-1}(SO_k) \to \pi_{i-1}(H_k).$$

induced by the natural inclusions of SO_k in SO and SO_k in H_k, respectively. These maps induce maps on the cohomology,

$$S_{ik}^*\colon H^i(M_1, \pi_{i-1}(SO_k)) \to H^i(M_1, \pi_{i-1}(SO)),$$
$$j_{jk}^*\colon H^i(M_1, \pi_{i-1}(SO_k)) \to H^i(M_1, \pi_{i-1}(H_k)).$$

Now, by the last two propositions we have

$$S_{ik}^*(\sigma_i) = 0 \quad \text{and} \quad j_{jk}^*(\sigma_i) = 0.$$

Notice that, in the metastable range, the kernel of j_{ik} is contained in the kernel of J_{ik}, where

$$J_{ik}\colon \pi_{i-1}(SO_k) \to \pi_{i,k-1}(S^k)$$

is the Hopf-Whitehead homomorphism (see [5], P212). Observe that if $\sigma_1 = \sigma_2 = \dots = \sigma_{m-1} = 0$, then $\sigma_m \in H^m(M_1, \pi_{m-1}(SO_k)) \approx \pi_{m-1}(SO_k)$, and by above, $\sigma_m \in \ker J_{mk} \cap \ker S_{mk}$.

Remark. $\operatorname{Ker} J_{mk} \cap \ker S_{mk} = 0$ if $k \geqslant m-2$ or if $k \geqslant \frac{1}{2}(m+4)$, $m \leqslant 15$ (see [4]).

THEOREM. *Let f be a tangential equivalence between the 1-connected closed manifolds M_1^m and M_2^m. Then there exists a diffeomorphism F which makes the following diagram commutative iff $\sigma_i = 0$ for all i, $\frac{1}{2}(m+4) \leqslant k$, $m \geqslant 5$:*

$$\begin{array}{ccc} M_1^m \times D^k & \xrightarrow{F} & M_2^m \times D^k \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P} \\ M_1^m & \xrightarrow[f]{} & M_2^m \end{array}$$

Proof. Assume all the σ_i's vanish. Then the normal bundle ν_1 of M_1 in $M_2^m \times D^k$ is trivial. Then it follows easily from the h-cobordism theorem of Smale that $M_1^m \times D^k \equiv M_2^m \times D^k \equiv$ stands for diffeomorphism.

Conversely, assume there exists such a diffeomorphism

$$F\colon M_1^m \times D^k \to M_2^m \times D^k.$$

Then

$$M_1 \xrightarrow{i} M_1^m \times D^k \xrightarrow{F} M_2^m \times D^k$$

implies that the normal bundle of M_1^m in $M_2^m \times D^k$ is trivial and therefore all the σ_i's vanish.

Next we prove the following proposition.

PROPOSITION 3. *If M_1^m and M_2^m are tangentially equivalent 1-connected manifolds such that*

$$M_1^m \times D^{k+1} \equiv M_2^m \times D^{k+1}.$$

Then the obstructions σ_i to have

$$M_1^m \times D^k \equiv M_2^m \times D^k$$

vanish for $i < k$.

Proof. Let φ be a classifying map for the normal bundle ν_1 (we are still using the same notation). Then we have the following diagram:

$$\begin{array}{ccccc} M_1^m & \xrightarrow{\varphi} & BSO_k & \xrightarrow{j} & BH_k \\ & & \downarrow{\scriptstyle i} & & \\ & & BSO_{k+1} & & \end{array}$$

From assumption we get immediately $i \circ \varphi$ is null homotopic in BSO_{k+1}. But BSO_k is a bundle over BSO_{k+1} with fibre S^k, therefore by the lifting homotopy property, φ is homotopic to a map

$$\varphi'\colon M_1^m \to S^k \subseteq BSO_k,$$

and hence φ restricted to the $(k-1)$-skeleton of M_1 is null-homotopic and hence $\sigma_i = 0$ for $i \leqslant k-1$.

COROLLARY. *If M_1^m and M_2^m are two 1-connected closed manifolds which are tangentially equivalent then:*

(1) *If* $5 \leqslant m \leqslant 15$, *then* $M_1^m \times D^k \equiv M_2^m \times D^k$, $k \geqslant \frac{1}{2}(m+4)$,

(2) $M_1^m \times D^k \equiv M_2^m \times D^k$, $k \geqslant m-2$, $m \geqslant 5$.

Proof. The corollary follows from the theorem, Proposition 3 and the remark.

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Idempotent generated algebras and Boolean pairs

by

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Abstract. Let R be a commutative ring with unit. In this paper we introduce the category of Boolean R-pairs and obtain a full faithful functor $\mathcal{B}$ from the category of idempotent generated R-algebras to the category of Boolean R-pairs. We also determine an adjoint for $\mathcal{B}$. Results are given to point out some applications of these functors.

0. Introduction. If A is a commutative algebra over a commutative ring R, then it is well-known that the set of idempotents of A can be made into a Boolean ring. However this functor is not full. We consider the category, R-IGA, of commutative idempotent generated R-algebras and obtain a full faithful functor $\mathcal{B}$ on this category to the category of Boolean R-pairs (defined below) which contains full subcategories isomorphic to the category of Boolean rings. This result is then applied to the recent problem of finding categories in which the objects are determined (up to isomorphism) by monoids of endomorphisms. For related results on this problem see [3], [4], [5], [7], [8], [9], [10].

For the particular case of torsion free idempotent generated rings, George M. Bergman (see [1]) indicates a left adjoint for the functor $\mathcal{B}$. That is, given any Boolean ring B he constructs an idempotent generated ring $Z[B]$ with torsion free additive group such that the Boolean ring of idempotents of $Z[B]$ is isomorphic to B. Here we present the construction of such an adjoint for the category R-IGA of commutative idempotent generated R-algebras. Upon restricting to a certain subcategory of pairs, we obtain an equivalence with R-IGA which contains results of McCrea [5] and Stringall [10] as special cases.

Conventions. In this paper, all rings will be associative, commutative, with unit and all algebras will be unitary. For an R-algebra A, let

$$\operatorname{End}_R A = \{f\colon A \to A \mid (a+b)f = af+bf,\ (ab)f = afbf,\ (raf) = r(af),\ a, b \in A,\ r \in R\}.$$

We give a short outline of the paper. In Section 1, we show that the Boolean ring of any R-algebra is determined by $\operatorname{End}_R A$. In Section 2, the category of Boolean R-pairs is defined and the functor $\mathcal{B}$ is constructed. Applications of these results give the results of McCrea [5], Smith and Luh [8], and Stringall [10]. In Section 3, the left adjoint of $\mathcal{B}$ is constructed and in Section 4 an equivalence between R-IGA and a certain subcategory of pairs is obtained which generalizes the work of McCrea [5] and Stringall [10].