# A homological characterization of graded complete intersections, II

by

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**Abstract.** Two theorems are proved in support of the following conjecture:

Let $A$ be a graded commutative connected algebra over a field $K$. If the homology $\mathrm{Tor}^A_{**}(K, K)$ is a finitely generated bigraded algebra with divided powers over $K$, then $A$ is a graded complete intersection.

§ **1.** Throughout this paper the term "$K$-algebra" will mean an algebra over a field $K$, graded (by natural numbers $0, 1, \ldots$), connected and commutative.

The commutativity of a graded algebra $A$ means that the relations

$$ab = (-1)^{\deg a \cdot \deg b}\, ba\,,$$
$$a^2 = 0 \quad \text{if } \deg a \text{ is odd}\,,$$

hold for all homogeneous $a$, $b$ in $A$ (deg is the abbreviation for degree).

When dealing with a bigraded object $\{X_{p,q}\}$ we use the following notation: if $x \in X_{p,q}$, then $p = \deg_H(x)$ is called the *homological degree* of $x$ and $q = \deg_I(x)$ — the *internal degree* of $x$, the sum $\deg_H(x) + \deg_I(x)$ is called the *total degree* of $x$ and is denoted by $\deg_T(x)$.

The object of this note is a study of a homology $H_{**}(A)$ of a $K$-algebra $A$. It is defined as

$$H_{**}(A) = \mathrm{Tor}^A_{**}(K, K)\,,$$

where $K$ is regarded as a graded $A$-module via an augmentation mapping $A \to K$. The kernel of $A \to K$ will be denoted by $I$ and called the *augmentation ideal* of $A$; so we have $I = \bigoplus_{i>0} A_i$.

It is known that $H_{**}(A)$ admits the structure of a bigraded $\Gamma$-algebra over $K$, [3], and we just want to investigate it from that point of view.

To recall known results in this respect we need some definitions.

Let $A$ be a bigraded $\Gamma$-algebra over a field $K$ and $I(A)$ the augmentation ideal of $A$. The bigraded $K$-module

$$\bar{Q}(A) = I(A)/\big(I(A)^2 + J(A)\big)\,,$$

where $J(A)$ is an ideal generated by all divided powers $\gamma_k(x)$, $k > 1$, $\deg_H(x) > 0$, $\deg_T(x)$ even, is called the *module of indecomposable elements* of $A$.

A bigraded $\Gamma$-algebra $A$ is said to be *finitely generated* if $\bar{Q}(A)_{p*} = 0$ for all large $p$.

Further we recall that a *free* $K$-algebra $S(V)$ on a positively graded $K$-vector space $V$ is defined as an algebra isomorphic to $E(V^-)\otimes P(V^+)$ where $E(V^-)$ denotes the exterior algebra generated by the odd part $V^-$ of $V$ and $P(V^+)$ denotes the polynomial algebra generated by the even part $V^+$ of $V$ ($V^- = \bigoplus_{i\,\mathrm{odd}} V_i$, $V^+ = \bigoplus_{i\,\mathrm{even}} V_i$).

Let $A$ be a $K$-algebra. We quote from [3] and [4]:

I. $H_{**}(A)$ is generated as a bigraded $\Gamma$-algebra over $K$ by $H_{1*}(A)$ (i.e., $\bar{Q}(H_{**}(A))_{p*} = 0$ for $p>1$) if and only if $A$ is a free $K$-algebra.

II. $H_{**}(A)$ is generated as a bigraded $\Gamma$-algebra over $K$ by $H_{1*}(A)$ and $H_{2*}(A)$ (i.e., $\bar{Q}(H_{**}(A))_{p*} = 0$ for $p>2$) if and only if $A$ is a graded complete intersection.

In this note we will give evidence (Theorems 1 and 2) for the following conjecture.

Conjecture. Let $A$ be a $K$-algebra. If the homology $H_{**}(A)$ is a finitely generated bigraded $\Gamma$-algebra over $K$, then $A$ is a graded complete intersection.

Our main tool is the theory of the Tate resolutions as developed in papers [3] and [4], to which we refer for basic definitions and notations.

§ 2. For a $K$-algebra $A$ we denote by $Q(A)$ the $K$-vector space $I(A)/I(A)^2$.

Theorem 1. *Let $A$ be a finitely generated $K$-algebra satisfying $Q(A)^+ = 0$ and let* $\mathbf{E}$ *denote the Koszul complex of $A$. Then the following conditions are equivalent*:

(1) *$A$ is a free $K$-algebra, i.e., $A\simeq E(V^-)$ for some $V$;*

(2) *$H_{p*}(\mathbf{E}) = 0$ for some $p>0$;*

(3) *$\mathbf{E}$ is a free resolution of $K$ over $A$;*

(4) *$\bar{Q}(H_{**}(A))_{p*} = 0$ for $p>1$;*

(5) *$\bar{Q}(H_{**}(A))_{p*} = 0$ for all sufficiently large $p$, i.e., $H_{**}(A)$ is finitely generated as a bigraded $\Gamma$-algebra over $K$.*

Corollary 1. *If $\operatorname{char} K = 0$, then each property of Theorem 1 is equivalent to $H_{**}(A)$ being finitely generated as a bigraded algebra over $K$.*

Lemma 1. *Let $A$ be a finitely generated $K$-algebra and* $\mathbf{X}$ *a differential $\Gamma$-algebra over $A$ satisfying*

(i) *$H_{0*}(\mathbf{X}) = A/I$,*

(ii) *$Z_{p*}(\mathbf{X})\subset I\mathbf{X}_{p*}$ for all $p>0$.*

*If $\dim_K Q(A) = n$, then for each $\sigma\in H_{i*}(\mathbf{X})$, $i>0$, we have $\sigma^{n+1} = 0$.*

Proof. Let $s$ be a cycle representing $\sigma$ and let $x_1, \ldots, x_k$, $t_{k+1}, \ldots, t_n$ be a minimal set of generators of $I$, $\deg x_i$ even, $\deg t_i$ odd. By (ii) we have $s = s'+s''$ where

$$s' = \sum_{i=1}^{k} x_i u_i, \qquad s'' = \sum_{i=k+1}^{n} t_i v_i,$$

for some $u_i, v_i\in \mathbf{X}$.



By (i) $x_i$ are boundaries in $\mathbf{X}$, and so there exist elements $T_i\in \mathbf{X}$, $\deg_H T_i = 1$, such that $dT_i = x_i$, $i = 1, \ldots, k$. Write $s''' = \sum_{i=1}^{k} T_i du_i$, $\alpha = \sum_{i=1}^{k} T_i u_i$. Since $s'-s''' = d\alpha$, an element $\bar{s} = s'''+s''$ is a cycle homologous to $s$. But $\bar{s}^{n+1} = 0$ because of $T_i^2 = 0$, $t_i^2 = 0$; hence $\sigma^{n+1} = 0$.

Definition. Let $\mathbf{X}$ be a differential $\Gamma$-algebra. We define $q(\mathbf{X})$ as the largest integer $p$ such that $H_{p*}(\mathbf{X}) \neq 0$. If there is no such integer, we define $q(\mathbf{X}) = \infty$.

Lemma 2. *Let $A$ be a $K$-algebra and let* $\mathbf{X}$ *be a differential $\Gamma$-algebra over $A$. Let $s$ be a homogeneous cycle of positive homological degree in* $\mathbf{X}$ *such that its homology class is nilpotent. Put*

$$\mathbf{Y} = \mathbf{X}\langle S;\, dS = s\rangle.$$

*Then $q(\mathbf{Y})<\infty$ implies $q(\mathbf{X})<\infty$.*

The proof of Lemma 2 is similar to that of Lemma 2 in [2] and we omit it.

We denote by (#) the following property of a differential $\Gamma$-algebra $\mathbf{X}$:

(#) If $H_{p*}(\mathbf{X}) = 0$ for some $p>0$, then $H_{i*}(\mathbf{X}) = 0$ for all $i>0$.

Lemma 3. *Let* $\mathbf{X}$ *be a differential $\Gamma$-algebra over $A$ and put $\mathbf{Y} = \mathbf{X}\langle S\rangle$ where $\deg_T S$ is even and $\deg_H S = 1$. If* $\mathbf{X}$ *has the property* (#), *so does* $\mathbf{Y}$.

Proof. We have an exact sequence of complexes

$$0\to \mathbf{X}\to \mathbf{Y}\to \mathbf{Y}\to 0$$

and an associated long exact homology sequence

$$\ldots\to H_{p*}(\mathbf{Y})\to H_{p*}(\mathbf{X})\to H_{p*}(\mathbf{Y})\to H_{p-1*}(\mathbf{Y})\to \ldots$$

If $H_{p*}(\mathbf{Y}) = 0$ for some $p>0$, then also $H_{p*}(\mathbf{X}) = 0$. But $\mathbf{X}$ has the property (#); hence $H_{i*}(\mathbf{X}) = 0$ for all $i>0$, and from the homology sequence we get $H_{i*}(\mathbf{Y}) = 0$ for all $i>0$.

Corollary 2. *Let $A$ be a finitely generated $K$-algebra satisfying $Q(A)^+ = 0$. The Koszul complex* $\mathbf{E}$ *of $A$ has the property* (#).

Proof of Theorem 1. (4) $\Rightarrow$ (5) is trivial.

(5) $\Rightarrow$ (2). The assumption that $H_{**}(A)$ is finitely generated as a bigraded $\Gamma$-algebra means, by the minimality of the Tate resolution $\mathbf{X}$ of $A$, [3], that $F_r\mathbf{X} = \mathbf{X}$ for some $r>0$ (we recall that $\{F_i\mathbf{X}\}_{i=0,1,\ldots}$ is the canonical filtration of $\mathbf{X}$, and $\mathbf{E} = F_1\mathbf{X}$, see [3]). Since $A$ is a finitely generated $K$-algebra, each $F_p\mathbf{X}$ can be obtained from $F_{p-1}\mathbf{X}$ by an adjunction of a finite number of variables. Thus in order to obtain $\mathbf{X}$ we have to adjoin to $\mathbf{E}$ only a finite number of variables $W_1, \ldots, W_t$, say, i.e.,

$$\mathbf{X} = \mathbf{E}\langle W_1, \ldots, W_t\rangle = \mathbf{E}\langle W_1\rangle\ldots\langle W_t\rangle.$$

It is obvious that $q(\mathbf{X})<\infty$ because $\mathbf{X}$ is acyclic. Since any $\Gamma$-subalgebra of $\mathbf{X}$ of the form $\mathbf{E}\langle W_1, \ldots, W_j\rangle$ possesses properties (i), (ii) of Lemma 1, by a repeated application of Lemma 2 we obtain $q(\mathbf{E})<\infty$; hence $H_{p*}(\mathbf{E}) = 0$ for some $p>0$.

(2) $\Rightarrow$ (3) follows from Corollary 2.
(3) $\Rightarrow$ (4) follows from the minimality of $E$.
(1) $\Leftrightarrow$ (3) is a special case of Theorem (5.2) in [3].

§ 3. To state Theorem 2 we need the following property (# #) of a homogeneous ideal $\mathfrak{A}$ in a $K$-algebra $A$.

(# #) Among maximal regular sequences of elements of even degree in $\mathfrak{A}$ there exists at least one which is part of a minimal set of generators for $\mathfrak{A}$.

Remark 1. There exist homogeneous ideals which do not satisfy condition (# #).

THEOREM 2. *Let $A$ be a finitely generated $K$-algebra and suppose that there exist a free finitely generated $K$-algebra $A'$ and a homogeneous ideal $\mathfrak{A}'$ in $A'$ satisfying*

$1^0$ $A'/\mathfrak{A}' \simeq A$,

$2^0$ $\mathfrak{A}' \subset I'^2$,

$3^0$ $\mathfrak{A}'$ *has the property* (# #).

*Then the homology $H_{**}(A)$ is a finitely generated bigraded $\Gamma$-algebra over $K$ if and only if $A$ is a graded complete intersection.*

PROPOSITION 1 (Auslander–Buchsbaum, [1]). *Let $A$ be a finitely generated $K$-algebra and $M$ a finitely generated graded $A$-module with a finite free resolution. The following properties are equivalent*:

(i) $\mathrm{Ann}(M)$ *contains a homogeneous non-zero divisor*;

(ii) $\mathrm{Ann}(M) \neq 0$;

(iii) *The Euler characteristic of $M$ is zero.*

The proof proceeds essentially as in [5; p. 29] for Noetherian (ungraded) rings but some notions and results have to be adapted to the graded case. We state them below.

(1) Localization of a graded commutative algebra $C$ with respect to a homogeneous prime ideal $P$ consists in making invertible all homogeneous elements outside $P$. The usual procedure applies because homogeneous elements outside $P$ have even degrees. This gives rise to an algebra $C_P$ graded by all integers.

(2) Let $C$ be a ring graded by all integers and commutative (see § 1). $C$ is said to be a *graded local ring* if there exists a homogeneous ideal $\mathfrak{M}$ such that every homogeneous element outside $\mathfrak{M}$ is invertible.

Let $(C, \mathfrak{M})$ be a graded local ring and assume that $\mathfrak{M}$ is annihilated by some homogeneous element in $\mathfrak{M}$. Then every finitely generated graded $C$-module with a finite resolution by finitely generated free modules is free.

(3) Let $A$ be a $K$-algebra. An element in $A$ is said to be a *zero-divisor* if it is annihilated by some non-zero homogeneous element in $A$. Denote by $\mathfrak{z}(A)$ the set of all zero-divisors in $A$. As in the Noetherian case,

$$\mathfrak{z}(A) = P_1 \cup \ldots \cup P_n$$



where $P_i$ are homogeneous prime ideals and $P_i = \mathrm{Ann}(a_i)$ for some homogeneous $a_i \in A$.

(4) Let $A$ be a $K$-algebra. For a homogeneous ideal $\mathfrak{A}$ in $A$ we denote by $h(\mathfrak{A})$ the set of all homogeneous elements in $\mathfrak{A}$.

If $h(\mathfrak{A}) \subset P_1 \cup \ldots \cup P_k$ and $P_1, \ldots, P_k$ are homogeneous prime ideals, then $\mathfrak{A} \subset P_i$ for some $i$.

Proof of Theorem 2. The "if" part of the Theorem follows from II in § 1. We will now prove the "only if" part. The idea of the proof comes from [2].

By assumptions $1^0$-$3^0$ of the Theorem we have a free finitely generated $K$-algebra $A'$, a surjective $K$-algebra homomorphism $\varphi\colon A' \to A$ with $\mathfrak{A}' = \mathrm{Ker}\,\varphi \subset I'^2$, and a maximal regular sequence $\xi'_1, \ldots, \xi'_k$ in $\mathfrak{A}'$ which is part of a minimal set of generators for $\mathfrak{A}'$. If we denote by $J'$ the ideal generated by $\xi'_1, \ldots, \xi'_k$, then the mapping $J' \otimes K \to \mathfrak{A}' \otimes K$ induced by inclusion is injective.

Put $A'' = A'/J'$ and let $\psi\colon A'' \to A$ be a homomorphism induced by $\varphi$. To prove the Theorem it is sufficient to show that $A$ has finite projective dimension as an $A''$-module. Indeed, by construction every homogeneous element of $\mathrm{Ker}\,\psi$ is a zero-divisor in $A''$, and so Proposition 1 and the hypothesis that $\mathrm{pd}_{A''} A < \infty$ imply that $\mathrm{Ker}\,\psi = 0$, i.e., $A$ is a graded complete intersection.

We will now prove that if $H_{**}(A)$ is finitely generated as a bigraded $\Gamma$-algebra, then an $A''$-module $A$ is of finite projective dimension.

Let $E''$ be the Koszul complex of some minimal set of generators of the augmentation ideal $I''$ of $A''$ (Koszul complex of $A''$, for short). Since $J' \subset \mathfrak{A}' \subset I'^2$, $E = E'' \otimes_{A''} A$ is the Koszul complex of $A$. Because of the isomorphism $H_{1*}(E'') \simeq J' \otimes K$ there exist cycles $s''_1, \ldots, s''_k$ representing a basis of the $K$-vector space $H_{1*}(E'')$. Since $A''$ is a graded complete intersection, a differential $\Gamma$-algebra

$$X'' = F_2 X'' = E''\langle S_1, \ldots, S_k;\, dS_i = s''_i\rangle$$

is a minimal resolution of $K$ over $A''$. Consider differential $\Gamma$-algebras $F = X'' \otimes_{A''} A$ and $F_2 X$ over $A$. We recall that $F_2 X$ is defined as a differential $\Gamma$-algebra obtained from $F_1 X = E$ by an adjunction of variables which kill cycles representing a basis of $H_{1*}(E)$. Observe that the diagram

$$\begin{array}{ccc} H_{1*}(E'') & \simeq & J' \otimes K \\ \downarrow & & \downarrow \\ H_{1*}(E) & \simeq & \mathfrak{A}' \otimes K \end{array}$$

commutes where the left vertical map is induced by the obvious map $E'' \to E$ and the right vertical map is injective by construction. This implies that the set $s''_1, \ldots, s''_k$ of cycles representing a basis of $H_{1*}(E'')$ can be extended to a set $s''_1, \ldots, s''_k, s_{k+1}, \ldots, s_l$ of cycles representing a basis of $H_{1*}(E)$; hence

$$F_2 X = F\langle S_{k+1}, \ldots, S_l;\, dS_i = s_i\rangle.$$

Suppose now that $H_{**}(A)$ is finitely generated as a bigraded $\Gamma$-algebra. By the minimality of the Tate resolution $X$ of $A$, [3], this means that $F_r X = X$ for some $r>0$. If $r = 1$ or 2, then $A$ is a graded complete intersection by I and II of § 1. So assume that $r>2$. Proceeding as in the proof of Theorem 1 (implication (5) $\Rightarrow$ (2)), we get $X = F\langle W_1, \ldots, W_t\rangle$ for a finite number of variables $W_1, \ldots, W_t$. By Lemma 2 we infer that $q(F)<\infty$. But $H_{**}(F) = \mathrm{Tor}_{**}^{A''}(K, A)$, and hence $A$ has finite projective dimension over $A''$.

PROPOSITION 2. *If a minimal generating set of the ideal $\mathfrak{A}'$ (notation as in Theorem 2) is concentrated in a single degree and the field $K$ is infinite, then $\mathfrak{A}'$ has the property* (# #).

Proof. Suppose that $\mathfrak{A}'$ contains a homogeneous non-zero divisor. We will prove that such an element can be chosen from some minimal set of generators of $\mathfrak{A}'$. Let $V$ be a $K$-vector space generated by some fixed minimal set of generators of the ideal $\mathfrak{A}'$. If every homogeneous element from $\mathfrak{A}'\setminus I'\mathfrak{A}'$ is a zero-divisor in $A'$, then $V'\subset\mathfrak{z}(A') = P_1'\cup\ldots\cup P_k'$ because of the assumption that $V'$ is concentrated in a single degree. This implies $V' = \bigcup V'\cap P_i$ and consequently $V' = V'\cap P_i$ for some $i$ since a finite-dimensional vector space over an infinite field cannot be a set-theoretic union of a finite number of its proper subspaces. Thus $\mathfrak{A}'\subset P_i$ and we get a contradiction of the fact that $\mathfrak{A}'$ contains a homogeneous non-zero divisor.

We complete the proof by induction with respect to the length of a minimal regular sequence in $\mathfrak{A}'$.

**Added in proof.** When the paper had been submitted for publication I learned that the following result follows from Lemma 3.7 of [6].

PROPOSITION 3. *If $A$ is a free finitely generated $K$-algebra generated by elements of the same even degree and the field $K$ is infinite, then every non-zero homogeneous ideal of $A'$ has the property* (# #).

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# The André-Quillen homology of commutative graded algebras

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**Abstract.** The paper presents an extension of André-Quillen (co-)homology theory of commutative rings to graded commutative algebras.

Basic definitions and properties are given and characterization of free algebras and graded complete intersections is obtained in terms of the André-Quillen (co-)homology.

**§ 1. Introduction.** The aim of this paper is to present an extension of the André-Quillen (co-)homology theory of commutative rings (see [1], [2], [8]) to commutative graded algebras.

We consider only algebras graded by natural numbers and such an algebra $A = \{A^p\}_{p=0,1,\ldots}$ is said to be *commutative* if

$$x\cdot y = (-1)^{\deg x\cdot \deg y}\, y\cdot x\,,$$
$$x^2 = 0 \text{ when } \deg x \text{ is odd}\,,$$

for homogeneous elements $x, y$ in $A$.

§ 2 contains the definitions and basic properties of graded derivations and differentials.

In § 3 we give the definition of homology and cohomology modules for graded algebras and state their main general properties.

§ 4 is of an auxiliary nature and introduces some results on classical Tor-homology needed in the sequel.

In § 5 we compute low-dimensional (co-)homology modules and Theorem (5.5) gives the basic relation between multiplication in Tor-homology and the second André-Quillen homology module.

In § 6 we prove the Vanishing Theorem (6.1), characterizing regular sequences in a commutative graded algebra in terms of the André-Quillen (co-)homology and its consequences.

**§ 2. Graded derivations and differentials.** Let $K$ be a commutative ring with an identity. We denote by $K$-Alg the category of graded (by natural numbers 0, 1, ...) commutative algebras over $K$ and their homomorphisms of degree zero. Objects of $K$-Alg will be called "graded $K$-algebras" for short. For a graded