On weaker forms of choice in second order arithmetic

by

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Abstract. In the paper it is proved that the scheme of choice in second order arithmetic for formulas involving parameters is strictly stronger than the analogical scheme for parameter-free formulas. The case of the scheme of dependent choices is different; both forms turn out to be equivalent.

In the present paper we shall discuss some implications between weaker forms of the schema of choice and the schema of dependent choices in second order arithmetic. The system of second order arithmetic (abbreviated by A_2^-) is usually formulated in a language with two sorts of variables — for natural numbers and for sets of them. The axioms are as follows:

1. Peano's axioms for numbers with induction as one statement.
2. Extensionality for sets.
3. Comprehension schema:

$$(\mathrm{E}X)(y)[y \in X \equiv \varphi]$$

for every formula φ in which the variable X does not occur. In φ we allow parameters.

We shall also discuss the following schemas:

4. Choice:

$$(x)(\mathrm{E}Y)\varphi(x, Y) \to (\mathrm{E}Y)(x)\varphi(x, (Y)_x),$$

where $(Y)_x = \{y\colon J(x, y) \in Y\}$ and J is an arithmetical pairing function.

5. Dependent choices:

$$(X)(\mathrm{E}Y)\varphi(X, Y) \to (\mathrm{E}X)(y)\varphi\big((X)_y, (X)_{y+1}\big).$$

We assume that φ in both 4 and 5 satisfies the necessary restrictions about variables occurring in it. The schema 4 will be abbreviated to AC if we do not allow parameters in φ and to **AC** if we allow them. The schema 5 will be abbreviated to DC and **DC**, respectively.

The following implications are obvious:

$$A_2^- \vdash \mathbf{AC} \to \mathrm{AC}, \quad A_2^- \vdash \mathbf{DC} \to \mathrm{DC}.$$

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We shall study the implications

$$A_2^- \vdash \mathrm{DC}\to\mathbf{DC}, \quad A_2^- \vdash \mathrm{AC}\to\mathbf{AC}.$$

The most important results of this paper are that the first of the above implications holds, while the second does not. The proof that $A_2^- \nvdash \mathrm{AC}\to\mathbf{AC}$ will use the method of forcing and will be carried out under the assumption of the existence of a standard model of ZF-set theory. It will be easily seen from the proof that the assumption

(*) there exists a standard model of $\mathrm{ZF}^- + \mathrm{V} = \mathrm{L} +$ "there exist at least ω_1 cardinal numbers"

is sufficient. The statement (*) can be proved in ZF.

Fact 1. (In A_2^-) $\mathrm{DC}\to\mathrm{AC}$ and $\mathbf{DC}\to\mathbf{AC}$.

An easy proof is left to the reader.

Sometimes the following schema is called the schema of dependent choices:

$$\mathrm{DC}'\colon \quad (X)(\mathrm{E}Y)\varphi(X,Y)\to(X)(\mathrm{E}Y)[(Y)_0 = X\,\&\,(z)\varphi((Y)_z,(Y)_{z+1})].$$

One can easily verify that

Fact 2. (In A_2^-) $\mathbf{DC}\equiv\mathbf{DC}'$.

THEOREM 3. *(In A_2^-)* $\mathrm{DC}\to\mathbf{DC}$.

Proof. Assume that DC holds while **DC** does not. Then there exist a formula φ and a set C such that

$$(X)(\mathrm{E}Y)\varphi(X,Y,C)\,\&\,(X)(\mathrm{E}n)\neg\varphi((X)_n,(X)_{n+1},C).)$$

Let us denote the above formula by $\Sigma(C)$. Thus we have $(\mathrm{E}C)\Sigma(C)$. Let us consider the following formula ψ:

$$\psi(X,Y)\equiv[\Sigma((X)_0)\,\&\,(Y)_0=(X)_0\,\&\,\varphi((X)_1,(Y)_1,(X)_0)]\vee$$
$$\vee[\neg\Sigma((X)_0)\,\&\,\Sigma((Y)_0)\,\&\,(Y)_1=0].$$

ψ is a formula without parameters. We shall show that $(X)(\mathrm{E}Y)\psi(X,Y)$.

Let us take an arbitrary X. There are two possible cases:

1. $\Sigma((X)_0)$. Then $(Z)(\mathrm{E}T)\varphi(Z,T,(X)_0)$ and we can find T such that $\varphi((X)_1,T,(X)_0)$. We take Y so that $(Y)_0=(X)_0$ and $(Y)_1=T$.

2. $\neg\Sigma((X)_0)$. Then we take Y so that $\Sigma((Y)_0)$ and $(Y)_1=0$.

Therefore we can find X such that $(n)\psi((X)_n,(X)_{n+1})$, because we have assumed DC. W.l.o.g. we can assume $\Sigma(((X)_0)_0)$. By induction we can easily prove that $((X)_n)_0=((X)_0)_0$. By the definition of ψ, for every n we have

$$\varphi(((X)_n)_1,((X)_{n+1})_1,C)\quad\text{where}\quad C=((X)_n)_0=((X)_0)_0.$$

We put Y so that $(Y)_n=((X)_n)_1$ and we obtain at the same time $(n)\varphi((Y)_n,(Y)_{n+1},C)$ and $\Sigma(C)$, a contradiction. ■

COROLLARY 4. *(In A_2^-)* $\mathrm{DC}\equiv\mathbf{DC}\equiv\mathbf{DC}'\equiv\mathrm{DC}'$.

The proof is immediate.

Now we are going to prove that an analogous theorem for AC does not hold. This will be done in two steps: the first is Theorem 5 and the second is Theorem 19.

THEOREM 5. *Let M be a countable transitive model of $\mathrm{ZF}+\mathrm{V}=\mathrm{L}$. Then there exists a model N of $\mathrm{ZF}+$ "$\omega_1=\omega^L_{\omega_1^L}$" such that the continuum of N forms a model of* AC.

Proof. The proof is technical and rather long, so we first give a brief idea of it. We add ω_1^M generic functions collapsing ω_α^M, $\alpha<\omega_1^M$, onto ω to M in order to obtain N. If $N\models(n)(\mathrm{E}y)\varphi(n,y)$ and φ is parameter-free, then $0\Vdash^*(n)(\mathrm{E}y)\varphi(n,y)$. Then for every $n\in\omega$ we can find a term t_n such that $p_n\Vdash\varphi(\underline{n},t_n)$ for some p_n. We can find an ordinal $\gamma<\omega_1^M$ such that all generic collapsing functions involved in any of the t_n's collapse cardinals smaller than ω_γ^M. Using a symmetry argument we shall show that the set

$$D_n=\{p\colon \text{for some term } t \text{ depending only on the } \gamma\text{th collapsing function,}$$
$$p\Vdash\varphi(\underline{n},t)\,\&\,t\subseteq\underline{\omega}\}$$

is dense. Therefore for every n we can find an example constructible from only the γth collapsing function and therefore we can find a sequence $\langle y_n\colon n\in\omega\rangle$ such that $(n)\varphi(n,y_n)$ holds in N. Now let us turn to the details.

We introduce a forcing language with new predicates $F_{\alpha,n}(\cdot,\cdot,\cdot)$ for $\alpha<\omega_1^M$ and $n\in\omega$. We put $\omega^M_{\omega_1}$ as the ordinal rank of these predicates. We define formulas and terms of the forcing language and their ordinal ranks as usual. It should be noticed that only a finite number of new predicates can occur in a formula or a term of the forcing language.

We define the order of a formula or a term as the maximum of ordinals α such that for some $n\in\omega$ the predicate $F_{\alpha,n}$ occurs in that formula or term.

A condition will be a function with a finite domain contained in $\omega_1^M\times\omega\times\omega_1^M\times\omega$ and the range contained in the class of ordinals such that

$$\langle\alpha,n,\beta,m\rangle\in\mathrm{dom}(p)\to\beta\leqslant\alpha\,\&\,p(\alpha,n,\beta,m)<\omega_\beta^M.$$

A condition p is stronger than a condition q (we write $p\leqslant q$) iff $p\supseteq q$.

The order of a condition p is the maximum of ordinals α such that $\langle\alpha,n,\beta,m\rangle\in\mathrm{dom}(p)$ for some n,β,m.

The forcing relation is defined as usual with the following initial clause:

$$p\Vdash F_{\alpha,n}(c_1,c_2,c_3)\text{ iff there exist }\beta\leqslant\alpha,\ m\in\omega\text{ and }\xi\in\omega_\beta^M\text{ such that }p(\alpha,n,\beta,m)$$
$$=\xi\text{ and }p\Vdash c_1\simeq\underline{\beta},\ p\Vdash c_2\simeq\underline{m},\ p\Vdash c_3\simeq\underline{\xi}.$$

The definition of $\simeq$, as well as the other clauses of the definition of forcing, can be found in [Fe].

We choose a complete sequence of conditions so that it intersects all those dense subsets of the set of conditions which belong to M. The resulting model will be denoted by N. By the general theory, $N\models\mathrm{ZF}$.

Now we shall prove that $N \models \omega_1 = \omega_{\omega_1^L}^L$. Since $M \models V = L$, $\omega_\alpha^N = \omega_\alpha^M$ in the model N. Let $\pi = \langle \pi_\alpha \colon \alpha < \omega_1^M \rangle$ be a sequence of permutations of ω which belongs to M. Then π acts on conditions, formulas and terms as follows:

$$\pi(p)(\alpha, n, \beta, m) = p(\alpha, \pi_\alpha^{-1}(n), \beta, m),$$

$$\pi(F_{\alpha,n}) = F_{\alpha,\pi_\alpha(n)}$$

and in other clauses as in [Fe]. As in [Fe] we prove the following lemmas:

LEMMA 6. *If $p \Vdash \varphi$, then $\pi(p) \Vdash \pi(\varphi)$.*

LEMMA 7. *If $I \subseteq \omega_1^M \times \omega$ is a set consisting of pairs $\langle \alpha, n \rangle$ such that $F_{\alpha,n}$ occurs in φ, then*

$$p \Vdash \varphi \rightarrow p \upharpoonright \operatorname{dom}(p) \cap I \times \omega_1^M \times \omega \ \Vdash^* \varphi .$$

LEMMA 8. *If c is a term, I is a set of pairs $\langle \alpha, n \rangle$ such that $F_{\alpha,n}$ occurs in c, $f_{\alpha,n}$ is the valuation of $F_{\alpha,n}$ in N and $\operatorname{val}_N(c) \subseteq M$, then $\operatorname{val}_N(c)$ is constructible from $\langle f_{\alpha,n} \colon \langle \alpha, n \rangle \in I \rangle$.*

From Lemma 8 it follows that $\omega_{\omega_1^M}^M$ is an uncountable cardinal in N. Namely, if $f\colon \omega \rightarrow \omega_{\omega_1^M}^M$ and $f \in N$, then f is constructible from finitely many functions $f_{\alpha,n}$. Let α_0 be the maximum of α's occurring as indices in the above functions. Then $\omega_{\alpha_0+1}^M$ is uncountable in $M[\langle f_{\alpha,n} \colon \langle \alpha, n \rangle \in I \rangle]$, and hence $\operatorname{rg}(f) \subseteq \omega_{\alpha_0+1}^M$.

On the other hand, $f_{\alpha,n}$ establishes a 1-1 correspondence between ω and ω_α^M, and hence $\omega_{\omega_1^M}^M$ is the least uncountable cardinal in N.

Now we have only to prove that AC holds in the continuum of N. We prove even a little more. Namely, if φ is a set-theoretical formula with only constructible parameters and

$$N \models (n)(\mathrm{E}x)[x \subseteq \omega \ \& \ \varphi(n, x)],$$

then there exists $f_{\gamma,0}$ such that

$$N \models (n)(\mathrm{E}x)[x \subseteq \omega \ \& \ x \in L[f_{\gamma,0}] \ \& \ \varphi(n, x)].$$

Since $L[f_{\gamma,0}]$ can be well ordered in N, we can define a sequence $\langle x_n \colon n \in \omega \rangle$ such that for all $n \in \omega$, $x_n \in L[f_{\gamma,0}]$, $x_n \subseteq \omega$ and $N \models \varphi(n, x_n)$. This sequence can be coded by a single subset of ω and hence the proof of AC will be completed.

Now let φ be such a formula. Then

$$0 \Vdash^* (n)(\mathrm{E}x)[x \subseteq \omega \ \& \ \varphi(n, x)]$$

because φ does not allow nonconstructible parameters. Therefore for every $n \in \omega$ we can find a condition p_n and a term c_n such that $p_n \Vdash \varphi(n, c_n)$ and $p_n \Vdash c_n \subseteq \omega$.

Let α_n be the order of p_n, let β_n be that of c_n and let us put $\gamma = \sup_{n \in \omega} \{\alpha_n, \beta_n\}$. Obviously $\gamma < \omega_1^M$.

Fix $n_0 \in \omega$.

It is enough to prove that the set D_{n_0} is dense, where $D_{n_0} = \{p\colon$ there exists a term c depending only on $F_{\gamma,0}$ such that $p \Vdash^* \varphi(\underline{n_0}, c)$ and $p \Vdash^* c \subseteq \underline{\omega}\}$.

At first we shall show that for every $k \in \omega$ there exist a condition $p_{n_0,k}$ and a term $c_{n_0,k}$ depending only on $F_{\gamma,k}$ such that $p_{n_0,k} \Vdash^* \varphi(\underline{n_0}, c_{n_0,k})$ and $p_{n_0,k} \Vdash^* c_{n_0,k} \subseteq \underline{\omega}$. By Lemma 7 we can also require that $p_{n_0,k}$ depend only on $\langle \gamma, k \rangle$, i.e., if $\alpha \neq \gamma$ or $m \neq k$, then $\langle \alpha, m, \beta, j \rangle \notin \operatorname{dom}(p_{n_0,k})$.

To simplify the notation we assume that $k = 0$ and the term c_{n_0} depends only on $F_{\alpha,0}$, $\alpha < \gamma$. The general case can be obtained by iterating the process described below. For the proof we shall construct an automorphism H of the set of conditions and we shall extend it to the set of formulas and terms so that $H(c_{n_0})$ depends only on $F_{\gamma,0}$ and for every condition p and every sentence φ, $p \Vdash^* \varphi \rightarrow H(p) \Vdash^* H(\varphi)$. Then we can put $p_{n_0,k} = H(p_{n_0})$ and $c_{n_0,k} = H(c_{n_0})$.

DEFINITION 9. We define a mapping H: Cond $\rightarrow$ Cond as follows:

$$H(p)(\xi, n, \eta, m) = p(\xi, n, \eta, m) \quad \text{if} \quad \xi \neq \alpha \ \& \ \xi \neq \gamma \text{ or } \xi = \gamma \ \& \ n \neq 0,$$

$$H(p)(\alpha, n, \eta, m) = p(\alpha, n+1, \eta, m),$$

$$H(p)(\gamma, 0, \eta, m) = p(\gamma, 0, \eta, m) \quad \text{if} \quad \alpha < \eta \leqslant \gamma,$$

$$H(p)(\gamma, 0, \eta, m) = p\big(\gamma, 0, \eta, \tfrac{1}{2}(m-1)\big) \quad \text{if} \quad \eta \leqslant \alpha \text{ and } 2 \nmid m,$$

$$H(p)(\gamma, 0, \eta, m) = p(\alpha, 0, \eta, \tfrac{1}{2}m) \quad \text{if} \quad \eta \leqslant \alpha \text{ and } 2 \mid m.$$

LEMMA 10. *The mapping H is an automorphism of the set of conditions.*

The easy proof is left to the reader.

We extend H to act on formulas and terms of the forcing language in the following way:

$$H(F_{\xi,n}(x, y, z)) = F_{\xi,n}(H(x), H(y), H(z)) \quad \text{if} \quad \xi \neq \alpha \ \& \ \xi \neq \gamma \text{ or } \xi = \gamma \ \& \ n \neq 0,$$

$$H(F_{\alpha,n}(x, y, z)) = F_{\alpha,n-1}(H(x), H(y), H(z)) \quad \text{if} \quad n \neq 0,$$

$$H(F_{\alpha,0}(x, y, z)) = \neg \underline{\alpha} \varepsilon H(x) \ \& \ (\mathrm{E}u)^\omega[F_{\gamma,0}(H(x), u, H(z)) \ \& \ \& \ u \varepsilon \underline{\omega} \ \& \ H(y) \varepsilon \underline{\omega} \ \& \ u =_\omega \underline{2} \cdot H(y)],$$

$$H(F_{\gamma,0}(x, y, z)) = [\underline{\alpha} \varepsilon H(x) \ \& \ F_{\gamma,0}(H(x), H(y), H(z))] \vee \vee [\neg \underline{\alpha} \varepsilon H(x) \ \& \ (\mathrm{E}u)^\omega[F_{\gamma,0}(H(x), u, H(z)) \ \& \ \& \ u \varepsilon \underline{\omega} \ \& \ H(y) \varepsilon \underline{\omega} \ \& \ u =_\omega \underline{2} \cdot H(y) + \underline{1}]],$$

where $x =_\omega y$ denotes $(z)^\omega[z \varepsilon x \equiv z \varepsilon y]$. Observe that both $+$ and $\cdot$ are always applied to natural numbers in $H(F_{\xi,n})$ and also in $G(F_{\xi,n})$ below.

The other clauses are as follows:

$H(x) = x$ if x is a variable or a constant term, i.e., $x = \underline{a}$ for some $a \in M$,

$H(E^\xi x \psi(x)) = E^\xi x H(\psi(x))$,

H is a homomorphism w.r.t. logical connectives, quantifiers and ε.

It should be observed that $H(c)$ has the same ordinal rank as c and that if c depends only on $F_{\alpha,0}$, then $H(c)$ depends only on $F_{\gamma,0}$.

Now it is enough to prove the symmetry lemma. However, we must first prove the following

LEMMA 11. *H is an automorphism on terms, i.e., there exists a mapping G on terms such that for every term c, $0 \Vdash H(G(c)) \simeq c$.*

Proof. We define G by induction as follows:

$$G(F_{\xi,n}(x,y,z)) = F_{\xi,n}(G(x),G(y),G(z)) \quad \text{if} \quad \xi \neq \alpha \,\&\, \xi \neq \gamma \text{ or } \xi = \gamma \,\&\, n \neq 0,$$

$$G(F_{\alpha,n}(x,y,z)) = F_{\alpha,n+1}(G(x),G(y),G(z)),$$

$$\begin{aligned} G(F_{\gamma,0}(x,y,z)) = {} & [\underline{\alpha}\,\varepsilon G(x) \,\&\, F_{\gamma,0}(G(x),G(y),G(z))] \vee \\ & \vee [\neg \underline{\alpha}\,\varepsilon G(x) \,\&\, (\mathrm{E}w)^{\omega}[F_{\gamma,0}(G(x),w,G(z)) \,\& \\ & \quad \& \, w \varepsilon \underline{\omega} \,\&\, G(y)\varepsilon\underline{\omega} \,\&\, G(y) =_{\omega} \underline{2}\cdot w + \underline{1}]] \vee \\ & \vee [\neg \underline{\alpha}\,\varepsilon G(x) \,\&\, (\mathrm{E}w)^{\omega}[F_{\alpha,0}(G(x),w,G(z)) \,\& \\ & \quad \& \, w \varepsilon \underline{\omega} \,\&\, G(y)\varepsilon\underline{\omega} \,\&\, G(y) =_{\omega} \underline{2}\cdot w]]. \end{aligned}$$

The other clauses are similar to those of H.

We assume that for every term c of the ordinal rank less than ξ, $0 \Vdash HG(c) \simeq c$. By induction on the length of the formula ψ of rank not bigger than ξ we prove that if $c_1, \ldots, c_n$ are terms of rank less than ξ then for every condition p

$$(**) \qquad p \Vdash^* \psi(c_1, \ldots, c_n) \equiv p \Vdash^* HG(\psi(c_1, \ldots, c_n)).$$

I. If ψ is an atomic formula $x \varepsilon y$ or $F_{\xi,n}$ for $\langle \xi, n\rangle \neq \langle \gamma, 0\rangle$, the proof is trivial.

II. $p \Vdash^* F_{\gamma,0}(c_1,c_2,c_3) \equiv p \Vdash^* HG(F_{\gamma,0}(c_1,c_2,c_3))$.

Let us take a condition p and assume the left-hand side. Let $q \leqslant p$ and let r be a condition such that $r \leqslant q$ and $r \Vdash^* F_{\gamma,0}(c_1,c_2,c_3)$.

Then there exist $\beta \leqslant \gamma$, $m \in \omega$ and $\xi \in \omega_\beta^M$ such that $r(\gamma, 0, \beta, m) = \xi$ and

$$r \Vdash c_1 \simeq \underline{\beta}, \qquad r \Vdash c_2 \simeq \underline{m} \quad \text{and} \quad r \Vdash c_3 \simeq \underline{\xi}.$$

Case 1. $\alpha < \beta$.

We shall show that

$$r \Vdash H(\underline{\alpha}\,\varepsilon G(c_1) \,\&\, F_{\gamma,0}(G(c_1),G(c_2),G(c_3))).$$

By the assumption $0 \Vdash HG(c_i) \simeq c_i$, $i = 1,2,3$. Since $\alpha < \beta$, $0 \Vdash \underline{\alpha}\,\varepsilon \underline{\beta}$. From the above $r \Vdash c_1 \simeq \underline{\beta}$. Hence $r \Vdash \underline{\alpha}\,\varepsilon HG(c_1)$. To show that

$$r \Vdash H(F_{\gamma,0}(G(c_1),G(c_2),G(c_3)))$$

it is enough to show that

$$r \Vdash \underline{\alpha}\,\varepsilon HG(c_1) \,\&\, F_{\gamma,0}(HG(c_1), HG(c_2), HG(c_3)).$$

As above $r \Vdash^* \underline{\alpha}\,\varepsilon HG(c_1)$. By the assumption, $r \Vdash F_{\gamma,0}(c_1,c_2,c_3)$ and $0 \Vdash HG(c_i) \simeq c_i$, $i = 1,2,3$. Hence

$$r \Vdash^* F_{\gamma,0}(HG(c_1), HG(c_2), HG(c_3)),$$

so r weakly (and hence strongly) forces the conjunction.

Case 2. $\alpha \geqslant \beta$, $2 \nmid m$.

We shall show that

$$r \Vdash H(\neg \underline{\alpha}\,\varepsilon G(c_1) \,\&\, (\mathrm{E}w)^{\omega}[F_{\gamma,0}(G(c_1),w,G(c_3)) \,\&\, w\varepsilon\underline{\omega} \,\& \,\& \, G(c_2)\varepsilon\underline{\omega} \,\&\, G(c_2) =_{\omega} \underline{2}\cdot w + \underline{1}]).$$

As before, we show that $r \Vdash^* \neg \underline{\alpha}\,\varepsilon HG(c_1)$. Since $2 \nmid m$, there exists $k \in \omega$ such that $m = 2k+1$. We show that

$$r \Vdash H(F_{\gamma,0}(G(c_1), \underline{k}, G(c_3))) \,\&\, \underline{k}\,\varepsilon\underline{\omega} \,\&\, HG(c_2)\varepsilon\underline{\omega} \,\&\, HG(c_2) =_{\omega} \underline{2}\cdot\underline{k} + \underline{1}.$$

By the assumption $0 \Vdash HG(c_2) \simeq c_2$. Hence $r \Vdash^* HG(c_2) \simeq \underline{m}$, and so $r \Vdash^* HG(c_2) =_{\omega} \underline{2}\cdot\underline{k}+\underline{1}$. It is trivial to show that $r \Vdash \underline{k}\,\varepsilon\underline{\omega}$ and $r \Vdash HG(c_2)\varepsilon\underline{\omega}$. We have only to show that

$$r \Vdash H(F_{\gamma,0}(G(c_1), \underline{k}, G(c_3))).$$

To do it we shall show that

$$r \Vdash \neg \underline{\alpha}\,\varepsilon HG(c_1) \,\&\, (\mathrm{E}u)^{\omega}[F_{\gamma,0}(HG(c_1), u, HG(c_3)) \,\&\, u\varepsilon\underline{\omega} \,\& \,\& \, \underline{k}\,\varepsilon\underline{\omega} \,\&\, u =_{\omega} \underline{2}\cdot\underline{k}+\underline{1}].$$

We put $u = \underline{m}$ and as before we can easily show that r forces all conjuncts in square brackets above.

Case 3. $\alpha \geqslant \beta$, $2 | m$.

In this case we proceed as before to show that r forces the third disjunct in the disjunction $HG(F_{\gamma,0})$.

Thus the proof of the implication to the right is completed.

Now let us take a condition p and assume the right-hand side. Let $q \leqslant p$ and let us take $r \leqslant q$ so that $r \Vdash HG(F_{\gamma,0}(c_1,c_2,c_3))$.

The proof will be finished when we find $s \leqslant r$ such that $s \Vdash^* F_{\gamma,0}(c_1,c_2,c_3)$.

Case 1. $r \Vdash H(\underline{\alpha}\,\varepsilon G(c_1) \,\&\, F(G(c_1),G(c_2),G(c_3)))$.

In this case $r \Vdash^* \underline{\alpha}\,\varepsilon HG(c_1)$ and $r \Vdash^* H(F_{\gamma,0}(G(c_1),G(c_2),G(c_3)))$.

Thus we can find a condition $t \leqslant r$ such that

$$t \Vdash H(F_{\gamma,0}(G(c_1),G(c_2),G(c_3))).$$

Subcase 1a. $t \Vdash \underline{\alpha}\,\varepsilon HG(c_1) \,\&\, F_{\gamma,0}(HG(c_1), HG(c_2), HG(c_3))$. Since $0 \Vdash HG(c_i) \simeq c_i$, $i = 1,2,3$, we have $t \Vdash^* F_{\gamma,0}(c_1,c_2,c_3)$ and we can put $s = t$.

Subcase 1b. $t \Vdash \neg \underline{\alpha}\,\varepsilon HG(c_1) \,\&\, (\mathrm{E}w)^{\omega}[\ldots]$. This subcase is impossible because then $t \Vdash^* \neg \underline{\alpha}\,\varepsilon HG(c_1)$ and we have already shown that $r \Vdash^* \underline{\alpha}\,\varepsilon HG(c_1)$ and $t \leqslant r$.

Case 2. $r \Vdash H(\neg \underline{\alpha}\,\varepsilon G(c_1) \,\&\, (\mathrm{E}w)^{\omega}[F_{\gamma,0}(G(c_1),w,G(c_3)) \,\&\, w\varepsilon\underline{\omega} \,\&\, G(c_2)\varepsilon\underline{\omega} \,\& \,\& \, G(c_2) =_{\omega} \underline{2}\cdot w+\underline{1}])$.

Now it follows that there exist a condition $t \leqslant r$ and a term c_4 of ordinal rank less than ω such that

$$t \Vdash \neg \underline{\alpha}\,\varepsilon HG(c_1), \qquad t \Vdash H(F_{\gamma,0}(G(c_1), c_4, G(c_3))),$$

$$t \Vdash c_4 \varepsilon \underline{\omega}, \qquad t \Vdash HG(c_2)\varepsilon\underline{\omega} \quad \text{and} \quad t \Vdash HG(c_2) =_{\omega} \underline{2}\cdot c_4 + \underline{1}.$$

It follows that there exist $k, m \in \omega$ such that

$$t \Vdash c_4 \simeq \underline{k} \quad \text{and} \quad t \Vdash HG(c_2) \simeq \underline{m}\,.$$

Since $t \Vdash HG(c_2) =_\omega \underline{2}\cdot c_4 + \underline{1}$, $m = 2k+1$.

There are two subcases:

Subcase 2a. $t \Vdash \underline{\alpha}\, \varepsilon\, HG(c_1) \,\&\, F_{\gamma,0}(HG(c_1), c_4, HG(c_3))$. This subcase is impossible, because we have already proved that $t \Vdash \neg \underline{\alpha}\, \varepsilon\, HG(c_1)$.

Subcase 2b.

$$t \Vdash \neg \underline{\alpha}\, \varepsilon\, HG(c_1) \,\&\, (\mathrm{E}u)^\omega [F_{\gamma,0}(HG(c_1), u, HG(c_3)) \,\&\, u\, \varepsilon\, \underline{\omega} \,\&\, H(c_4)\, \varepsilon\, \underline{\omega} \,\& \\ \& \, u =_\omega H(c_4)\cdot \underline{2} + \underline{1}]\,.$$

But $H(c_4) = c_4$, because no new predicates can occur in c_4 (the rank of c_4 is less than ω). Thus there exist a condition $t_1 \leqslant t$ and a term c_5 of ordinal rank less than ω such that

$$t_1 \Vdash F_{\gamma,0}(HG(c_1), c_5, HG(c_3))\,, \quad t_1 \Vdash c_4\, \varepsilon\, \underline{\omega}\,,$$
$$t_1 \Vdash c_5\, \varepsilon\, \underline{\omega} \quad \text{and} \quad t_1 \Vdash c_5 =_\omega \underline{2}\cdot c_4 + \underline{1}\,.$$

Therefore there exist $k_1, m_1 \in \omega$ such that $t_1 \Vdash c_4 \simeq \underline{k_1}$ and $t_1 \Vdash c_5 \simeq \underline{m_1}$. Hence $t_1 \Vdash^* \underline{m_1} =_\omega \underline{2}\cdot \underline{k_1} + \underline{1}$, so $m_1 = 2k_1 + 1$. Since $t \Vdash c_4 \simeq \underline{k}$ and $t_1 \Vdash c_4 \simeq \underline{k_1}$, $k = k_1$ and consequently $m = m_1$. Hence $t_1 \Vdash^* c_5 \simeq HG(c_2)$, so $t_1 \Vdash c_5 \simeq c_2$. By the assumption $0 \Vdash HG(c_i) \simeq c_i$, $i = 1, 2, 3$. Therefore $t_1 \Vdash^* F_{\gamma,0}(c_1, c_2, c_3)$ and so there is $s \leqslant t_1$ which strongly forces $F_{\gamma,0}(c_1, c_2, c_3)$.

Case 3. $r \Vdash H(\neg \underline{\alpha}\, \varepsilon\, G(c_1) \,\&\, (\mathrm{E}w)^\omega [F_{\alpha,0}(G(c_1), w, G(c_3)) \,\&\, w\, \varepsilon\, \underline{\omega} \,\&\, G(c_2)\, \varepsilon\, \underline{\omega} \,\& \,\& \, G(c_2) =_\omega \underline{2}\cdot w])$.

In this case we can proceed as before, so we omit the proof.

In this way the proof of II is completed.

III. $\psi \equiv \neg \psi_1$.

IV. $\psi \equiv \psi_1 \vee \psi_2$.

These two cases are trivial and we omit the proofs.

V. $\psi(x_1, \ldots, x_n) \equiv (\mathrm{E}x_0)^\eta \psi_1(x_0, x_1, \ldots, x_n)$.

We have to prove that if $c_1, \ldots, c_n$ are terms of ranks less than ξ then for any p $p \Vdash^* \psi(c_1, \ldots, c_n) \equiv p \Vdash^* HG(\psi(c_1, \ldots, c_n))$.

Let us take a condition p and assume the left-hand side. Let $q \leqslant p$ and r be such that $r \leqslant q$ and $r \Vdash \psi(c_1, \ldots, c_n)$.

Then there exists c_0 of ordinal rank less than η such that $r \Vdash \psi_1(c_0, c_1, \ldots, c_n)$. Since the rank of c_0 is less than ξ, our assumption about c_0 is fulfilled. By the assumption about ψ_1, $r \Vdash HG(\psi_1(c_0, c_1, \ldots, c_n))$, i.e.,

$$r \Vdash HG(\psi_1)(HG(c_0), HG(c_1), \ldots, HG(c_n)).$$

Therefore there exists a condition $r_1 \leqslant r$ such that

$$r_1 \Vdash (\mathrm{E}x_0) HG(\psi_1(x_0, x_1, \ldots, x_n))(HG(c_1), \ldots, HG(c_n))$$

(because $HG(c_0)$ has the same rank as c_0) and consequently $r_1 \Vdash HG(\psi(c_1, \ldots, c_n))$.

The proof of the converse implication is similar, and so we omit it.

Therefore the proof of (**) is completed.

Now let c be a term of rank ξ. We have to prove that $0 \Vdash HG(c) \simeq c$.

The only interesting case is when $c = E^\xi x \psi(x)$, where ψ is a formula of rank not bigger than ξ. Then $HG(c) = E^\xi x\, HG(\psi(x))$. If c_1 is any term of rank less than ξ, then $0 \Vdash \psi(c_1) \equiv HG(\psi(c_1))$. Hence

$$0 \Vdash \psi(c_1) \equiv HG(\psi(x))(HG(c_1))\,,$$

and so $0 \Vdash \psi(c_1) \equiv HG(\psi(x))(c_1)$, because $0 \Vdash HG(c_1) \simeq c_1$. Therefore $0 \Vdash c \simeq HG(c)$. ■

Corollary 12. *For every term c of rank ξ there exists a term c_1 of the same rank such that* $0 \Vdash H(c_1) \simeq c$.

The proof is immediate.

Lemma 13. *If p is a condition and ψ is a sentence of the forcing language then* $p \Vdash^* \psi \equiv H(p) \Vdash^* H(\psi)$.

Proof. By induction on the ordinal rank of ψ.

When ψ is a negation or a disjunction the proof is trivial. In the cases of quantifiers we use Corollary 12 (for the implication to the left).

Thus we shall only prove the case of atomic sentences. Sentences of the form $c\, \varepsilon\, \underline{a}$, $c\, \varepsilon\, E^\xi x \psi_1(x)$ and $F_{\xi,n}(c_1, c_2, c_3)$ for $\langle \xi, n\rangle \neq \langle \alpha, 0\rangle$ or $\langle \xi, n\rangle \neq \langle \gamma, 0\rangle$ are easy to check.

I. $\psi = F_{\alpha,0}(c_1, c_2, c_3)$.

Let p be a condition and assume that $p \Vdash^* F_{\alpha,0}(c_1, c_2, c_3)$. Let $q \leqslant H(p)$ and take $r \leqslant H^{-1}(q)$ such that $r \Vdash F_{\alpha,0}(c_1, c_2, c_3)$. Then there exist $\beta \leqslant \alpha$, $m \in \omega$ and $\xi \in \omega_\beta^M$ such that $r(\alpha, 0, \beta, m) = \xi$, and

$$r \Vdash c_1 \simeq \underline{\beta}\,, \quad r \Vdash c_2 \simeq \underline{m} \quad \text{and} \quad r \Vdash c_3 \simeq \underline{\xi}\,.$$

Since $\alpha \geqslant \beta$, $0 \models \neg \underline{\alpha}\, \varepsilon\, \underline{\beta}$. By the inductional hypothesis $(r) \Vdash^* H(c_1) \simeq \underline{\beta}$, $H$and hence $H(r) \Vdash \neg \underline{\alpha}\, \varepsilon\, H(c_1)$.

Now it will be sufficient to prove that

$$H(r) \Vdash F_{\gamma,0}(H(c_1), \underline{2m}, H(c_3))\,, \quad H(r) \Vdash H(c_2)\, \varepsilon\, \underline{\omega}\,,$$
$$H(r) \Vdash \underline{2m}\, \varepsilon\, \underline{\omega} \quad \text{and} \quad H(r) \Vdash \underline{2m} =_\omega \underline{2}\cdot H(c_2)\,.$$

(a) Since $r(\alpha, 0, \beta, m) = \xi$, $H(r)(\gamma, 0, \beta, 2m) = \xi$. By the induction hypothesis $H(r) \Vdash^* H(c_1) \simeq \underline{\beta}$ and $H(r) \Vdash^* H(c_3) \simeq \underline{\xi}$, and hence $H(r)$ also strongly forces these statements. Thus

$$H(r) \Vdash F_{\gamma,0}(H(c_1), \underline{2m}, H(c_3))\,.$$

(b) By the induction hypothesis $H(r) \Vdash H(c_2) \simeq \underline{m}$, and so $H(r) \Vdash H(c_2)\, \varepsilon\, \underline{\omega}$.

(c) Trivial.

(d) Since $H(r) \Vdash \underline{2m} =_\omega \underline{2}\cdot \underline{m}$ and $H(r) \Vdash H(c_2) \simeq \underline{m}$, also

$$H(r) \Vdash \underline{2m} =_\omega \underline{2}\cdot H(c_2)\,.$$

For the proof of the implication to the left, let us assume that $H(p) \Vdash^* \Vdash^* H(F_{\alpha,0}(c_1, c_2, c_3))$.

Let $q \leqslant p$ and take $r_1 \leqslant H(q)$ so that $r_1 \Vdash H(F_{\alpha,0}(c_1, c_2, c_3))$. Then $r_1 = H(r)$ for some $r \leqslant q$. Moreover, we can demand that $H(r) \Vdash \neg \underline{\alpha}\, \varepsilon H(c_1)$ and that there exist a term c_4 of rank less than ω such that:

$$H(r) \Vdash F_{\gamma,0}(H(c_1), c_4, H(c_3)), \quad H(r) \Vdash H(c_2)\, \varepsilon\, \underline{\omega},$$
$$H(r) \Vdash c_4\, \varepsilon\, \underline{\omega} \quad \text{and} \quad H(r) \Vdash c_4 =_\omega \underline{2}\cdot H(c_2).$$

Thus there exist $\beta \leqslant \gamma$, $m \in \omega$, $\xi < \omega_\beta^M$ and $k \in \omega$ such that:

$$H(r)(\gamma, 0, \beta, m) = \xi, \quad H(r) \Vdash H(c_1) \simeq \underline{\beta}, \quad H(r) \Vdash c_4 \simeq \underline{m},$$
$$H(r) \Vdash H(c_3) \simeq \underline{\xi} \quad \text{and} \quad H(r) \Vdash H(c_2) \simeq \underline{k}.$$

Since $H(r) \Vdash \neg \underline{\alpha}\, \varepsilon H(c_1)$ and $H(r) \Vdash H(c_1) \simeq \underline{\beta}$, we obtain $\alpha \geqslant \beta$. Since $H(r) \Vdash c_4 =_\omega \underline{2}\cdot H(c_2)$, we have $H(r) \Vdash \underline{m} =_\omega \underline{2}\cdot \underline{k}$, so $m = 2k$.

Therefore $r(\alpha, 0, \beta, k) = \xi$. By the induction hypothesis

$$r \Vdash c_1 \simeq \underline{\beta}, \quad r \Vdash c_2 \simeq \underline{k} \quad \text{and} \quad r \Vdash c_3 \simeq \underline{\xi}.$$

Hence $r \Vdash F_{\alpha,0}(c_1, c_2, c_3)$, which completes the proof of this case.

II. $\psi = F_{\gamma,0}(c_1, c_2, c_3)$.

The proof is similar, and so it will be omitted.

The proof of Lemma 13 is now completed. ■

COROLLARY 14. *For every $n, k \in \omega$ there exist a condition $p_{n,k}$ such that $\langle \xi, m, \eta, l\rangle \in \mathrm{dom}(p_{n,k}) \to \xi = \gamma \;\&\; m = k$ and a term $c_{n,k}$ depending only on $F_{\gamma,k}$ such that $p_{n,k} \Vdash^* \varphi(\underline{n}, c_{n,k}) \;\&\; c_{n,k} \subseteq \underline{\omega}$.*

The proof is trivial by Lemma 13.

Now we are going to show that the set D_{n_0} (defined on the page 134) is dense. Let q be any condition. Let $k \in \omega$ be a natural number such that if $\langle \gamma, i, \eta, j\rangle \in \mathrm{dom}(q)$, then $i, j < k$. Let us consider $p_{n_0,k}$ and $c_{n_0,k}$. We shall find an automorphism J of the set of conditions and extend it to the set of terms so that $J(c_{n_0,k})$ will depend only on $F_{\gamma,0}$, $J(p_{n_0,k})$ and q will have disjoint domains and $J(p_{n_0,k}) \Vdash^* \Vdash^* \varphi(\underline{n}, J(c_{n_0,k})) \;\&\; J(c_{n_0,k}) \subseteq \underline{\omega}$. This will be enough, because every common extension of $J(p_{n_0,k})$ and q belongs to D_{n_0}.

We could, of course, find one automorphism instead of two: H and J, but then the definition of it would be very complicated. So, in order to facilitate the reading of this paper and to make the idea clearer, we have decided to consider each of them separately.

DEFINITION 15. We define a mapping J in the following way:

$$\begin{array}{lll} J(p)(\xi, i, \eta, j) = p(\xi, i, \eta, j) & \text{if} & \xi \neq \gamma \text{ or} \\ & & \xi = \gamma \text{ and } 0 < i < k, \\ J(p)(\gamma, i, \eta, j) = p(\gamma, i+1, \eta, j) & \text{if} & i \geqslant k, \\ J(p)(\gamma, 0, \eta, j) = p(\gamma, 0, \eta, j) & \text{if} & j < k, \\ J(p)(\gamma, 0, \eta, j) = p(\gamma, 0, \eta, \tfrac{1}{2}(j+k)) & \text{if} & j \geqslant k \text{ and } 2 \mid j+k, \\ J(p)(\gamma, 0, \eta, j) = p(\gamma, k, \eta, \tfrac{1}{2}(j-k-1)) & \text{if} & j \geqslant k \text{ and } 2 \nmid j-k. \end{array}$$

LEMMA 16. *J is an automorphism of the set of conditions such that $\mathrm{dom}(J(p_{n_0,k})) \cap \mathrm{dom}(q) = 0$.*

The proof is left to the reader.

We extend J to act on formulas and terms of the forcing language in the following way:

$$\begin{array}{ll} J(F_{\xi,i}(x, y, z)) = F_{\xi,i}(J(x), J(y), J(z)) & \text{if } \xi \neq \gamma \text{ or } \xi = \gamma \;\&\; 0 < i < k, \\ J(F_{\gamma,j}(x, y, z)) = F_{\gamma,j-1}(J(x), J(y), J(z)) & \text{if } j > k, \\ J(F_{\gamma,k}(x, y, z)) = (\mathrm{E}u)^\omega[F_{\gamma,0}(J(x), u, J(z)) \;\&\; u\,\varepsilon\,\underline{\omega} \;\&\; J(y)\,\varepsilon\,\underline{\omega} \;\&\; & \\ \qquad \& \; u =_\omega \underline{2}\cdot J(y) + \underline{k} + \underline{1}], & \\ J(F_{\gamma,0}(x, y, z)) = [J(y)\,\varepsilon\,\underline{k} \;\&\; F_{\gamma,0}(J(x), J(y), J(z))] \vee & \\ \qquad \vee [J(y)\,\varepsilon\,\underline{\omega} - \underline{k} \;\&\; (\mathrm{E}u)^\omega[F_{\gamma,0}(J(x), u, J(z)) \;\&\; & \\ \qquad \& \; u\,\varepsilon\,\underline{\omega} \;\&\; u =_\omega \underline{2}\cdot J(y) - \underline{k}]]. & \end{array}$$

Other clauses are the same as those of H, given on the page 135.

It should be observed that $J(c_{n_0,k})$ depends only on $F_{\gamma,0}$. To finish the proof of Theorem 5 it is enough to prove that $p \Vdash^* \psi \equiv J(p) \Vdash^* J(\psi)$ for any condition p and any sentence ψ of the forcing language.

The proof is similar to that for H. We must first prove the following

LEMMA 17. *For every term c there exists a term c_1 of the same rank such that $0 \Vdash J(c_1) \simeq c$.*

Proof. We define a converse mapping:

$$\begin{array}{ll} K(F_{\xi,i}(x, y, z)) = F_{\xi,i}(K(x), K(y), K(z)) & \text{if } \xi \neq \gamma \text{ or } \xi = \gamma \;\&\; 0 < i < k, \\ K(F_{\gamma,i}(x, y, z)) = F_{\gamma,i+1}(K(x), K(y), K(z)) & \text{if } i \geqslant k, \\ K(F_{\gamma,0}(x, y, z)) = [K(y)\,\varepsilon\,\underline{k} \;\&\; F_{\gamma,0}(K(x), K(y), K(z))] \vee & \\ \qquad \vee [K(y)\,\varepsilon\,\underline{\omega} - \underline{k} \;\&\; (\mathrm{E}w)^\omega[F_{\gamma,k}(K(x), w, K(z)) \;\&\; & \\ \qquad \& \; w\,\varepsilon\,\underline{\omega} \;\&\; K(y) =_\omega \underline{2}\cdot w + \underline{k} + \underline{1}]] \vee & \\ \qquad \vee [K(y)\,\varepsilon\,\underline{\omega} - \underline{k} \;\&\; (\mathrm{E}w)^\omega[F_{\gamma,0}(K(x), w, K(z)) \;\&\; & \\ \qquad \& \; w\,\varepsilon\,\omega \;\&\; K(y) =_\omega \underline{2}\cdot w - \underline{k}]]. & \end{array}$$

Other clauses are as on page 135.

By induction on the rank of c we prove that $0 \Vdash JK(c) \simeq c$. Since the proof is similar to that of Lemma 11, we shall omit it. ■

LEMMA 18. *If p is a condition and ψ is a sentence of the forcing language, then $p \Vdash^* \psi \equiv J(p) \Vdash^* J(\psi)$.*

Proof. Since the proof is similar to that of Lemma 13, we shall prove, as an example, one of the initial cases:

$$p \Vdash^* F_{\gamma,k}(c_1, c_2, c_3) \equiv J(p) \Vdash^* J(F_{\gamma,k}(c_1, \bar{c}_2, c_3)).$$

Let us take a condition p.

I. Assume the left-hand side and take any $q_1 \leqslant J(p)$. Then $q_1 = J(q)$ for some $q \leqslant p$. We can take $r \leqslant q$ so that $r \Vdash F_{\gamma,k}(c_1, c_2, c_3)$.

Then there exist $\beta \leqslant \gamma$, $m \in \omega$ and $\xi < \omega_\beta^M$ such that $r(\gamma, k, \beta, m) = \xi$ and

$$r \Vdash c_1 \simeq \underline{\beta}\,, \quad r \Vdash c_2 \simeq \underline{m} \quad \text{and} \quad r \Vdash c_3 \simeq \underline{\xi}\,.$$

Take $l = 2m+k+1$. It is enough to show that

$$J(r) \Vdash F_{\gamma,0}(J(c_1), \underline{l}, J(c_3))\,, \quad J(r) \Vdash \underline{l} \varepsilon \underline{\omega}\,,$$
$$J(r) \Vdash J(c_2) \varepsilon \underline{\omega} \quad \text{and} \quad J(r) \Vdash \underline{l} =_\omega \underline{2} \cdot J(c_2) + \underline{k} + \underline{1}\,.$$

By the induction hypothesis $J(r) \Vdash J(c_2) \simeq \underline{m}$, and hence

$$J(r) \Vdash J(c_2) \varepsilon \underline{\omega} \quad \text{and} \quad J(r) \Vdash \underline{l} =_\omega \underline{2} \cdot J(c_2) + \underline{k} + 1\,.$$

It is obvious that $J(r) \Vdash \underline{l} \varepsilon \underline{\omega}$.

By the induction hypothesis

$$J(r) \Vdash J(c_1) \simeq \underline{\beta} \quad \text{and} \quad J(r) \Vdash J(c_3) \simeq \underline{\xi}\,.$$

Moreover, $J(r)(\gamma, 0, \beta, 1) = r(\gamma, k, \beta, m) = \xi$, since $2m+k+1 \geqslant k$ and $2 \not| 2m+ +k+1-k$. Hence

$$J(r) \Vdash F_{\gamma,0}(J(c_1), \underline{l}, J(c_3))\,.$$

On the other hand, let $J(p) \Vdash^* J(F_{\gamma,k}(c_1, c_2, c_3))$. Let us take $q \leqslant p$ and $r_1 \leqslant J(q)$ so that $r_1 \Vdash J(F_{\gamma,k}(c_1, c_2, c_3))$. Then for some $r \leqslant q$, $r_1 = J(r)$. It is enough to show that $s \Vdash F_{\gamma,k}(c_1, c_2, c_3)$ for some $s \leqslant r$. By the above there exists a term c_4 of rank less than ω such that

$$J(r) \Vdash F_{\gamma,0}(J(c_1), c_4, J(c_3))\ \&\ c_4 \varepsilon \underline{\omega}\ \&\ J(c_2) \varepsilon \underline{\omega}\ \&\ c_4 =_\omega \underline{2} \cdot J(c_2) + \underline{k} + \underline{1}\,.$$

Let s be such that $s \leqslant r$ and:

$$J(s) \Vdash F_{\gamma,0}(J(c_1), c_4, J(c_3))\,, \quad J(s) \Vdash c_4 \varepsilon \underline{\omega}\,,$$
$$J(s) \Vdash J(c_2) \varepsilon \underline{\omega} \quad \text{and} \quad J(s) \Vdash c_4 =_\omega \underline{2} \cdot J(c_2) + \underline{k} + \underline{1}\,.$$

Then there exist $\beta \leqslant \gamma$, $m \in \omega$, $\xi \in \omega_\beta^M$ and $l \in \omega$ such that:

$$J(s)(\gamma, 0, \beta, m) = \xi\,, \quad J(s) \Vdash J(c_1) \simeq \underline{\beta}\,, \quad J(s) \Vdash c_4 \simeq \underline{m}\,,$$
$$J(s) \Vdash J(c_3) \simeq \underline{\xi} \quad \text{and} \quad J(s) \Vdash J(c_2) \simeq \underline{l}\,.$$

Hence $J(s) \Vdash \underline{m} =_\omega \underline{2} \cdot \underline{1} + \underline{k} + \underline{1}$, and so $m = 2l+k+1$. Therefore $s(\gamma, k, \beta, l) = \xi$ and by the induction hypothesis

$$s \Vdash c_1 \simeq \underline{\beta}\,, \quad s \Vdash c_2 \simeq \underline{1} \quad \text{and} \quad s \Vdash c_3 \simeq \underline{\xi}\,,$$

and so $s \Vdash F_{\gamma,k}(c_1, c_2, c_3)$, which completes the proof.

The proof of Lemma 18 and therefore of Theorem 5 is thus completed. ■

Now we have to prove that the continuum of the model N of Theorem 5 is not a model of **AC**.

THEOREM 19. *Assume* ZF + "$\omega_1 = \omega^L_{\omega_1^L}$". *Then* **AC** *does not hold for some* Π^1_2*-formula.*

Proof. We follow the proof from [Le]. Let $a \subseteq \omega^2$ be a well-ordering of ω of type ω_1^L. Such an a exists because $\omega_1^L < \omega^L_{\omega_1^L} = \omega_1$.

By $a \upharpoonright n$ we shall denote the initial segment of a consisting of the a-predecessors of n.

By $(a \upharpoonright n)+1$ we denote the initial segment of a consisting of the a-predecessors of n together with n.

Let us consider the following formula Φ:

$\Psi(n, x, a) \equiv x \subseteq \omega^2\ \&\ x$ is a well-ordering of ω of type at least ω_α^L, where α is the type of $a \upharpoonright n$.

It is easy to see that the countable axiom of choice fails for Φ with the parameter a.

Using the argument of Lévy ([Le], p. 133, 134) we can show that the statement

$\Psi(n, x, a) \equiv (y)(z)[x$ is a well-ordering of natural numbers & y codes a well-founded binary relation $\in'$ on ω & $(\sigma\ \&\ \psi)$ holds in the model $\langle \omega, \in' \rangle$ & (k) [k is an ordinal in $\langle \omega, \in' \rangle$ & z is an isomorphism of the field of x on the ordinals of $\langle \omega, \in' \rangle$ which stand in the relation $\in'$ to $k \to$ there exists an embedding t of the field of $(a \upharpoonright n)+1$ into cardinal numbers of $\langle \omega, \in' \rangle$ such that for every i from the field of $(a \upharpoonright n)+1$, $t(i) \in' k$ or $t(i) = k$]]

is equivalent to Φ and is a Π^1_2-formula.

As in [Le], σ denotes the axiom of extensionality and ψ denotes a formula such that if $\langle A, \in \rangle \Vdash \psi$, then $A = L_\alpha$ for a limit ordinal α and conversely. ■

From Theorem 3 it follows that DC also fails in model N, but the proof of that theorem does not give a good estimation of the class of a formula φ, for which it fails.

THEOREM 20. *Assume* ZF + "$\omega_1 = \omega^L_{\omega_1^L}$". *Then* DC *fails for some* Π^1_2*-formula.*

Proof. Consider the following formula:

$\Phi(x, y) \equiv [x = 0\ \&\ (y)_1$ is a well-ordering of ω of type $\geqslant \omega_1^L\ \&\ (y)_2 = 1\ \&\ (y)_3$ is a well-ordering of ω of type $\geqslant \omega_\alpha^L$, where α is the type of $(y)_1 \upharpoonright 1] \vee$ $\vee [(x)_1$ is a well-ordering of ω of type $\geqslant \omega_1^L\ \&\ (x)_2$ is a singleton say $\{n\}\ \&\ (x)_3$ is a well-ordering of ω of type $\geqslant \omega_\alpha^L$, where α is the type of $(x)_1 \upharpoonright m\ \&\ (y)_1 = (x)_1\ \&\ (y)_2 = \{n+1\}\ \&\ (y)_3$ is a well-ordering of ω of type $\geqslant \omega_\beta^L$, where β is the type of $(y)_1 \upharpoonright n+1] \vee [x$ is not as in the above cases & $y = 0]$.

It is easy to check that DC does not hold for Φ and that Φ is a Σ_3^1-formula. Hence Σ_3^1-DC fails and therefore so does Π_2^1-DC. ■

Final remarks. In the paper we have shown that in second order arithmetic DC ≡ **DC**, AC ↮ **AC**, and AC ↮ DC.

The only remaining question is whether AC→**DC**. This problem has been answered negatively by S. G. Simpson ([Si]). Simpson's proof, however is not known to the author of this paper.

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Movability and shape-connectivity

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Abstract. THEOREM 1. *If* (X, x) *is a uniformly movable pointed continuum with* $\underline{\pi}_n(X, x) = 0$ *for all* n, *then* (X, x) *has trivial shape.* From this and the fact that a metric continuum X is approximately 1-connected if and only if it is the inverse limit of a sequence of simply connected ANR's, one obtains the corollary: *An approximately* 1-*connected movable metric continuum* X *with* $\underline{\pi}_n(X) = 0$, *for all* n, *has the shape of a point.* Another corollary is that the concept of uniform movability introduced in [12] is stronger than movability.

Introduction. In this paper we obtain a special case of a shape version of the Whitehead Theorem without a dimension restriction.

THEOREM 1. *If* (X, x) *is a uniformly movable pointed continuum with* $\underline{\pi}_n(X, x) = 0$, *for all* n, *then* (X, x) *has trivial shape.*

Uniform movability here is taken in the sense of Kozlowski–Segal [12] which is a generalization of the concept of uniform movability defined by M. Moszyńska in [17] and which coincides for metric compacta with K. Borsuk's concept of movability [3]. As a corollary of Theorem 1 we show in Section 3 that a certain compact connected topological group is movable but not uniformly movable. This example is inspired by and heavily depends on the work of J. Keesling. As another application we have

COROLLARY. *An approximately* 1-*connected movable metric continuum* X *with* $\underline{\pi}_n(X) = 0$, *for all* n, *has the shape of a point.*

In this paper a compactum means a compact Hausdorff space, continuum means a connected compactum. All ANR's are understood to be compact. As a reference for the ANR-system approach to shape see [15]. *We assume that when we deal with a continuum the* ANR-*system associated with it is composed of connected* ANR'*s*. As a reference for the shape groups $\underline{\pi}_n$ see [16] where their isomorphism with the limit homotopy groups is established. Here we deal with only the latter groups which we accordingly take as the definition of the $\underline{\pi}_n$'s: if the ANR-system $\{(X_\alpha, x_\alpha), p_{\alpha\alpha'}, \mathscr{A}\}$ is associated with (X, x), then $\underline{\pi}_n(X, x)$ is defined to be $\underleftarrow{\lim}\{\pi_n(X_\alpha, x_\alpha), p_{\alpha\alpha'\#}, \mathscr{A}\}$. *In dealing with maps between* ANR'*s and their induced homomorphisms between homotopy groups we shall omit reference to base-points.*

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