A note on "Atomic compactness in $\aleph_1$-categorical Horn theories" by John T. Baldwin

by

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Abstract. Every model of an ω_1-categorical Horn theory is atomic compact.

In paper [2] (mentioned in the title above) Baldwin has proved the following theorem:

THEOREM 0. *If T is a theory satisfying the following three conditions:*
(a) *T is almost strongly minimal,*
(b) *T is an $\forall\exists$-theory,*
(c) *T is a Horn theory,*
then every model of T is atomic compact.

In this note we shall show that (a) can be replaced by a weaker assumption of ω_1-categoricity of T, (see [1]); (b) can be omitted; and (c) can be replaced by the assumption that *the class of all models of T is closed under direct products*. Hence we get the following theorem:

THEOREM 1. *If T is an ω_1-categorical theory such that the class of all models of T is closed under direct products, then every model of T is atomic compact.*

In the proof of Theorem 1, we shall use the following proposition (see e.g. [5], Th. 4) which results from a theorem in [4].

PROPOSITION 2. *Let I be an infinite set and $\mathscr{F}_I$ a filter on $I\times I$ generated by all the equivalence relations ϱ on I such that I/ϱ is finite and all but one ϱ-equivalence classes are finite. Then for every structure $\mathfrak{A}$ we have $\mathfrak{A}^I|\mathscr{F}_I \prec \mathfrak{A}^I$. Moreover* $\mathrm{Card}(\mathfrak{A}^I|\mathscr{F}_I) = \mathrm{Card}(\mathfrak{A})\cdot\mathrm{Card}(I)$ (¹).

Proof of Theorem 1. Let $\mathfrak{A}$ be a model of T and let $\Sigma = \Sigma(\bar{c},\bar{x})$ be a set of atomic formulas with a sequence $\bar{c}$ of parameters from $\mathfrak{A}$ and a sequence $\bar{x}$ of variables. Suppose that $\Sigma(\bar{c},\bar{x})$ is finitely satisfiable in $\mathfrak{A}$. Take I such that $\mathrm{Card}(I)\geqslant\max(\mathrm{Card}(\mathfrak{A}),\mathrm{Card}(\Sigma),\omega_1)$ and consider $\mathfrak{A}^I|\mathscr{F}_I$. Let d be the diagonal embedding of $\mathfrak{A}$ into $\mathfrak{A}^I|\mathscr{F}_I$. Of course $\Sigma(d(\bar{c}),\bar{x})$ is finitely satisfiable in $\mathfrak{A}^I|\mathscr{F}_I$.

(¹) $\mathfrak{A}^I|\mathscr{F}$ denotes the limit power of $\mathfrak{A}$, for more details see e.g. [4].

By Proposition 2, $\mathfrak{A}^I|\mathscr{F}_I \prec \mathfrak{A}^I$ holds. Let r be a mapping of $\mathfrak{A}^I$ onto $\mathfrak{A}$ such that $rd(x) = x$ for all $x \in A$.

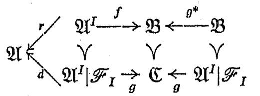
Now take $\mathfrak{B} \succ \mathfrak{A}^I|\mathscr{F}_I$ which has the following properties:

(1) $\Sigma(d(\bar{c}), \bar{x})$ *is satisfiable in* $\mathfrak{B}$;

(2) $\mathrm{Card}(\mathfrak{B}) = \mathrm{Card}(\mathfrak{A}^I)$;

(3) $\mathrm{Card}(\mathfrak{B}) \geqslant \omega_1$.

Since $\mathrm{Card}(\mathfrak{B}) = \mathrm{Card}(\mathfrak{A}^I) \geqslant \omega_1$, and T is ω_1-categorical, there is an isomorphism f of $\mathfrak{A}^I$ onto $\mathfrak{B}$. Since $\mathfrak{A}^I|\mathscr{F}_I \prec \mathfrak{A}^I$, f maps $\mathfrak{A}^I|\mathscr{F}_I$ onto an elementary submodel of $\mathfrak{B}$, say $\mathfrak{C}$. Let g be an isomorphism of $\mathfrak{A}^I|\mathscr{F}_I$ onto $\mathfrak{C}$ such that $g = f \upharpoonright (A^I|\mathscr{F}_I)$. Now $\mathfrak{B}$ is an uncountable model of T. Consequently $\mathfrak{B}$ is saturated, whence homogeneous. Therefore there is an authomorphism g^* of $\mathfrak{B}$ such that $g \subseteq g^*$. In this way we get the following diagram:



Now, $\Sigma(d(\bar{c}), \bar{x})$ is satisfied in $\mathfrak{B}$ by a sequence $\bar{a}$ of elements of $\mathfrak{B}$. So $\Sigma(d(\bar{c}), a)$ holds in $\mathfrak{B}$.

Since g^* is an authomorphism of $\mathfrak{B}$, the set $\Sigma(g^*d(\bar{c}), g^*(\bar{a}))$ holds in $\mathfrak{B}$ too. If we pass from $\mathfrak{B}$ to $\mathfrak{A}^I$ by f^{-1}, we see that $\Sigma(f^{-1}g^*d(\bar{c}), f^{-1}g^*(\bar{a}))$ holds in $\mathfrak{A}^I$. But $f^{-1}g^*d(x) = d(x)$ for each $x \in A$. So $\Sigma(d(\bar{c}), f^{-1}g^*(\bar{a}))$ holds in $\mathfrak{A}$. Finally, applying r, we come back to $\mathfrak{A}$ and we see that $\Sigma(rd(\bar{c}), rf^{-1}g^*(\bar{a}))$ holds in $\mathfrak{A}$. But $rd(x) = x$ for each $x \in A$; consequently $\Sigma(\bar{c}, \bar{x})$ is satisfied in $\mathfrak{A}$ by the sequence $rf^{-1}g^*(\bar{a})$. Thus $\mathfrak{A}$ is atomic compact. Q.E.D.

Remark 1. Baldwin's Theorem and the Theorem just proved deal with theories in countable languages. This is not an essential restriction. Indeed, if the language $\mathscr{L}$ of T is of the cardinality $\lambda \geqslant \omega$, then we can use Shelah's Categoricity Theorem (see e.g. [3]) for T to get (by the same proof with ω_1 — replaced by λ^+) the following theorem:

THEOREM 3. *Let T be a complete theory in a language of the cardinality $\lambda \geqslant \omega$. If T is categorical in some cardinality $\varkappa > \lambda$ and the class of all models of T is closed under direct products, then every model of T is atomic compact.*

Remark 2 (²). No assumption can be omitted in Theorem 1. Indeed, the theory of algebraicaly closed fields of the characteristic 0 is ω_1-categorical, but does not have an atomic compact model.

To check that ω_1-categoricity is essential, take the lattice $\mathfrak{A} = \langle A, \wedge, \vee, 0, 1\rangle$ such that A is infinite and elements of $A - \{0, 1\}$ are pairwise incomparable. Take $T = \mathrm{Th}(\mathfrak{A}^{\omega}_{\mathscr{F}})$, where $\mathscr{F}$ is the Fréchet filter. It is easy to see that the class of all models of T is closed under direct products but T does not have an atomic compact model.

Remark 3. The proof of Theorem 1, yields the following *representation theorem*:

(²) Remarks 2 and 3 answer questions raised by L. Pacholski.

THEOREM 4. *If T is an ω_1-categorical theory and the class of all models of T is closed under direct products, then for each countable model $\mathfrak{B}$ of T, every uncountable model $\mathfrak{A}$ of T is isomorphic with $\mathfrak{B}^I|\mathscr{F}_I$, where* $\mathrm{Card}(I) = \mathrm{Card}(\mathfrak{A})$.

Unfortunately the statement above is not true if $\mathfrak{A}$ is countable. Indeed, let T be the theory of countably many distinct individual constants. If $\mathfrak{B}$ is a model of T, then $\mathfrak{B}^2$ is saturated. Consequently other countable models of T are not products of a fixed model of T.

References

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