A note on "Atomic compactness in $\kappa$-categorical Horn theories" by John T. Baldwin

by

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Abstract. Every model of an $\omega_1$-categorical Horn theory is atomic compact.

In paper [2] (mentioned in the title above) Baldwin has proved the following theorem:

THEOREM 0. If $T$ is a theory satisfying the following three conditions:
(a) $T$ is almost strongly minimal,
(b) $T$ is an $\forall_3$-theory,
(c) $T$ is a Horn theory,
then every model of $T$ is atomic compact.

In this note we shall show that (a) can be replaced by a weaker assumption of $\omega_1$-categoricity of $T$, (see [1]); (b) can be omitted; and (c) can be replaced by the assumption that the class of all models of $T$ is closed under direct products. Hence we get the following theorem:

THEOREM 1. If $T$ is an $\omega_1$-categorical theory such that the class of all models of $T$ is closed under direct products, then every model of $T$ is atomic compact.

In the proof of Theorem 1, we shall use the following proposition (see e.g. [5], Th. 4) which results from a theorem in [4].

PROPOSITION 2. Let $I$ be an infinite set and $\mathcal{F}$ a filter on $I \times I$ generated by all the equivalence relations $E$ on $I$ such that $|E|$ is finite and all but one $E$-equivalence classes are finite. Then for every structure $\mathcal{A}$ we have $\mathcal{A}[\mathcal{F}] <^* \mathcal{A}$. Moreover $\text{Card}(\mathcal{A}[\mathcal{F}]) = \text{Card}(\mathcal{A}) \cdot \text{Card}(I)$.

Proof of Theorem 1. Let $\mathcal{A}$ be a model of $T$ and let $\Sigma = \Sigma(\mathcal{A}, \mathcal{X})$ be a set of atomic formulas with a sequence $\mathcal{X}$ of parameters from $\mathcal{A}$ and a sequence $\mathcal{X}$ of variables. Suppose that $\Sigma(\mathcal{A}, \mathcal{X})$ is finitely satisfiable in $\mathcal{A}$. Take $I$ such that $\text{Card}(I) \geq \max \{ \text{Card}(\mathcal{A}), \text{Card}(\mathcal{X}), \omega_1 \}$ and consider $\mathcal{A}[\mathcal{F}]$. Let $d$ be the diagonal embedding of $\mathcal{A}$ into $\mathcal{A}[\mathcal{F}]$. Of course $\Sigma(d(\mathcal{A}, \mathcal{X}))$ is finitely satisfiable in $\mathcal{A}[\mathcal{F}]$. ($\mathcal{A}[\mathcal{F}]$ denotes the limit power of $\mathcal{A}$, for more details see e.g. [4].

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By Proposition 2, \( W_1 | F_1 < W' \) holds. Let \( r \) be a mapping of \( W' \) onto \( W \) such that 
\( r(x) = x \) for all \( x \in A \).
Now take \( B > W_1 | F_1 \), which has the following properties:
1. \( \Sigma(d(\bar{v}, \bar{x})) \) is satisfiable in \( B \);
2. \( \text{Card}(\bar{v}) = \text{Card}(W'_1) \);
3. \( \text{Card}(\bar{v}) > \omega_1 \).
Since \( \text{Card}(\bar{v}) = \text{Card}(W'_1) > \omega_1 \), and \( T \) is \( \omega_1 \)-categorical, there is an isomorphism \( f \) of \( W' \) onto \( W_1 \). Since \( W_1 | F_1 < W_1 \), \( f \) maps \( W_1 | F_1 \) onto an elementary submodel of \( B \), say \( C \). Let \( g \) be an isomorphism of \( W_1 | F_1 \) onto \( C \) such that \( g = f \restriction (A_1 | F_1) \). Now \( B \) is an uncountable model of \( T \). Consequently \( B \) is saturated, whence homogeneous. Therefore there is an automorphism \( g^* \) of \( B \) such that 
\( g \preceq g^* \). In this way we get the following diagram:

\[
\begin{array}{ccc}
W & \xrightarrow{f} & W_1 | F_1 \\
\downarrow{r} & & \downarrow{r} \\
B & \xrightarrow{g} & C \\
\end{array}
\]

Now, \( \Sigma(d(\bar{v}, \bar{x})) \) is satisfiable in \( B \) by a sequence \( \bar{a} \) of elements of \( B \). So \( \Sigma(d(\bar{v}, a)) \) holds in \( B \).
Since \( g^* \) is an automorphism of \( B \), the set \( \Sigma(g^*d(\bar{v}, g^*(\bar{a})) \) holds in \( B \) too.
If we pass from \( B \) to \( W'_1 \) by \( f^{-1} \), we see that \( \Sigma(f^{-1}g^*d(\bar{v}, f^{-1}g^*(\bar{a})) \) holds in \( W' \). But \( f^{-1}g^*d(\bar{v}, f^{-1}g^*(\bar{a})) = d(\bar{v}) \) for each \( x \in A \). So \( \Sigma(d(\bar{v}, f^{-1}g^*(\bar{a})) \) holds in \( A \). Finally, applying \( r \), we come back to \( W \) and we see that \( \Sigma(\bar{r}(\bar{v}), r^{-1}g^*(\bar{a})) \) holds in \( W \).
Thus \( W \) is atomic compact. Q.E.D.

Remark 1. Baldwin's theorem and the Theorem just proved deal with theories in countable languages. This is not an essential restriction. Indeed, if the language \( \mathcal{L} \) of \( T \) is of the cardinality \( |\mathcal{L}| \geq \omega_1 \), then we can use Shelah's categoricity theorem (see e.g. [3]) for \( T \) to get (by the same proof with \( \omega_1 \) replaced by \( \lambda^+ \)) the following theorem:

**Theorem 4.** If \( T \) is an \( \omega_1 \)-categorical theory and the class of all models of \( T \) is closed under direct products, then for each countable model \( B \) of \( T \), every uncountable model \( W \) of \( T \) is isomorphic with \( W_1 | F_1 \), where \( \text{Card}(F_1) = \text{Card}(W_1) \).

Unfortunately the statement above is not true if \( A \) is countable. Indeed, let \( T \) be the theory of countably many distinct individual constants. If \( B \) is a model of \( T \), then \( B \) is saturated. Consequently other countable models of \( T \) are not products of a fixed model of \( T \).

References


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