Locally bounded topologies on the rational field *

by

M. E. Shanks (Lafayette, Ind.) and **Seth Warner** (Durham, N. C.)

Abstract. We resolve a long-standing problem in the theory of topological rings by showing that the only locally bounded ring topologies on the field Q of rationals are the known ones.

Let P be the set of prime numbers, and let $P' = P \cup \{\infty\}$, ∞ referring to the "infinite prime", that is, the ordinary archimedean absolute value on the rational field Q. Each subset of P' determines a locally bounded ring topology on Q in a manner described below. Our purpose here is to show that conversely, every locally bounded ring topology on Q is determined in this way, thus answering a long-standing question in the theory of topological rings. As early as 1948, Kaplansky [6, p. 813] posed a special case of the problem: What are the topologies of type V (which are necessarily locally bounded) on Q? This problem was solved by Kowalsky and Dürbaum [7] and Fleischer [5], who showed that a topology of type V on any field was given either by an absolute value or a valuation. Results of Correl [3] in 1958 yielded a solution of the problem for the case where the given locally bounded topology was a field topology (that is, multiplicative inversion is continuous) for which the open additive subgroups form a neighborhood basis at zero; in that case the topology is determined by a finite subset of P. The general problem is implicit in Mutylin's work [8] in 1966. He showed that any nondiscrete locally bounded topology on Q not described by a subset of P' had to be stronger than the ordinary archimedean topology (that corresponding to ∞) but weaker than the topology determined by a proper subset of P'. The general problem is also explicitly mentioned in [4].

A *nucleus* of a topological ring is a neighborhood of zero. A subset B of a topological ring is *bounded* if for every nucleus V there is a nucleus W such that $BW \subseteq V$ and $WB \subseteq V$. A topological ring is *locally bounded* if there is a bounded nucleus. Locally compact rings are locally bounded, since a compact subset of a topological ring is bounded; normable topological rings are also clearly locally bounded.

A subset U of a field K is an *almost order* if $0, 1 \in U$, $-U = U$, $U \neq K$, $UU = U$, $a(U+U) \subseteq U$ for some nonzero $a \in K$, and $K = \{ab^{-1}\colon a, b \in U, b \neq 0\}$. If $\mathcal{T}$ is a nondiscrete locally bounded ring topology on K, then there is an almost

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order U on K that is a bounded nucleus for $\mathcal{T}$, and $\{Ux: x \neq 0\}$ is a fundamental system of nuclei for $\mathcal{T}$. Conversely, if U is an almost order on K, there is a unique nondiscrete locally bounded ring topology on K for which U is a bounded nucleus [1, Exercise 20, p. 120; 7, Theorems 5 and 6].

For each $p \in P$, let $\boldsymbol{Q}_p$ be the field of p-adic numbers, $\boldsymbol{Z}_p$ the ring of p-adic integers, $|\ldots|_p$ the associated p-adic absolute value. Let $\boldsymbol{Q}_\infty$ and $\boldsymbol{Z}_\infty$ both denote the topological field $\boldsymbol{R}$ of real numbers, let $|\ldots|_\infty$ be the ordinary archimedean absolute value and $\mathcal{T}_\infty$ the ordinary archimedean topology on $\boldsymbol{Q}$. For each subset R of P', let $O(R) = \{x \in \boldsymbol{Q}: |x|_p \leqslant 1 \text{ for all } p \in R\}$. If R is a proper subset of P', $O(R)$ is clearly an almost order on $\boldsymbol{Q}$; we define $\mathcal{T}_R$ to be the unique locally bounded ring topology on $\boldsymbol{Q}$ for which $O(R)$ is a bounded nucleus. Since $O(P') = \{0, 1, -1\}$, we also define $\mathcal{T}_{P'}$ to be the discrete topology on $\boldsymbol{Q}$. Note that $\mathcal{T}_\emptyset$ is the trivial topology whose only open sets are $\emptyset$ and $\boldsymbol{Q}$.

We shall prove that the only locally bounded ring topologies on $\boldsymbol{Q}$ are the topologies $\mathcal{T}_R$ where R is a subset of P'. In particular, the only Hausdorff, additive generated, locally bounded ring topology on $\boldsymbol{Q}$ is $\mathcal{T}_\infty$ (a ring topology is additively generated if there are no proper open additive subgroups). Our proof will depend on the following two results of Mutylin:

LEMMA 1 [8, Lemma 10]. *If $\mathcal{T}$ is a Hausdorff, locally bounded ring topology on $\boldsymbol{Q}$ that is not stronger than $\mathcal{T}_\infty$, then every nucleus for $\mathcal{T}$ contains all integral multiples of some nonzero rational.*

LEMMA 2 [8, Corollary 5]. *If $\mathcal{T}$ is a nondiscrete, locally bounded ring topology on $\boldsymbol{Q}$ that is stronger then $\mathcal{T}_\infty$, then for some prime p, $\mathcal{T}$ is weaker than $\mathcal{T}_{P'-\{p\}}$.*

Actually, Lemma 1 marks the halfway point in Mutylin's proof that under the hypotheses of Lemma 1, $\mathcal{T}$ is $\mathcal{T}_R$ for some nonempty subset R of P. This result, of course, is a consequence of our theorem.

For each nonempty subset R of P', we define A_R to be the local direct product of the topological rings $(\boldsymbol{Q}_p)_{p\in R}$ with respect to the open subrings $(\boldsymbol{Z}_p)_{p\in R}$. Thus $A_R = \{(x_p) \in \prod_{p\in R} \boldsymbol{Q}_p: x_p \in \boldsymbol{Z}_p \text{ for all but finitely many } p \in R\}$, topologized as follows: Let $B_R = \prod_{h\in R} \boldsymbol{Z}_p$, topologized with the cartesian product topology; we topologize A_R by declaring B_R to be a nucleus. Thus topologized, A_R is a locally compact ring. We define $\Delta_R: \boldsymbol{Q} \to A_R$ by $\Delta_R(r) = (r_p)_{p\in R}$, where $r_p = r$ for all $p \in R$. Note that if $S \subseteq R$, then the canonical injection from A_S into A_R (defined by $(x_p)_{p\in S} \mapsto (y_p)_{p\in R}$, where $y_p = x_p$ if $p \in S$, $y_p = 0$ if $p \in R-S$) is a topological isomorphism from A_S onto a subring of A_R. Hence we shall consider A_S a subring of A_R by means of this identification.

The following lemma is an immediate consequence of the Strong Approximation Theorem (see, for example, [4, Theorem 9]).

LEMMA 3. *If R is a nonempty proper subset of P', then Δ_R is a topological isomorphism from $\boldsymbol{Q}$, equipped with $\mathcal{T}_R$, onto a dense subring of A_R.*

THEOREM 1. *If $\mathcal{T}$ is a Hausdorff, locally bounded ring topology on $\boldsymbol{Q}$ that is weaker than $\mathcal{T}_R$ for some nonempty proper subset R of P', then $\mathcal{T} = \mathcal{T}_S$ for some nonempty subset S of R.*

Proof. Let E be the completion of $\boldsymbol{Q}$ for $\mathcal{T}$. Our hypothesis implies that $\boldsymbol{Q}; \mathcal{T}$ is a topological algebra over $\boldsymbol{Q}; \mathcal{T}_R$. Identifying the completion of $\boldsymbol{Q}; \mathcal{T}_R$ with A_R and extending by continuity the scalar multiplication of the $(\boldsymbol{Q}; \mathcal{T}_R)$-algebra $\boldsymbol{Q}; \mathcal{T}$ to $A_R \times E$, we conclude that E is a topological algebra over A_R under a scalar multiplication . satisfying

$$\Delta_R(r) \,.\, s = rs \tag{1}$$

for all $r, s \in \boldsymbol{Q}$.

For each $p \in R$, let in_p denote the canonical injection of $\boldsymbol{Q}_p$ into A_R, and let $e_p = \mathrm{in}_p(1)$. Thus $\mathrm{in}_p(\boldsymbol{Q}_p) = A_R e_p$. Clearly

$$\mathrm{in}_p(r) = \Delta_R(r) e_p \tag{2}$$

for all $r \in \boldsymbol{Q}$, $p \in R$. By restricting scalar multiplication to $(A_R e_p) \times (e_p \,.\, E)$ and replacing $A_R e_p$ with $\boldsymbol{Q}_p$, we obtain a continuous scalar multiplication $_{(p)}$ from $\boldsymbol{Q}_p \times e_p \,.\, E$ into $e_p \,.\, E$ defined by

$$\lambda_{(p)} x = \mathrm{in}_p(\lambda) \,.\, x$$

for all $\lambda \in \boldsymbol{Q}_p$, $x \in e_p \,.\, E$. With scalar multiplication so defined, $e_p \,.\, E$ is a topological algebra over $\boldsymbol{Q}_p$, since if $x \in e_p \,.\, E$, then $x = e_p \,.\, x$, whence

$$1_{(p)} x = \mathrm{in}_p(1) \,.\, x = \Delta_R(1) e_p \,.\, x = e_p \,.\, x = x\,.$$

Let $S = \{p \in R: e_p \,.\, E \neq (0)\}$. If $e_p \,.\, 1 = 0$, clearly $e_p \,.\, \boldsymbol{Q} = (0)$, whence $e_p \,.\, E = (0)$, since $\boldsymbol{Q}$ is dense in E and so, by the continuity of $x \mapsto e_p \,.\, x$, $e_p \,.\, \boldsymbol{Q}$ is dense in $e_p \,.\, E$. Thus $p \in S$ if and only if $e_p \,.\, 1 \neq 0$. Restricting scalar multiplication to $A_S \times E$ where A_S is canonically identified with a subring of A_R, we conclude that E is a topological algebra over A_S, since for each $x \in E$,

$$x = \Delta_R(1) \,.\, x = \Big(\Delta_S(1) + \sum_{p\in R-S} e_p\Big) \,.\, x = \Delta_S(1) \,.\, x + \sum_{p\in R-S} e_p \,.\, x = \Delta_S(1) \,.\, x\,.$$

In particular, we conclude that $S \neq \emptyset$. More generally, if $r \in \boldsymbol{Q}$ and if $x \in E$,

$$\Delta_R(r) \,.\, x = \Delta_S(r) \,.\, x \tag{3}$$

since $\Delta_R(r) \,.\, x = \Delta_R(r) \,.\, (\Delta_S(1) \,.\, x) = \Delta_R(r)\Delta_S(1) \,.\, x = \Delta_S(r) \,.\, x$. For each $p \in S$ and each $r \in \boldsymbol{Q}$,

$$r_{(p)}(e_p \,.\, 1) = e_p \,.\, r \tag{4}$$

since $r_{(p)}(e_p \,.\, 1) = \Delta_R(r) e_p \,.\, (e_p \,.\, 1) = \Delta_R(r) e_p \,.\, 1 = e_p \,.\, (\Delta_R(r) \,.\, 1) = e_p \,.\, r$ by (1) and (2). Consequently, the one-dimensional subspace of the $\boldsymbol{Q}_p$-vector space $e_p \,.\, E$ generated by $e_p \,.\, 1$ contains $e_p \,.\, \boldsymbol{Q}$. But $\boldsymbol{Q}_{p(p)}(e_p \,.\, 1)$ is closed in $e_p \,.\, E$ and $e_p \,.\, \boldsymbol{Q}$ is dense in

$e_p . E$; thus $\{e_p . 1\}$ is a basis of the Q_p-vector space $e_p . E$. Consequently, $\varphi_p: \lambda \mapsto \lambda_{(p)}(e_p . 1)$ is a topological isomorphism from Q_p onto $e_p . E$.

Let U be an almost order defining the topology of E. Then by continuity, $\varphi_p^{-1}(e_p . U)$ is a bounded, multiplicatively closed subset of Q_p and hence is contained in Z_p. Let $G: E \to A_S$ be defined by

$$G(x) = (\varphi_p^{-1}(e_p . x))_{p \in S} .$$

For each $x \in E$, $G(x)$ does indeed belong to A_S, for as $(e_p)_{p \in S}$ is summable in A_S, by continuity $(e_p . x)_{p \in S}$ is summable in E, whence for all but finitely many $p \in S$, $e_p . x \in U$; but if $e_p . x \in U$, then $e_p . x = e_p^2 . x = e_p . (e_p . x) \in e_p . U$, so $\varphi_p^{-1}(e_p . x) \in Z_p$. For each $p \in S$, $x \mapsto \varphi_p^{-1}(e_p . x)$ is continuous from U into Z_p; hence the restriction of G to U is continuous from U into $\prod_{p \in S} Z_p$. By definition of the topology of A_S and since U is a nucleus, therefore, G is a continuous homomorphism from E into A_S.

Let $F: A_S \to E$ be defined by $F(\lambda) = \lambda . 1$. Then $G \circ F: A_S \to A_S$ is continuous and agrees with the identity mapping on $\Delta_S(Q)$, for if $r \in Q$, then $\varphi_p^{-1}(e_p . r) = r$ for all $p \in S$ by (4), whence

$$G(r) = \Delta_S(r) \tag{5}$$

and therefore $(G \circ F)(\Delta_S(r)) = G(\Delta_S(r) . 1) = G(\Delta_R(r) . 1) = G(r) = \Delta_S(r)$ by (3). Consequently, as $\Delta_S(Q)$ is dense in A_S, $G \circ F$ is the identity mapping of A_S.

Similarly, $F \circ G: E \to F$ is continuous and agrees with the identity mapping on Q, for if $r \in Q$, then by (5), (3), and (1),

$$(F \circ G)(r) = F(\Delta_S(r)) = \Delta_S(r) . 1 = \Delta_R(r) . 1 = r .$$

Thus $F \circ G$ is the identity map on E. Therefore $G^{-1} = F$, so G is a topological isomorphism from E onto A_S satisfying $G(r) = \Delta_S(r)$ for all $r \in \mathbf{Q}$. Thus $\mathcal{T} = \mathcal{T}_S$.

THEOREM 2. *The only locally bounded ring topologies on Q are the topologies $\mathcal{T}_R$ where R is a subset of P'. In particular, the only Hausdorff, locally bounded, additively generated ring topology on Q is $\mathcal{T}_\infty$. Thus $R \mapsto \mathcal{T}_R$ is an isomorphism from the lattice of all subsets of P' onto the lattice of all locally bounded ring topologies on Q.*

Proof. Let $\mathcal{T}$ be a Hausdorff, locally bounded ring topology on Q. First, assume that $\mathcal{T}$ is not stronger than $\mathcal{T}_\infty$. By Lemma 1, if U is a nucleus for $\mathcal{T}$, then $U \supseteq Za$ for some nonzero rational a; but Za is open for $\mathcal{T}_P$ since $Z = O(P)$ is open for $\mathcal{T}_P$. Thus $\mathcal{T}$ is weaker than $\mathcal{T}_P$. By Theorem 1, therefore, the assertion follows. If $\mathcal{T}$ is stronger than $\mathcal{T}_\infty$, the assertion also follows from Lemma 2 and Theorem 1.

COROLLARY 1. *The only locally compact rings containing Q densely are the rings A_R, where R is a nonempty proper subset of P', and the ring Q equipped with the discrete topology.*

Proof. If A is a locally compact ring properly containing Q as a dense subset,

then A is the completion of Q for its induced topology, which is necessarily locally bounded. The result therefore follows from Theorem 2.

The following corollary generalizes a theorem of Mutylin [5, Theorem 3].

COROLLARY 2. *If A is a Hausdorff, complete, locally bounded ring containing Q, then either Q is discrete, or the closure of Q is A_R for some nonempty proper subset R of P'. In particular, if A is a Hausdorff, complete, locally bounded field of characteristic zero, either Q is discrete, or the closure of Q in A is either $\mathbf{R}$ or Q_p for some prime p.*

The second statement follows from the first, since A_R contains proper zero divisors if R is a subset of P' containing more than one element.

References

[1] N. Bourbaki, *Topologie Générale*, Ch. 3–4, Paris 1960.
[2] — *Algèbre commutative*, Ch. 7, Paris 1965.
[3] E. Correl, *On topologies for fields*, Ph. D. dissertation, Purdue University, Lafayette, Indiana, 1958.
[4] M. Endo, *On the topologies of the rational number field*, Comment. Math. Univ. St. Paul 16 (1967–68), pp. 11–20.
[5] I. Fleischer, *Sur les corps topologiques et les valuations*, C. R. Acad. Sci. Paris 236 (1953), pp. 1320–1322.
[6] I. Kaplansky, *Topological rings*, Bull. Amer. Math. Soc. 54 (1948), pp. 809–826.
[7] H. J. Kowalsky und H. Dürbaum, *Arithmetische Kennzeichnung von Körpertopologien*, J. Reine Angew. Math. 191 (1953), pp. 135–152.
[8] А. Ф. Мутылин, *Пример нетривиальной топологизации поля рациональных чисел. Полные локально ограниченные поля*, Известия АН СССР, сер. матем. 30 (1966), pp. 873–890; 32 (1968), pp. 245–246.

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