A hereditarily indecomposable non-metric Hausdorff continuum

by

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Abstract. Recently there were given examples (e.g. Bellamy's paper [1] and Bellamy and Rubin [2]) of indecomposable non-metric Hausdorff continua. However these continua are not hereditarily indecomposable. The aim of this note is to give a series of such examples based on author's construction given in [4].

Construction. Let X be an arbitrary non-degenerated metric continuum. For each $x \in X$, let M_x be a metric non-degenerated continuum and let $T_x\colon M_x \overset{\text{onto}}{\to} X$ be a continuous map. Let $S = \bigcup \{\{x\}xT_x^{-1}(x)\colon x \in X\}$. For each $x \in X$ and an open subset U of M_x which intersects $T_x^{-1}(x)$, let $R(x, U)$ denote the subset of S to which (t, P) belongs iff either $t = x$ and P is in $U \cap T_x^{-1}(x)$, or P is in $T_t^{-1}(t)$ and $T_x^{-1}(t) \subset U$. The collection of all such subsets of S generates a topology in S. Let π denote a map (projection) of S onto X such that $\pi^{-1}(x) = \{x\} \times T_x^{-1}(x)$.

LEMMA 1. *There is no countable base in S.*

Proof. Let β be an arbitrary base in S. The collection $\mathscr{P}$ of all subset of S of the form $R(x, U)$, being multiplicative, is a basis of the topology in S. Thus there is subfamilly $\mathscr{P}'$ of $\mathscr{P}$ such that $\mathscr{P}'$ is a basis in S and $\operatorname{card}\mathscr{P}' \leqslant \operatorname{card}\beta$. Let $x \in X$ and let $R(x, U)$ be an open subset of S such that $T_x^{-1}(x) - R(x, U) \neq \emptyset$ and that $R(x, U) \cap (\{x\} \times T_x^{-1}(x)) \neq \emptyset$. Then there is an R_x in $\mathscr{P}'$ such that $R_x \subset R(x, U)$ and that $R_x \cap (\{x\} \times T_x^{-1}(x)) \neq \emptyset$. Hence $T_x^{-1}(x) - R_x \neq \emptyset$. This implies that $R_x \neq R_y$ for $x \neq y$. Hence $\operatorname{card}\mathscr{P}' \geqslant \operatorname{card} X \geqslant \mathfrak{c}$. This ends the proof.

Note 1. In [4] the author showed that

(i) if for each $x \in X$, $\lim_{t \to x} \operatorname{diam} T_x^{-1}(t) = 0$ and $T_x^{-1}(x)$ is connected then S is a separable first countable continuum,

(ii) π is an atomic map,

(iii) for each $x \in X$ there exist M_x and $T_x\colon M_x \overset{\text{onto}}{\to} X$ such that $\lim_{t \to x} \operatorname{diam} T_x^{-1}(t) = 0$ and $T_x^{-1}(x)$ is a given arbitrary metric continuum.

Note 2. It is known (cf. Cook [3]) that if $f\colon X \overset{\text{onto}}{\to} Y$ is an atomic map onto a hereditarily indecomposable continuum Y and the preimage under f of any point of Y is a hereditarily indecomposable continuum, then X is a hereditarily indecomposable continuum.

THEOREM. *There exist non-metric hereditarily indecomposable continua.*

Proof. Let, in the above construction, X and $T_x^{-1}(x)$ for each $x \in X$ be hereditarily indecomposable metric continua; e.g. pseudo-arcs. By Lemma 1, Note 1 and Note 2, we infer that S in this construction is a non-metric hereditarily indecomposable continuum.

References

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Paracompactness of topological completions

by

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Abstract. Let X be a completely regular T_2 space, and $\mu(X)$ a topological completion of X (that is, a completion of X with respect to its finest uniformity agreeing with the topology of X). If $\mu(X)$ is paracompact, then X is said to be *pseudo-paracompact*. In this paper some remarkable properties of pseudo-paracompact spaces are studied.

1. Introduction. The purpose of this paper is to give detailed proofs for the author's abstract [6]. Throughout this paper all spaces are assumed to be completely regular T_2. For every space X, we denote by μ its finest uniformity agreeing with the topology of X, that is, μ is the family of all normal open coverings of X. Concerning pseudo-paracompactness, the following results are known.

THEOREM 1.1 (Morita [13]). *For every M-space X $\mu(X)$ is a paracompact M-space.*

THEOREM 1.2 (Howes [5]). *A space X is pseudo-paracompact if and only if every weakly Cauchy filter in X with respect to μ is contained in some Cauchy filter with respect to μ.*

Let $\{\mathfrak{U}_\lambda | \lambda \in \Lambda\}$ be the family of all normal open coverings of a space X. A filter $\mathfrak{F} = \{F_\alpha\}$ in X is weakly Cauchy with respect to μ if for any $\lambda \in \Lambda$ there exists $U \in \mathfrak{U}_\lambda$ such that $U \cap F_\alpha \neq \emptyset$ for every $F_\alpha \in \mathfrak{F}$. In other words, a filter $\mathfrak{F}$ is weakly Cauchy with respect to μ if for any $\lambda \in \Lambda$ there exists a filter $\mathfrak{F}_\lambda$ stronger than $\mathfrak{F}$ such that $L \subset U$ for some $U \in \mathfrak{U}_\lambda$ and $L \in \mathfrak{F}_\lambda$. In this paper we shall study further results related to pseudo-paracompactness. § 2 contains other characterizations of pseudo-paracompact spaces and another proof of Howes's theorem. Furthermore it is shown by an example that there exists a strongly normal (i.e., countably paracompact and collectionwise normal) space which is not pseudo-paracompact. § 3 is concerned with the following:

(1) The sum theorems of pseudo-paracompact spaces.

(2) The sucffient conditions for the preimage X of a paracompact space (or a paracompact q-space [10]) Y under a closed map f to be pseudo-paracompact.

(3) The invariance of strongly normal pseudo-paracompactness under a perfect map.

(4) Characterizations of pseudo-locally-compact and pseudo-paracompact spaces.