A note on tangentially equivalent manifolds

by

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Abstract. Let $M^n_k$, $M^n_m$ be two closed 1-connected smooth manifolds which are tangentially equivalent. B. Mazur proved that, for large values of $k$, $k \geq m+2$, there exists a diffeomorphism

$$F: M^n_k \times D^k \rightarrow M^n_m \times D^k.$$ 

In this note we define an obstruction theory for the existence of such a diffeomorphism in the metaspline range, $k \geq \frac{1}{2}(m+4)$ and for $m \geq 5$.

Recall that two closed oriented smooth manifolds $M^n_k$ and $M^n_m$ are called tangentially equivalent if there exists a smooth homotopy equivalence

$$f: M^n_k \rightarrow M^n_m,$$

such that $f^* \varphi(M_k) = \varphi(M_m)$, where $\varphi(M_i)$ is the stable tangent bundle of $M_i (i = 1, 2)$. B. Mazur in [6] proved that if $M^n_k$ and $M^n_m$ are two closed simply connected tangentially equivalent manifolds, then for large $k$, $k \geq m+2$, there exists a diffeomorphism

$$F: M^n_k \times D^k \rightarrow M^n_m \times D^k$$

such that the following diagram is commutative up to homotopy:

$$\begin{array}{ccc}
M^n_k \times D^k & \xrightarrow{f} & M^n_m \times D^k \\
\downarrow & & \downarrow \\
M^n_k & \xrightarrow{f^*} & M^n_m
\end{array}$$

where the vertical maps are projections on the first factor, and $f$ the tangential equivalence. In this note we define an obstruction theory for the existence of the diffeomorphism for values of $k$ in the metaspline range, i.e. for $k \geq \frac{1}{2}(m+4)$ and for 1-connected manifolds, $m \geq 5$.

Let $f: M^n_k \rightarrow M^n_m$ be a tangential equivalence between the 1-connected closed manifolds $M_k$ and $M_m$. Consider the composition map $i \circ f = f^*$$

$$
M^n_k \rightarrow M^n_m \rightarrow M^n_m \times D^k,
$$

where $i$ is the inclusion map, $f^*$ induces an isomorphism between the homotopy groups in all dimensions, hence by Haefliger theorem, for $k \geq \frac{1}{2}(m+4)$, [2]. $f^*$ can be approximated, within its homotopy class, by an imbedding

$$g: M^n_k \rightarrow M^n_m \times D^k.$$
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Now, by the last two propositions we have

\[ S^2(\sigma_i) = 0 \quad \text{and} \quad J^3(\sigma_i) = 0. \]

Notice that, in the metastable range, the kernel of \( J_k \) is contained in the kernel of \( J_k \), where

\[ J_k : \pi_{k-1}(SO_k) \to \pi_k(SO_k). \]

is the Hopf-Whitehead homomorphism (see [5], F212). Observe that if \( \sigma_i = \sigma_1 = \ldots = \sigma_{i-1} = 0 \), then \( \alpha_{i-1} \in H^4(M_i, \pi_{i-1}(SO_k)) \approx \pi_{i-1}(SO_k) \), and by above,

\[ \alpha_{i-1} \in \ker J_k \cap \ker S_i. \]

**Remark.** \( \ker J_k \cap \ker S_i = 0 \) if \( k > m - 2 \) or if \( k > 4(m+4), m < 15 \) (see [4]).

**Theorem.** Let \( f \) be a tangential equivalence between the \( 1 \)-connected closed manifolds \( M_i^k \) and \( M_i^k \). Then there exists a diffeomorphism \( F \) which makes the following diagram commutative if \( \sigma_i = 0 \) for all \( i, \frac{1}{2}(m+4) < k, m > 5 \):

\[
\begin{array}{ccc}
M_i^k \times D^k & \xrightarrow{F} & M_i^k \times D^k \\
\alpha_{i-1} \downarrow \quad \downarrow \alpha_{i-1} & & \downarrow \alpha_{i-1} \\
M_i^k \xrightarrow{J_k} & \xrightarrow{J_k} & M_i^k \\
\end{array}
\]

**Proof.** Assume all the \( \sigma_i \)'s vanish. Then the normal bundle \( \nu_{i} \) of \( M_i \) in \( M_i^k \times D^k \) is trivial. Then it follows easily from the \( h \)-cobordism theorem of Smale that \( M_i^k \times D^k \approx M_i^k \times D^k \), stands for diffeomorphism.

Conversely, assume there exists such a diffeomorphism

\[
F : M_i^k \times D^k \to M_i^k \times D^k.
\]

Then

\[
M_i^k \times D^k \approx M_i^k \times D^k.
\]

implies that the normal bundle of \( M_i^k \) in \( M_i^k \times D^k \) is trivial and therefore all the \( \sigma_i \)'s vanish.

Next we prove the following proposition.

**Proposition 3.** If \( M_i^k \) and \( M_i^k \) are tangentially equivalent \( 1 \)-connected manifolds such that

\[
M_i^k \times D^{k+1} \approx M_i^k \times D^{k+1}.
\]

Then the obstructions \( \sigma_i \) to have

\[
M_i^k \times D^k = M_i^k \times D^k
\]

vanish for \( i < k \).

**Proof.** Let \( \phi \) be a classifying map for the normal bundle \( \nu_i \) (we are still using the same notation). Then we have the following diagram:

\[
\begin{array}{ccc}
M_i^k \times BSO_k & \xrightarrow{\phi} & BH_{k} \\
\downarrow \quad \downarrow \alpha_{i-1} & & \downarrow \alpha_{i-1} \\
BSO_{k+1} & \approx & BH_k
\end{array}
\]
From assumption we get immediately \( i \circ \varphi \) is null homotopic in \( BSO_{k+1} \). But \( BSO_k \) is a bundle over \( BSO_{k+1} \) with fibre \( S^k \), therefore by the lifting homotopy property, \( \varphi \) is homotopic to a map
\[
\varphi : M^k_t \times D^k \simeq BSO_k,
\]
and hence \( \varphi \) restricted to the \((k-1)\)-skeleton of \( M_t \) is null-homotopic and hence \( \sigma_i = 0 \) for \( i \leq k-1 \).

**Corollary.** If \( M^k_t \) and \( M^k_s \) are two 1-connected closed manifolds which are tangentially equivalent then:

1. If \( 5 \leq m \leq 15 \), then \( M^k_t \times D^m = M^k_s \times D^m \), \( k \geq \frac{1}{2} (m+4) \),
2. \( M^k_t \times D^m = M^k_s \times D^m \), \( k \geq m-2 \), \( m \geq 5 \).

**Proof.** The corollary follows from the theorem, Proposition 3 and the remark.

**References**


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**Idempotent generated algebras and Boolean pairs**

by

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Abstract. Let \( R \) be a commutative ring with unit. In this paper we introduce the category of Boolean \( R \)-pairs and obtain a full faithful functor \( S \) from the category of idempotent generated \( R \)-algebras to the category of Boolean \( R \)-pairs. We also determine an adjoint for \( S \). Results are given to point out some applications of these functors.

**0. Introduction.** If \( A \) is a commutative algebra over a commutative ring \( R \), then it is well-known that the set of idempotents of \( A \) can be made into a Boolean ring. However this functor is not full. We consider the category, \( R \)-IGA, of commutative idempotent generated \( R \)-algebras and obtain a full faithful functor \( \# \) on this category to the category of Boolean \( R \)-pairs (defined below) which contains full subcategories isomorphic to the category of Boolean rings. This result is then applied to the recent problem of finding categories in which the objects are determined (up to isomorphism) by monoids of endomorphisms. For related results on this problem see [3], [4], [5], [7], [9], [9], [10].

For the particular case of torsion free idempotent generated rings, George M. Bergman (see [1]) indicates a left adjoint for the functor \( \# \). That is, given any Boolean ring \( B \) he constructs an idempotent generated ring \( Z[B] \) with torsion free additive group such that the Boolean ring of idempotents of \( Z[B] \) is isomorphic to \( B \). Here we present the construction of such an adjoint for the category \( R \)-IGA of commutative idempotent generated \( R \)-algebras. Upon restricting to a certain subcategory of pairs, we obtain an equivalence with \( R \)-IGA which contains results of McCrea [5] and Stringall [10] as special cases.

Conventions. In this paper, all rings will be associative, commutative, with unit and all algebras will be unitary. For an \( R \)-algebra \( A \), let \( \text{End}_R(A) = \{ f : A \to A | (a+b)f = af+bf, (ab)f = a(fb), (raf) = r(af), a, b \in A, r \in R \} \).

We give a short outline of the paper. In Section 1, we show that the Boolean ring of any \( R \)-algebra is determined by \( \text{End}_R(A) \). In Section 2, the category of Boolean \( R \)-pairs is defined and the functor \( \# \) is constructed. Applications of these results give the results of McCrea [5], Smith and Luh [8], and Stringall [10]. In Section 3, the left adjoint of \( \# \) is constructed and in Section 4 an equivalence between \( R \)-IGA and a certain subcategory of pairs is obtained which generalizes the work of McCrea [5] and Stringall [10].