Concerning the Whitehead Theorem for movable compacta

by

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Abstract. There was a gap in the proof of the Whitehead Theorem for movable compacta (Th. 4.3, [7]). In this paper the uniform movability of a shape map is defined and the Whitehead Theorem for uniformly movable shape maps is established.

Contents

The Whitehead Theorem for movable metric compacta was established in [7] (Theorem 4.3). However, as was noticed by Sibe Mardešić, there was a mistake in the proof because of an error in the statement 6.6 of [5] (in that proof 6.6 was used on page 261[11] of [7]). In 6.6 of [5], p. 144[4,5,6], $\hat{\mathscr{G}}^*$ should be replaced by $\hat{\mathscr{G}}^*_0$ — the full subcategory of uniformly movable inverse systems of groups.

The main purpose of the present note is to correct Theorem 4.3 [7] (here Theorem 6.2). Another form of that theorem was established in § 5 [7], Theorem 5.2, [7]. However, in 5.2 [7] the assumption of X and Y should also be replaced by the assumption of Theorem 6.2.

Independently James Keesling filled the gap in the proof of 4.3 [7] without any additional assumption (to appear in Fund. Math.). However, he makes essential use of the metrizability of X and Y, while the method presented here seems to be applicable to non-metric case.

1. Uniformly movable maps of inverse systems. Let us consider a category $\mathscr{K}$ with an equivalence relation $\sim$ in $\mathrm{Mor}_{\mathscr{K}}(X, Y)$ for any pair of objects X, Y. Assume that $\sim$ satisfies the condition

$$f\sim f' \wedge g\sim g' \Rightarrow gf\sim g'f'$$

whenever these compositions exist.

Let $\mathscr{K}^*_{\sim}$ be the category of inverse systems in $\mathscr{K}$ over closure finite directed sets, and let $\hat{\mathscr{K}}^*_{\sim}$ be the quotient category with respect to the following relation of similarity $\approx$ (see [5]). Take $X = (X_\alpha, p_\alpha^{\alpha'}, A)$ and $Y = (Y_\beta, q_\beta^{\beta'}, B)$ and let $f, f' \in \mathrm{Mor}_{\mathscr{K}^*_\sim}(X, Y)$, $f = (\varphi, f_\beta)$, $f' = (\varphi', f'_\beta)$. Then

$$f \approx f' \underset{\mathrm{Df}}{\Leftrightarrow} \bigwedge_{\beta} \bigvee_{\alpha \geqslant \varphi(\beta), \varphi'(\beta)} f_\beta p^\alpha_{\varphi(\beta)} \sim f'_\beta p^\alpha_{\varphi'(\beta)}.$$

For any $\alpha_0 \in A$ let

$$A^{(\alpha_0)} = \{\alpha \in A\colon \alpha \geqslant \alpha_0\} \quad \text{and} \quad X^{(\alpha_0)} = (X_\alpha, p_\alpha^{\alpha'}, A^{(\alpha_0)}).$$

According to [5], the system X is said to be *uniformly movable* in $\mathscr{K}^*_\sim$ whenever there exist

1° a collection of constant functions $(\chi^{(\alpha_0)}\colon A^{(\alpha_0)} \to A)_{\alpha_0 \in A}$, such that $\chi^{(\alpha_0)}$ is increasing with respect to α_0 and $\chi^{(\alpha_0)}(\alpha) = \hat{\alpha}_0 \geqslant \alpha_0$ for $\alpha \in A^{(\alpha_0)}$,

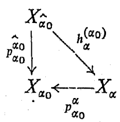
2° a collection of morphisms in $\mathscr{K}^*_\sim$, $(h^{(\alpha_0)}\colon X_{\hat{\alpha}_0} \to X^{(\alpha_0)})_{\alpha_0 \in A}$ such that

(i) $$h^{(\alpha_0)} = (\chi^{(\alpha_0)}, h_\alpha^{(\alpha_0)})$$

and

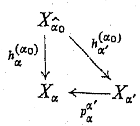
(ii) $$p^\alpha_{\alpha_0} h^{(\alpha_0)}_\alpha \sim p^{\hat{\alpha}_0}_{\alpha_0} \quad \text{for} \quad \alpha \geqslant \alpha_0,$$

i.e., the diagram



commutes up to $\sim$.

The collection $(h^{(\alpha_0)})_{\alpha_0}$ will be referred to as a *uniform movement* of X. Thus, for a uniform movement $(h^{(\alpha_0)})_{\alpha_0}$, the diagram

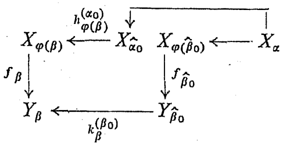


commutes up to $\sim$ for every $\alpha' \geqslant \alpha$.

Omitting this assumption (i.e. assuming $h^{(\alpha_0)}$ to be only a pseudomap in $\mathscr{K}^*_\sim$, see [7] p. 239) one gets a definition of movability in $\mathscr{K}^*_\sim$. In this case the collection $(h^{(\alpha_0)})_{\alpha_0 \in A}$ will be referred to as a *movement* of X.

Consider now two (uniformly) movable inverse systems, $X = (X_\alpha, p_\alpha^{\alpha'}, A)$ and $Y = (Y_\beta, q_\beta^{\beta'}, B)$, and a morphism $f = (\varphi, f_\beta)\colon X \to Y$ in $\mathscr{K}^*_\sim$. Let $(h^{(\alpha_0)})_{\alpha_0 \in A}$ and $(k^{(\beta_0)})_{\beta_0 \in B}$ be (uniform) movements of X and Y, respectively. Define the formula Φ as follows:

$$\Phi((h)), f, (k)) \underset{\mathrm{Df}}{\Leftrightarrow} \bigwedge_{\beta_0} \bigwedge_{\alpha_0 \geqslant \varphi(\beta_0)} \bigvee_{\alpha \geqslant \hat{\alpha}_0, \varphi(\hat{\beta}_0)} \bigwedge_{\beta \geqslant \beta_0} [\varphi(\beta) \geqslant \alpha_0 \Rightarrow f_\beta h^{(\alpha_0)}_{\varphi(\beta)} p^\alpha_{\hat{\alpha}_0} \sim k^{(\beta_0)}_\beta f_{\hat{\beta}_0} p^\alpha_{\varphi(\hat{\beta}_0)}]$$



A morphism $f\colon X \to Y$ is said to be (*uniformly*) *movable* whenever there exist (uniform) movements $(h^{(\alpha_0)})_{\alpha_0 \in A}$ and $(k^{(\beta_0)})_{\beta_0 \in B}$ for X and Y, respectively, such that $\Phi((h), f, (k))$.

Define the formula Φ_w as follows:

$$\Phi_w((h), f, (k)) \underset{\mathrm{Df}}{\Leftrightarrow} \bigwedge_{\beta_0} [\alpha_0 = \varphi(\beta_0) \Rightarrow \bigvee_{\alpha \geqslant \hat{\alpha}_0, \varphi(\hat{\beta}_0)} \bigwedge_{\beta \geqslant \beta_0} f_\beta h^{(\alpha_0)}_{\varphi(\beta)} p^\alpha_{\hat{\alpha}_0} \sim k^{(\beta_0)}_\beta f_{\hat{\beta}_0} p^\alpha_{\varphi(\hat{\beta}_0)}].$$

A morphism $f\colon X \to Y$ is said to be *weakly* (*uniformly*) *movable* whenever there are (uniform) movements $(h^{(\alpha_0)})$ and $(k^{(\beta_0)})$ of X and Y such that $\Phi_w((h), f, (k))$.

The class of uniformly movable maps will be denoted by UM, the class of weakly uniformly movable maps — by WUM. Obviously, $\mathrm{UM} \subset \mathrm{WUM}$.

1.1. Proposition. *Let* $(h^{(\alpha_0)})_{\alpha_0 \in A}$, $(k^{(\beta_0)})_{\beta_0 \in B}$ *and* $(l^{(\gamma_0)})_{\gamma_0 \in C}$ *be uniform movements of* X, Y *and* Z, *respectively, and let* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *be morphisms in* $\mathscr{K}^*_\sim$. *Then*

$$\Phi((h), f, (k)) \wedge \Phi_w((k), g, (l)) \Rightarrow \Phi((h), gf, (l)).$$

Proof. Take $X = (X_\alpha, p_\alpha^{\alpha'}, A)$, $Y = (Y_\beta, q_\beta^{\beta'}, B)$, $Z = (Z_\gamma, r_\gamma^{\gamma'}, C)$ and $f = (\varphi, f_\beta)$, $g = (\psi, g_\gamma)$. Let

$$\Phi((h), f, (k)) \quad \text{and} \quad \Phi_w((k), g, (l)).$$

Then, given $\beta_0 \in B$ and $\alpha_0 \geqslant \varphi(\beta_0)$, there is an $\alpha \geqslant \hat{\alpha}_0, \varphi(\hat{\beta}_0)$ such that

(1) $$f_\beta h^{(\alpha_0)}_{\varphi(\beta)} p^\alpha_{\hat{\alpha}_0} \sim k^{(\beta_0)}_\beta f_{\hat{\beta}_0} p^\alpha_{\varphi(\hat{\beta}_0)} \quad \text{for} \quad \beta \geqslant \beta_0,\ \varphi(\beta) \geqslant \alpha_0.$$

Given a $\gamma_0 \in C$ and $\beta_0 = \psi(\gamma_0)$, there is a $\beta' \geqslant \hat{\beta}_0, \psi(\hat{\gamma}_0)$ such that

(2) $$g_\gamma k^{(\beta_0)}_{\psi(\gamma)} q^{\beta'}_{\hat{\beta}_0} \sim l^{(\gamma_0)}_\gamma g_{\hat{\gamma}_0} q^{\beta'}_{\psi(\hat{\gamma}_0)} \quad \text{for} \quad \gamma \geqslant \gamma_0.$$

To prove $\Phi((h), gf, (l))$, take a $\gamma_0 \in C$ and let $\alpha_0 \geqslant \varphi\psi(\gamma_0)$. Take $\beta_0 = \psi(\gamma_0)$. There is a $\beta' \geqslant \hat{\beta}_0, \psi(\hat{\gamma}_0)$ satisfying (2) and there is an $\alpha \geqslant \hat{\alpha}_0, \varphi(\hat{\beta}_0), \varphi(\beta')$ satisfying (1). Let $\gamma \geqslant \gamma_0$ and $\varphi\psi(\gamma) \geqslant \alpha_0$. Then, setting $\beta = \psi(\gamma)$ in (1) we get

$$(g_\gamma f_{\psi(\gamma)}) h^{(\alpha_0)}_{\varphi\psi(\gamma)} p^\alpha_{\hat{\alpha}_0} \sim g_\gamma k^{(\beta_0)}_{\psi(\gamma)} f_{\hat{\beta}_0} p^\alpha_{\psi(\hat{\beta}_0)} = g_\gamma k^{(\beta_0)}_{\psi(\gamma)} f_{\hat{\beta}_0} p^{\varphi(\beta')}_{\varphi(\hat{\beta}_0)} p^\alpha_{\varphi(\beta')} \sim g_\gamma k^{(\beta_0)}_{\psi(\gamma)} q^{\beta'}_{\hat{\beta}_0} f_{\beta'} p^\alpha_{\varphi(\beta')}$$
$$\sim l^{(\gamma_0)}_\gamma g_{\hat{\gamma}_0} q^{\beta'}_{\psi(\hat{\gamma}_0)} f_{\beta'} p^\alpha_{\varphi(\beta')} \sim l^{(\gamma_0)}_\gamma (g_{\hat{\gamma}_0} f_{\psi(\hat{\gamma}_0)}) p^\alpha_{\varphi\psi(\hat{\gamma}_0)}.$$

Thus the proof is complete. ■

Let us prove

1.2. Proposition. *Let* X *and* Y *be two uniformly movable inverse systems over the same* $(A, \geqslant)$ *and let* $f = (\varphi, f_\beta)\colon X \to Y$ *and* $g = (\psi, g_\alpha)\colon Y \to X$ *be mutually inverse* *isomorphisms in* $\mathscr{K}^{*}_{\sim}$ *with* $\varphi(\alpha)\geqslant\alpha$ *and* $\psi(\alpha)\geqslant\alpha$ *for every* α. *Then for every uniform movement* $(\boldsymbol{h}^{(\alpha_0)})_{\alpha_0}$ *of* X *there is a uniform movement* $(\boldsymbol{k}^{(\beta_0)})_{\beta_0}$ *of* Y *such that*

$$\Phi_w((\boldsymbol{h}),\boldsymbol{f},(\boldsymbol{k})) \quad \textit{and} \quad \Phi((\boldsymbol{k}),\boldsymbol{g},(\boldsymbol{h}))\,.$$

Proof. Take $X=(X_\alpha, p_\alpha^{\alpha'}, A)$, $Y=(Y_\beta, q_\beta^{\beta'}, A)$, $f=(\varphi, f_\beta)$ and $g=(\psi, g_\alpha)$; let $\varphi(\alpha)\geqslant\alpha$, $\psi(\alpha)\geqslant\alpha$ for every $\alpha\in A$, and let $gf\approx 1_X$ and $fg\approx 1_Y$.

Let $(\boldsymbol{h}^{(\alpha_0)})_{\alpha_0\in A}$ be a uniform movement of X. Since g is a right inverse of f, following the proof of Theorem 3.9 [5], we can define a uniform movement $(\boldsymbol{k}^{(\beta_0)})_{\beta_0\in B}$ of Y by the formulae

(1) $$\boldsymbol{k}^{(\beta_0)}=(\varkappa^{(\beta_0)}, k_\beta^{(\beta_0)}),\quad \hat\beta_0=\varkappa^{(\beta_0)}(\beta)\underset{\mathrm{Df}}{=}\psi(\hat\alpha_0)\,,\quad k_\beta^{(\beta_0)}\underset{\mathrm{Df}}{=}f_\beta h^{(\alpha_0)}_{\varphi(\beta)}g_{\hat\alpha_0}\quad \text{for}\quad \beta\geqslant\beta_0,$$

where $\alpha_0=\varphi(\beta_0)$ and $\hat\alpha_0=\chi^{(\alpha_0)}(\alpha)$.

First, let us prove $\Phi_w((\boldsymbol{h}),\boldsymbol{f},(\boldsymbol{k}))$. Take β_0 and let $\alpha_0=\varphi(\beta_0)$. Since g is a left inverse of f, there is an $\alpha\geqslant\hat\alpha_0, \varphi\psi(\hat\alpha_0)$ such that

(2) $$g_{\hat\alpha_0}f_{\psi(\hat\alpha_0)}p^{\alpha}_{\varphi\psi(\hat\alpha_0)}\sim p^{\alpha}_{\hat\alpha_0}\,.$$

By (1) and (2), for every $\beta\geqslant\beta_0$

$$k_\beta^{(\beta_0)}f_{\hat\beta_0}p^{\alpha}_{\varphi(\hat\beta_0)}=f_\beta h^{(\alpha_0)}_{\varphi(\beta)}g_{\hat\alpha_0}f_{\psi(\hat\alpha_0)}p^{\alpha}_{\varphi\psi(\hat\alpha_0)}\sim f_\beta h^{(\alpha_0)}_{\varphi(\beta)}p^{\alpha}_{\hat\alpha_0}\,,$$

and thus $\Phi_w((\boldsymbol{h}),\boldsymbol{f},(\boldsymbol{k}))$.

Now let us prove $\Phi((\boldsymbol{k}),\boldsymbol{g},(\boldsymbol{h}))$. Take α_0 and let $\beta_0\geqslant\psi(\alpha_0)$. Let $\beta\underset{\mathrm{Df}}{=}\hat\beta_0$ and let $\alpha\geqslant\alpha_0$. Since g is a left inverse of f, there is an $\alpha'\geqslant\alpha, \varphi\psi(\alpha)$ such that

(3) $$g_\alpha f_{\psi(\alpha)}p^{\alpha'}_{\varphi\psi(\alpha)}\sim p^{\alpha'}_{\alpha}\,.$$

Since $\varphi\psi(\alpha')\geqslant\alpha'$, we get

(4) $$g_\alpha f_{\psi(\alpha)}p^{\varphi\psi(\alpha')}_{\varphi\psi(\alpha)}\sim p^{\varphi\psi(\alpha')}_{\alpha}\,.$$

By (1) and (4)

$$\begin{aligned} g_\alpha k^{(\beta_0)}_{\psi(\alpha)}q^{\beta}_{\hat\beta_0} &= g_\alpha f_{\psi(\alpha)}h^{(\alpha_0)}_{\varphi\psi(\alpha)}g_{\hat\alpha_0}q^{\alpha}_{\hat\beta_0}\sim g_\alpha f_{\psi(\alpha)}p^{\varphi\psi(\alpha')}_{\varphi\psi(\alpha)}h^{(\alpha_0)}_{\varphi\psi(\alpha')}g_{\hat\alpha_0}q^{\beta}_{\hat\beta_0}\\ &\sim p^{\varphi\psi(\alpha')}_{\alpha}h^{(\alpha_0)}_{\varphi\psi(\alpha')}g_{\hat\alpha_0}q^{\beta}_{\hat\beta_0}\sim h^{(\alpha_0)}_{\alpha}g_{\hat\alpha_0}q^{\beta}_{\psi(\hat\alpha_0)}\,.\end{aligned}$$

Thus $\Phi((\boldsymbol{k}),\boldsymbol{g},(\boldsymbol{h}))$. ■

Let us prove

1.3. Proposition. *Let* $f\approx f'\colon X\to Y$ *and let* $(\boldsymbol{h}^{(\alpha_0)})_{\alpha_0\in A}$ *and* $(\boldsymbol{k}^{(\beta_0)})_{\beta_0\in B}$ *be uniform movements of* X *and* Y, *respectively. Then*

$$\Phi((\boldsymbol{h}),\boldsymbol{f},(\boldsymbol{k}))\Rightarrow\bigvee_{(\boldsymbol{h}'^{(\alpha_0)})_{\alpha_0}}\Phi((\boldsymbol{h}'),\boldsymbol{f}',(\boldsymbol{k}))\,.$$

Proof. Take $f=(\varphi, f_\beta)$ and $f'=(\varphi', f'_\beta)$. Let $\Phi((\boldsymbol{h}),\boldsymbol{f},(\boldsymbol{k}))$, i.e.,

(1) $$\bigwedge_{\beta_0}\ \bigwedge_{\alpha_0\geqslant\varphi(\beta_0)}\ \bigvee_{\alpha\geqslant\hat\alpha_0,\,\varphi(\hat\beta_0)}\ \bigwedge_{\beta\geqslant\beta_0}[\varphi(\beta)\geqslant\alpha_0\Rightarrow f_\beta h^{(\alpha_0)}_{\varphi(\beta)}p^{\alpha}_{\hat\alpha_0}\sim k^{(\beta_0)}_\beta f_{\hat\beta_0}p^{\alpha}_{\varphi(\hat\beta_0)}]\,.$$

We are going to define a uniform movement $(\boldsymbol{h}'^{(\alpha_0)})_{\alpha_0}$ satisfying $\Phi((\boldsymbol{h}'),\boldsymbol{f}',(\boldsymbol{k}))$. Since $(A,\geqslant)$ is a directed set, given a pair of functions (φ,φ') we can find a function $\delta\colon A\to A$ satisfying the following condition:

(2) $$\alpha_0\geqslant\varphi'(\beta_0)\ \Rightarrow\ \delta(\alpha_0)\geqslant\alpha_0,\ \varphi(\beta_0)\,.$$

Since $(A,\geqslant)$ is closure finite, δ can be made increasing (see Lemma 5 of [3]). By (2), $\chi^{(\delta(\alpha_0))}\geqslant\chi^{(\alpha_0)}$. Let

(3) $$\hat{\hat\alpha}=\chi'^{(\alpha_0)}(\alpha)\underset{\mathrm{Df}}{=}\chi^{(\delta(\alpha_0))}(\alpha),\qquad h'^{(\alpha_0)}_\alpha\underset{\mathrm{Df}}{=}h^{(\alpha_0)}_\alpha p^{\hat{\hat\alpha}_0}_{\hat\alpha_0}\quad \text{for}\quad \alpha\geqslant\alpha_0$$

and let

$$\boldsymbol{h}'^{(\alpha_0)}\underset{\mathrm{Df}}{=}(\chi'^{(\alpha_0)}, h'^{(\alpha_0)}_\alpha)\,.$$

Notice that $\boldsymbol{h}'^{(\alpha_0)}$ is a morphism in $\mathscr{K}^{*}_{\sim}$; indeed, for $\alpha'\geqslant\alpha$

$$p^{\alpha'}_\alpha h'^{(\alpha_0)}_{\alpha'}=p^{\alpha'}_\alpha h^{(\alpha_0)}_{\alpha'}p^{\hat{\hat\alpha}_0}_{\hat\alpha_0}\sim h^{(\alpha_0)}_\alpha p^{\hat{\hat\alpha}_0}_{\hat\alpha_0}=h'^{(\alpha_0)}_\alpha\,.$$

The maps $\boldsymbol{h}'^{(\alpha_0)}$, $\alpha_0\in A$, form a movement of X; indeed, for $\alpha\geqslant\alpha_0$

$$p^{\alpha}_{\alpha_0}h'^{(\alpha_0)}_\alpha=p^{\alpha}_{\alpha_0}h^{(\alpha_0)}_\alpha p^{\hat{\hat\alpha}_0}_{\hat\alpha_0}\sim p^{\hat\alpha_0}_{\alpha_0}p^{\hat{\hat\alpha}_0}_{\hat\alpha_0}=p^{\hat{\hat\alpha}_0}_{\alpha_0}\,.$$

It remains to show that $\Phi((\boldsymbol{h}'),\boldsymbol{f}',(\boldsymbol{k}))$. Take $\beta_0\in B$ and let $\alpha_0\geqslant\varphi'(\beta_0)$; then, by (2), $\delta(\alpha_0)\geqslant\varphi(\beta_0)$. Thus, by (1), there is an $\alpha\geqslant\hat\alpha_0, \varphi(\hat\beta_0)$ such that

(4) $$f_\beta h^{(\delta(\alpha_0))}_{\varphi(\beta)}p^{\alpha}_{\hat{\hat\alpha}_0}\sim k^{(\beta_0)}_\beta f_{\hat\beta_0}p^{\alpha}_{\varphi(\hat\beta_0)}\quad \text{for every}\ \ \beta\geqslant\beta_0,\ \ \varphi(\beta)\geqslant\alpha_0.$$

Since $f'\approx f$, there is an $\alpha'\geqslant\varphi(\hat\beta_0), \varphi'(\hat\beta_0)$ such that

(5) $$f'_{\hat\beta_0}p^{\alpha'}_{\varphi'(\hat\beta_0)}\sim f_{\hat\beta_0}p^{\alpha'}_{\varphi(\hat\beta_0)}\,.$$

Of course, we can assume

(6) $$\alpha'\geqslant\alpha,\ \hat{\hat\alpha}_0\,.$$

Take $\beta\geqslant\beta_0$ with $\varphi(\beta)\geqslant\alpha_0$; there is an $\alpha''\geqslant\varphi(\beta), \varphi'(\beta)$ such that

(7) $$f'_\beta p^{\alpha''}_{\varphi'(\beta)}\sim f_\beta p^{\alpha''}_{\varphi(\beta)}\,.$$

Applying (3)-(7), we get

$$\begin{aligned} f'_\beta h'^{(\alpha_0)}_{\varphi'(\beta)}p^{\alpha'}_{\hat{\hat\alpha}_0} &= f'_\beta h^{(\alpha_0)}_{\varphi'(\beta)}p^{\alpha'}_{\hat\alpha_0}\sim f'_\beta p^{\alpha''}_{\varphi'(\beta)}h^{(\alpha_0)}_{\alpha''}p^{\alpha'}_{\hat\alpha_0}\sim f_\beta p^{\alpha''}_{\varphi(\beta)}h^{(\alpha_0)}_{\alpha''}p^{\alpha'}_{\hat\alpha_0}\\ &\sim f_\beta h^{(\alpha_0)}_{\varphi(\beta)}p^{\alpha'}_{\hat\alpha_0}\sim k^{(\beta_0)}_\beta f_{\hat\beta_0}p^{\alpha'}_{\varphi(\hat\beta_0)}\sim k^{(\beta_0)}_\beta f'_{\hat\beta_0}p^{\alpha'}_{\varphi'(\hat\beta_0)}\,.\end{aligned}$$

Thus $\Phi((\boldsymbol{h}'),\boldsymbol{f}',(\boldsymbol{k}))$. ■

1.4. COROLLARY. *If $f \approx f'$ and $f \in \mathrm{UM}$ then $f' \in \mathrm{UM}$.* ■

As a consequence of 1.2, 1.4 and the statement 3.8 of [5], we get.

1.5. COROLLARY. *Let X and Y be two uniformly movable inverse systems over A. If $g\colon Y \to X$ is an isomorphism in $\hat{\mathscr{K}}^*_{\approx}$, then $g \in \mathrm{UM}$.*

Proof. Let $g = (\psi, g_\alpha)\colon Y \to X$ be an isomorphism in $\hat{\mathscr{K}}^*_{\approx}$. By 3.8 of [5], there is a map $g' = (\psi', g'_\alpha)\colon Y \to X$ such that $g' \approx g$ and $\psi'(\alpha) \geqslant \alpha$ for every α. Obviously g' is an isomorphism again. By 1.2, g' is uniformly movable; thus, by 1.4, g is uniformly movable as well. ■

Let us establish the following law of composing with isomorphisms.

1.6. PROPOSITION. *Let X and X' (Y and Y') be two inverse systems over A (over B) and let $i'\colon X' \to X$ and $j\colon Y \to Y'$ be isomorphisms in $\hat{\mathscr{K}}^*_{\approx}$. Then for every $f\colon X \to Y$ the following implications hold:*

(a) $f \in \mathrm{WUM} \Rightarrow fi' \in \mathrm{UM}$,

(b) $f \in \mathrm{UM} \Rightarrow jf \in \mathrm{UM}$.

Proof. Let $j'\colon Y' \to Y$ and $i\colon X \to X'$ be inverse isomorphisms for j and i'. By 3.8 of [5] together with Corollary 1.4, i can be assumed to be of the form $i = (\tau, i_\alpha)$ with $\tau(\alpha) \geqslant \alpha$ for $\alpha \in A$; the same holds for i', j and j'.

(a) If $f \in \mathrm{WUM}$, then there are uniform movements $(h^{(\alpha_0)})_{\alpha_0}$ and $(k^{(\beta_0)})_{\beta_0}$ for X and Y such that $\Phi_w((h), f, (k))$. By Proposition 1.2 applied to the pair (i, i'), there is a uniform movement $(h'^{(\alpha_0)})_{\alpha_0}$ for X such that $\Phi((h)', i', (h))$. By 1.1,

$$\Phi((h'), i', (h)) \wedge \Phi_w((h), f, (k)) \Rightarrow \Phi((h'), fi', (k)).$$

Hence $fi' \in \mathrm{UM}$.

(b) If $f \in \mathrm{UM}$, there are $(h^{(\alpha_0)})_{\alpha_0}$ and $(k^{(\beta_0)})_{\beta_0}$ such that $\Phi((h), f, (k))$. By 1.2 applied to the pair (j, j'), there is a uniform movement $(k'^{(\beta_0)})_{\beta_0}$ of Y such that $\Phi_w((k), j, (k'))$. By 1.1,

$$\Phi((h), f, (k)) \wedge \Phi_w((k), j, (k')) \Rightarrow \Phi((h), jf, (k')).$$

Thus $jf \in \mathrm{UM}$. ■

As a direct consequence of 1.6 and 1.4, we get

1.7. COROLLARY. *Consider the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

i and j being isomorphisms in $\hat{\mathscr{K}}^*_{\approx}$ and X and X' (Y and Y') being inverse systems over the same A (the same B).

If the diagram is commutative up to $\approx$, then

$$f \in \mathrm{UM} \Rightarrow f' \in \mathrm{UM}. \ ■$$

Notice that

1.8. *Any covariant functor $\pi\colon (\mathscr{K}, \sim) \to (\mathscr{K}', \sim)$ preserves (uniform) movability of morphisms, i.e., for every $f \in \mathrm{Mor}_{\mathscr{K}^*_{\sim}}(X, Y)$, if f is (uniformly) movable in $\mathscr{K}^*_{\sim}$ then $\pi(f)$ is (uniformly) movable in $\mathscr{K}'^*_{\sim}$.* ■

2. Uniform movability of kernels. We are now interested in a category with kernels, $\mathscr{K}$. Fix a directed set $(A, \geqslant)$. Let $\sim$ be the identity relation $=$, and let $\mathscr{K}^* = \mathscr{K}^*_{=}$ and $\hat{\mathscr{K}}^* = \hat{\mathscr{K}}^*_{=}$ (see [5]). By Proposition 3.3 of [7], $\mathscr{K}^*$ and $\hat{\mathscr{K}}^*$ are again categories with kernels and for any morphism $f = (1_A, f_\alpha)\colon X \to Y$ in $\mathscr{K}^*$, $\operatorname{Ker} f$ is of the form

$$\operatorname{Ker} f = (N, j), \quad \text{where} \quad N = (N_\alpha, n_\alpha^{\alpha'}, A), \quad j = (1_A, j_\alpha) \quad \text{and} \quad (N_\alpha, j_\alpha) = \operatorname{Ker} f_\alpha.$$

We shall use the notation $\ker f$ for N. Let us prove

2.1. PROPOSITION. *If f is weakly uniformly movable in $\mathscr{K}^*$, then $\ker f$ is uniformly movable in $\mathscr{K}^*$.*

Proof. Take $X = (X_\alpha, p_\alpha^{\alpha'}, A)$, $Y = (Y_\alpha, q_\alpha^{\alpha'}, A)$ and let $\operatorname{Ker} f = (N, j)$, $\ker f = N = (N_\alpha, n_\alpha^{\alpha'}, A)$ and $j = (1_A, j_\alpha)\colon N \to X$. By 3.3 Section 1, [7], $(N_\alpha, j_\alpha) = \operatorname{Ker} f_\alpha$. Then

(1) the diagram
$$\begin{array}{ccc} N_\alpha & \xleftarrow{n_\alpha^{\alpha'}} & N_{\alpha'} \\ {\scriptstyle j_\alpha}\downarrow & & \downarrow{\scriptstyle j_{\alpha'}} \\ X_\alpha & \xleftarrow[p_\alpha^{\alpha'}]{} & X_{\alpha'} \end{array}$$
is commutative for $\alpha' \geqslant \alpha$.

Since $f \in \mathrm{WUM}$, there are uniform movements $(h^{(\alpha_0)})_{\alpha_0 \in A}$ and $(k^{(\alpha_0)})_{\alpha_0 \in A}$ for X and Y, respectively, such that

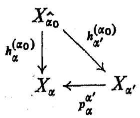
(2)
$$\bigwedge_{\alpha_0} \bigvee_{\alpha_0^* \geqslant \hat{\alpha}_0} \bigwedge_{\alpha \geqslant \alpha_0} f_\alpha h_\alpha^{(\alpha_0)} p_{\hat{\alpha}_0}^{\alpha_0^*} = k_\alpha^{(\alpha_0)} f_{\hat{\alpha}_0} p_{\hat{\alpha}_0}^{\alpha_0^*}$$

$$\begin{array}{ccccc} X_\alpha & \xleftarrow{h_\alpha^{(\alpha_0)}} & X_{\hat{\alpha}_0} & \xleftarrow{p_{\hat{\alpha}_0}^{\alpha_0^*}} & X_{\alpha_0^*} \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle f_{\hat{\alpha}_0}} & & \\ Y_\alpha & \xleftarrow[k_\alpha^{(\alpha_0)}]{} & Y_{\hat{\alpha}_0} & & \end{array}$$

Since $(h^{(\alpha_0)})_{\alpha_0 \in A}$ is a uniform movement, it follows that

(3) the diagram
$$\begin{array}{ccc} X_{\hat{\alpha}_0} & & \\ {\scriptstyle p_{\alpha_0}^{\hat{\alpha}_0}}\downarrow & \searrow{\scriptstyle h_\alpha^{(\alpha_0)}} & \\ X_{\alpha_0} & \xleftarrow[p_{\alpha_0}^{\alpha}]{} & X_\alpha \end{array}$$
is commutative for $\alpha \geqslant \alpha_0$

and

(4) the diagram  is commutative for $\alpha'\geqslant\alpha\geqslant\alpha_0$.

Take an $\alpha_0\in A$ and let $\delta(\alpha_0) \underset{\mathrm{Df}}{=} \alpha_0^*$. By Lemma 5 of [3], δ may be assumed to be increasing. Take $\alpha\geqslant\alpha_0$ and consider the map

$$h_\alpha^{(\alpha_0)} j_{\hat\alpha_0} n_{\hat\alpha_0}^{\alpha_0^*}\colon N_{\alpha_0^*}\to X_\alpha .$$

Since $N_{\hat\alpha_0}=\ker f_{\hat\alpha_0}$, we get by (1) and (2)

$$f_\alpha(h_\alpha^{(\alpha_0)} j_{\hat\alpha_0} n_{\hat\alpha_0}^{\alpha_0^*}) = f_\alpha h_\alpha^{(\alpha_0)} p_{\hat\alpha_0}^{\alpha_0^*} j_{\alpha_0^*} = k_\alpha^{(\alpha_0)} f_{\hat\alpha_0} p_{\hat\alpha_0}^{\alpha_0^*} j_{\alpha_0^*} = k_\alpha^{(\alpha_0)}(f_{\hat\alpha_0} j_{\hat\alpha_0}) n_{\hat\alpha_0}^{\alpha_0^*}$$
$$= k_\alpha^{(\alpha_0)}\omega_{N_{\hat\alpha_0}Y_{\hat\alpha_0}} n_{\hat\alpha_0}^{\alpha_0^*} = \omega_{N_{\alpha_0^*}Y_\alpha} .$$

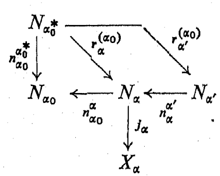
Thus, there is a map

$$r_\alpha^{(\alpha_0)}\colon N_{\alpha_0^*}\to N_\alpha$$

such that

(5) $$j_\alpha r_\alpha^{(\alpha_0)} = h_\alpha^{(\alpha_0)} j_{\hat\alpha_0} n_{\hat\alpha_0}^{\alpha_0^*} .$$

Consider the diagram



By (1), (4) and (5)

$$j_\alpha(n_\alpha^{\alpha'} r_{\alpha'}^{(\alpha_0)}) = p_\alpha^{\alpha'}(j_{\alpha'} r_{\alpha'}^{(\alpha_0)}) = p_\alpha^{\alpha'} h_{\alpha'}^{(\alpha_0)} j_{\hat\alpha_0} n_{\hat\alpha_0}^{\alpha_0^*} = h_\alpha^{(\alpha_0)} j_{\hat\alpha_0} n_{\hat\alpha_0}^{\alpha_0^*} = j_\alpha r_\alpha^{(\alpha_0)} .$$

Since j_α is a monomorphism, it follows that

(6) $$n_\alpha^{\alpha'} r_{\alpha'}^{(\alpha_0)} = r_\alpha^{(\alpha_0)} .$$

Let

$$\varrho^{(\alpha_0)}(\alpha) \underset{\mathrm{Df}}{=} \delta(\alpha_0) \quad \text{for} \quad \alpha\geqslant\alpha_0$$

and let

$$\boldsymbol{r}^{(\alpha_0)} \underset{\mathrm{Df}}{=} (\varrho^{(\alpha_0)}, r_\alpha^{(\alpha_0)}) .$$

By (6), $\boldsymbol{r}^{(\alpha_0)}$ is a morphism of $\mathcal{K}^*$. By (1), (3) and (5)

$$j_{\alpha_0}(n_{\alpha_0}^\alpha r_\alpha^{(\alpha_0)}) = p_{\alpha_0}^\alpha(j_\alpha r_\alpha^{(\alpha_0)}) = p_{\alpha_0}^\alpha h_\alpha^{(\alpha_0)} j_{\hat\alpha_0} n_{\hat\alpha_0}^{\alpha_0^*} = p_{\alpha_0}^{\hat\alpha_0} j_{\hat\alpha_0} n_{\hat\alpha_0}^{\alpha_0^*} = j_{\alpha_0} n_{\alpha_0}^{\alpha_0^*} .$$

Since j_{α_0} is a monomorphism, it follows that

(7) $$n_{\alpha_0}^\alpha r_\alpha^{(\alpha_0)} = n_{\alpha_0}^{\alpha_0^*} ,$$

whence $(\boldsymbol{r}^{(\alpha_0)})_{\alpha_0\in A}$ is a uniform movement of N. ■

3. Relative monomorphisms. Let $\mathcal{K}_0$ be a subcategory of a category $\mathcal{K}$ and let $f\in \mathrm{Mor}_{\mathcal{K}}(X, Y)$;

$$f \text{ is a monomorphism in } (\mathcal{K}, \mathcal{K}_0) \underset{\mathrm{Df}}{\Leftrightarrow} \bigwedge_{Z\in \mathrm{Ob}_{\mathcal{K}_0}} \ \bigwedge_{v,v'\in \mathrm{Mor}_{\mathcal{K}}(Z,X)} [fv = fv' \Rightarrow v = v'] .$$

We are going to replace some statements of [5] and [7] concerning monomorphisms by stronger statements concerning relative monomorphisms. First, statement 1.5, Section 1 of [7] can be replaced by the following

3.1. PROPOSITION. *If f is a monomorphism in $(\mathcal{K}, \mathcal{K}_0)$ and $\ker f\in \mathrm{Ob}_{\mathcal{K}_0}$ then* $\ker f = 0$.

Proof. Let $\mathrm{Ker} f = (N, j)$. By the assumption, $N\in \mathrm{Ob}_{\mathcal{K}_0}$ and f is a monomorphism in $(\mathcal{K}, \mathcal{K}_0)$. Since $j, \omega_{NX}\in \mathrm{Mor}_{\mathcal{K}}(N, X)$ and $fj = \omega_{NY} = f\omega_{NX}$, it follows that $j = \omega_{NX}$. Take an arbitrary $\varphi\colon N\to N$. We have $j\varphi = \omega_{NX}\varphi = \omega_{NX} = j\omega_{NN}$, where j is a monomorphism; so $\varphi = \omega_{NN}$. Thus $N = 0$. ■

Secondly Proposition 2.3, Section 1 of [7] can be replaced by the following

3.2. PROPOSITION. *Given an exact diagram*

$$X\xrightarrow{\tau}Y\xrightarrow{\xi}Z\xrightarrow{\partial}X'\xrightarrow{\tau'}Y'$$

in a weak additive category $\mathcal{K}$ with zero objects, let τ be an epimorphism in $\mathcal{K}$ and τ' a monomorphism in $(\mathcal{K}, \mathcal{K}_0)$. If $\ker\tau'\in \mathrm{Ob}_{\mathcal{K}_0}$ then $Z = 0$.

Proof. Since τ' is a monomorphism in $(\mathcal{K}, \mathcal{K}_0)$ and $\ker\tau'\in \mathrm{Ob}_{\mathcal{K}_0}$, it follows by Proposition 3.1 that $\ker\tau' = 0$ and thus $\mathrm{Ker}\tau' = (0, \omega_{OX'})$. Since τ is an epimorphism, it follows by 1.6, Section 1 of [7] that $\mathrm{Coker}\tau = (0, \omega_{YO})$. Then, by 1.7, Section 1, [7], $\mathrm{Im}\tau = (N_p, j_p)$, j_p being an isomorphism. By the exactness of the diagram, $\mathrm{Ker}\xi = \mathrm{Im}\tau = (N_p, j_p)$, and thus by 1.8, Section 1, [7], $\mathrm{Im}\xi = (0, \omega_{OZ})$. By the exactness, $\mathrm{Ker}\partial = \mathrm{Im}\xi = (0, \omega_{OZ})$ and $\mathrm{Im}\partial = \mathrm{Ker}\tau' = (0, \omega_{OX'})$. Hence, by 2.2, Section 1 of [7], we get $Z = 0$. ■

Now consider the category $\mathcal{K}^*$ of inverse systems in $\mathcal{K}$ and the subcategory $\mathcal{K}_0^*$ of uniformly movable inverse systems. Let $\hat{\mathcal{K}}^*$ and $\hat{\mathcal{K}}_0^*$ be the corresponding quotient categories with respect to the relation $\cong$ in $\mathrm{Mor}_{\mathcal{K}^*}$ (see [5]). Corollary 6.6 of [5] (as well as 4.3 and 4.4) can be replaced by the following

3.3. Proposition. *Let* $f \in \mathrm{Mor}_{\mathscr{K}^*}(X, Y)$ *and let* $f = \underline{\lim} f$.

(1) *If* $Y \in \mathrm{Ob}_{\hat{\mathscr{K}}_0^*}$ *and* f *is an epimorphism in* $\mathscr{K}$ *then* $[f]$ *is an epimorphism in* $\hat{\mathscr{K}}^*$.

(2) *If* f *is a monomorphism in* $\mathscr{K}$ *then* $[f]$ *is a monomorphism in* $(\hat{\mathscr{K}}^*, \hat{\mathscr{K}}_0^*)$.

Proof. See the proofs of 4.3 and 4.4 of [5]. ■

4. Uniformly movable maps of ANR systems. Let us now consider the category $\mathscr{R}$ of pointed pairs of ANR's with continuous maps as morphisms and the homotopy relation $\simeq$ as $\sim$. Identifying (X, x_0) with $(X, \{x_0\}, x_0)$, we can consider any (X, x_0) as an object of $\mathscr{R}$. Let us prove the following

4.1. Proposition. *Let* $(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{x}_0)$ *be an inverse system in* $\mathscr{R}$ *and let* $\boldsymbol{i}\colon (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{Z}, \boldsymbol{x}_0)$ *be the inclusion map. If* $(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{x}_0)$ *is uniformly movable in* $\mathscr{R}^*_{\simeq}$ *then* $\boldsymbol{i}$ *is weakly uniformly movable.*

Proof. Let $(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{x}_0) = \big((Z_\alpha, X_\alpha, x_\alpha), r_\alpha^{\alpha'}, A\big)$ and let $p_\alpha^{\alpha'} = r_\alpha^{\alpha'} | X_{\alpha'}$. Since $(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{x}_0)$ is uniformly movable, there is a uniform movement $(\hat{\boldsymbol{k}}^{(\alpha_0)})_{\alpha_0}$ of $(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{x}_0)$, $\hat{\boldsymbol{k}}^{(\alpha_0)} = (\varkappa^{(\alpha_0)}, \hat{k}_\alpha^{(\alpha_0)})$. Define

$$\boldsymbol{h}^{(\alpha_0)}\colon (X_{\hat{\alpha}_0}, x_{\hat{\alpha}_0}) \to (\boldsymbol{X}, \boldsymbol{x}_0)^{(\alpha_0)} \quad \text{and} \quad \boldsymbol{k}^{(\alpha_0)}\colon (Z_{\hat{\alpha}_0}, x_{\hat{\alpha}_0}) \to (\boldsymbol{Z}, \boldsymbol{x}_0)^{(\alpha_0)}$$

by the formulae

$$\boldsymbol{h}^{(\alpha_0)} = (\varkappa^{(\alpha_0)}, h_\alpha^{(\alpha_0)}) \quad \text{and} \quad \boldsymbol{k}^{(\alpha_0)} = (\varkappa^{(\alpha_0)}, k_\alpha^{(\alpha_0)}),$$

where

$$h_\alpha^{(\alpha_0)}(x) = \hat{k}_\alpha^{(\alpha_0)}(x) \text{ for } x \in X_{\hat{\alpha}_0} \quad \text{and} \quad k_\alpha^{(\alpha_0)}(z) = \hat{k}_\alpha^{(\alpha_0)}(z) \text{ for } z \in Z_{\hat{\alpha}_0}.$$

Evidently $(\boldsymbol{h}^{(\alpha_0)})_{\alpha_0}$ and $(\boldsymbol{k}^{(\alpha_0)})_{\alpha_0}$ are uniform movements. Let us prove $\Phi_w((\boldsymbol{h}), \boldsymbol{i}, (\boldsymbol{k}))$. Take an $\alpha_0 \in A$ and let $\alpha = \hat{\alpha}_0$. Then for every $\beta \geqslant \alpha_0$ and for every $x \in X_\alpha$

$$i_\beta h_\beta^{(\alpha_0)} p_{\hat{\alpha}_0}^{\alpha}(x) = k_\beta^{(\alpha_0)} i_{\hat{\alpha}_0} p_{\hat{\alpha}_0}^{\alpha}(x);$$

thus all the more

$$i_\beta h_\beta^{(\alpha_0)} p_{\hat{\alpha}_0}^{\alpha} \simeq k_\beta^{(\alpha_0)} i_{\hat{\alpha}_0} p_{\hat{\alpha}_0}^{\alpha}$$

i.e., $\Phi_w((\boldsymbol{h}), \boldsymbol{i}, (\boldsymbol{k}))$. ■

The notion of a mapping cylinder introduced in [4] has recently been modified by Mardešić, [2]. Contracting each segment $(x_\alpha) \times I$ in $X_\alpha \times I$ to a point, he obtains a mapping cylinder of a map of pointed ANR sequences. This mapping cylinder has the same properties in the pointed category as the cylinder defined in [4] in the non-pointed category; i.e. the inclusion $\boldsymbol{j}\colon (\boldsymbol{Y}, \boldsymbol{y}_0) \to (\boldsymbol{C_f}, \boldsymbol{x}_0)$ is a homotopy equivalence and $\boldsymbol{jf} \simeq \boldsymbol{i}$, where $\boldsymbol{i}\colon (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{C_f}, \boldsymbol{x}_0)$ is the inclusion.

4.2. Proposition. *Let* $\boldsymbol{C_f}$ *be a mapping cylinder of a map of* ANR *sequences,* $\boldsymbol{f}\colon (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{Y}, \boldsymbol{y}_0)$. *If* $(\boldsymbol{C_f}, \boldsymbol{Y}, \boldsymbol{x}_0)$ *is uniformly movable then* $\boldsymbol{f} \in \mathrm{UM}$.

Proof. Take the inclusions $\boldsymbol{i}\colon (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{C_f}, \boldsymbol{x}_0)$ and $\boldsymbol{j}\colon (\boldsymbol{Y}, \boldsymbol{y}_0) \to (\boldsymbol{C_f}, \boldsymbol{x}_0)$. By 4.1, $\boldsymbol{i} \in \mathrm{WUM}$. The inclusion $\boldsymbol{j}$ is a homotopy equivalence. Let $\boldsymbol{j}'$ be a homotopy inverse of $\boldsymbol{j}$. Since $\boldsymbol{fj} \simeq \boldsymbol{i}$, we have $\boldsymbol{f} \simeq \boldsymbol{ij}'$, where $\boldsymbol{j}'$ is an isomorphism in $\mathscr{R}^*_{\simeq}$ and $\boldsymbol{i} \in \mathrm{WUM}$. Thus, by 1.6, $\boldsymbol{ij}' \in \mathrm{UM}$, whence, by 1.4, $\boldsymbol{f} \in \mathrm{UM}$. ■

Remarks. 1. Propositions 4.1 and 4.2 apply automatically to non-pointed spaces.

2. These two propositions hold for inverse sequences of arbitrary topological spaces, not necessarily ANR's.

5. The Whitehead Theorem for uniformly movable maps of ANR sequences. Let $\mathscr{G}$ be the category of groups, $\mathscr{C}$ — the category of pointed sets, let $\mathscr{G}^*(\hat{\mathscr{G}}^*)$ and $\mathscr{C}^*(\hat{\mathscr{C}}^*)$ be corresponding categories of inverse systems (their quotient categories). Let $\hat{\mathscr{G}}_0^*$ and $\hat{\mathscr{C}}_0^*$ be full subcategories of uniformly movable inverse systems.

5.1. Theorem. *Let* $\boldsymbol{f} = (1_A, f_\alpha)\colon (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{Y}, \boldsymbol{y}_0)$ *be a uniformly movable map of inverse sequences of pointed arcwise connected spaces. Let* $\boldsymbol{f}_n\colon \pi_n(\boldsymbol{X}, \boldsymbol{x}_0) \to \pi_n(\boldsymbol{Y}, \boldsymbol{y}_0)$ *be the induced morphism of n-th homotopy systems. If* $[\boldsymbol{f}_n]$ *is a monomorphism in* $(\hat{\mathscr{C}}^*, \hat{\mathscr{C}}_0^*)$ *for* $n = 1, \ldots, n_0$, *an epimorphism in* $\hat{\mathscr{C}}^*$ *for* $n = 1$ *and in* $\hat{\mathscr{G}}^*$ *for* $n = 2, \ldots, n_0+1$, *then* $\pi_n(\boldsymbol{C_f}, \boldsymbol{X}, \boldsymbol{x}_0)$ *is a zero object in* $\hat{\mathscr{C}}^*$ *for* $n = 1, \ldots, n_0+1$.

Proof. Let $\boldsymbol{i}\colon (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{C_f}, \boldsymbol{x}_0)$ be the inclusion. By the assumption on $\boldsymbol{f}_n$ it follows that

(1) $[\boldsymbol{i}_n]$ is a monomorphism in $(\hat{\mathscr{G}}^*, \hat{\mathscr{G}}_0^*)$ for $n = 1, \ldots, n_0$ and

$$[\boldsymbol{i}_n] \text{ is an epimorphism in } \begin{cases} \hat{\mathscr{C}}^* & \text{for } n = 1, \\ \hat{\mathscr{G}}^* & \text{for } n = 2, \ldots, n_0+1. \end{cases}$$

Let $n = 1$. By 1.2, Section 2 [7], the diagram

$$\mathscr{D}_1\colon \ \pi_1(\boldsymbol{X}, \boldsymbol{x}_0) \xrightarrow{[\boldsymbol{i}_1]} \pi_1(\boldsymbol{C_f}, \boldsymbol{x}_0) \xrightarrow{[\boldsymbol{\xi}_1]} \pi_1(\boldsymbol{C_f}, \boldsymbol{X}, \boldsymbol{x}_0) \xrightarrow{[\boldsymbol{\partial}_1]} 0$$

is exact. Since, by (1), $[\boldsymbol{i}_1]$ is an epimorphism in $\hat{\mathscr{C}}^*$, by 1.9, Section 1 [7] it follows that

(2) $\pi_1(\boldsymbol{C_f}, \boldsymbol{X}, \boldsymbol{x}_0)$ is a zero object in $\hat{\mathscr{C}}^*$.

Let $2 \leqslant n \leqslant n_0+1$. By 1.2, Section 2 [7], the diagram

$$\mathscr{D}_n\colon \ \pi_n(\boldsymbol{X}, \boldsymbol{x}_0) \xrightarrow{[\boldsymbol{i}_n]} \pi_n(\boldsymbol{C_f}, \boldsymbol{x}_0) \xrightarrow{[\boldsymbol{\xi}_n]} \pi_n(\boldsymbol{C_f}, \boldsymbol{X}, \boldsymbol{x}_0) \xrightarrow{[\boldsymbol{\partial}_n]} \pi_{n-1}(\boldsymbol{X}, \boldsymbol{x}_0) \xrightarrow{[\boldsymbol{i}_{n-1}]} \pi_{n-1}(\boldsymbol{C_f}, \boldsymbol{x}_0)$$

is exact. Since $\boldsymbol{f} \in \mathrm{UM}$ and $\boldsymbol{i} \simeq \boldsymbol{jf}$, it follows by 1.4 and 1.6 that $\boldsymbol{i} \in \mathrm{UM}$. Thus, by 1.8 $[\boldsymbol{i}_{n-1}] \in \mathrm{UM}$, whence, by 2.1, $\ker [\boldsymbol{i}_{n-1}] \in \mathrm{Ob}_{\hat{\mathscr{G}}_0^*}$. By (1), $[\boldsymbol{i}_n]$ is an epimorphism in $\hat{\mathscr{G}}^*$ and $[\boldsymbol{i}_{n-1}]$ is a monomorphism in $(\hat{\mathscr{G}}^*, \hat{\mathscr{G}}_0^*)$; thus Proposition 3.2 implies

(3) $\pi_n(\boldsymbol{C_f}, \boldsymbol{X}, \boldsymbol{x}_0)$ is a zero object in $\hat{\mathscr{G}}^*$.

By (2) and (3), $\pi_n(\boldsymbol{C_f}, \boldsymbol{X}, \boldsymbol{x}_0)$ is a zero object for $n = 1, \ldots, n_0+1$. ■

As a consequence of Theorem 5.1 let us prove

5.2. Theorem. *Let* $\boldsymbol{f} = (1_N, f_\alpha)\colon (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{Y}, \boldsymbol{y}_0)$ *be a uniformly movable map of inverse sequences in the* PL*-category* $\mathscr{P}$, *all the spaces being connected and all the bonding maps being "onto". Let* $n_0 = \max\,(1+\dim \boldsymbol{X}, \dim \boldsymbol{Y}) < \infty$. *Let* $\boldsymbol{f}_n$ *be the induced morphism of n-th homotopy systems. If* $[\boldsymbol{f}_n]$ *is a monomorphism in* $(\hat{\mathscr{C}}^*, \hat{\mathscr{C}}_0^*)$ *for*

$n = 1, \ldots, n_0$, *an epimorphism in* $\hat{\mathscr{C}}^*$ *for* $n = 1$ *and in* $\hat{\mathscr{G}}^*$ *for* $n = 2, \ldots, n$ *then* f *is a homotopy equivalence.*

Proof [1]. By Theorem 5.1, $\pi_n(C_f, X, x_0) = 0$ for $n = 1, \ldots, n_0+1$. Thus, by Theorem 2 of [2], f is a (pointed) homotopy equivalence. ■

5.3. Remark. By 4.2 the assumption $f \in \mathrm{UM}$ in Theorems 5.1 and 5.2 can be replaced by the assumption of uniform movability of (C_f, X, x_0).

Notice that Propositions 4.2, 5.1 and 5.2 remain valid if we replace the map of inverse sequences and its mapping cylinder by a usual map of arbitrary inverse systems and its usual mapping cylinder (see [7], p. 253).

6. The Whitehead Theorem for uniformly movable shape maps. Consider two pointed compact Hausdorff spaces (X, x_0) and (Y, y_0) and a shape map $[\![f]\!]: (X, x_0) \to (Y, y_0)$ (see [1]).

$$[\![f]\!] \text{ is } \textit{uniformly movable} \underset{\mathrm{Df}}{\Leftrightarrow} \text{there exists a uniformly movable } f' \in [\![f]\!].$$

Corollary 1.7 implies

6.1. Proposition. *A shape map* $[\![f]\!]$ *is uniformly movable if and only if every representative of* $[\![f]\!]$ *is uniformly movable in* $\mathscr{R}^*_{\simeq}$. ■

Let us establish the main result:

6.2. Theorem. *Let* (X, x_0) *and* (Y, y_0) *be two pointed metric continua of finite shape dimension and let* $n_0 = \max(1+\mathrm{Fd}(X, x_0), \mathrm{Fd}(Y, y_0))$. *Let* $[\![f]\!]^*_n: \pi^*_n(X, x_0) \to \pi^*_n(Y, y_0)$ *be the homomorphism of n-th shape groups induced by a uniformly movable shape map* $[\![f]\!]: (X, x_0) \to (Y, y_0)$. *If* $[\![f]\!]^*_n$ *is an isomorphism for* $n \leqslant n_0$ *and is an epimorphism for* $n = n_0+1$, *then* $[\![f]\!]$ *is a shape equivalence.*

Proof. By the argument used in [7], pp. 260_{11}-261^1, (X, x_0) and (Y, y_0) are inverse limits of ANR sequences $(X, x_0) = ((X_\alpha, x_\alpha), p^{\alpha'}_\alpha, N)$ and $(Y, y_0) = ((Y_\alpha, y_\alpha), q^{\alpha'}_\alpha, N)$, X_α and Y_α being connected polyhedra, $\dim X_\alpha \leqslant n_0-1$, $\dim Y_\alpha \leqslant n_0$, and $p^{\alpha'}_\alpha$ and $q^{\alpha'}_\alpha$ being onto.

Take a representative $f = (1_N, f_\alpha): (X, x_0) \to (Y, y_0)$ of the shape map $[\![f]\!]$ and let $f_n: \pi_n(X, x_0) \to \pi_n(Y, y_0)$ be the induced morphism of nth homotopy systems. We have

$$\pi^*_n(X, x_0) = \underleftarrow{\lim}\, \pi_n(X, x_0) \quad \text{and} \quad \pi^*_n(Y, y_0) = \underleftarrow{\lim}\, \pi_n(Y, y_0).$$

Take $[\![f]\!]^*_n = \underleftarrow{\lim}\, f_n: \pi^*_n(X, x_0) \to \pi^*_n(Y, y_0)$. By the assumption

(1) $[\![f]\!]^*_n$ is a bimorphism in $\mathscr{C}$ for $n \leqslant n_0$ and is an epimorphism in $\mathscr{G}$ for $n = n_0+1$.

Thus

(2) $[\![f]\!]^*_n$ is a monomorphism in $\mathscr{C}$ for $n \leqslant n_0$, and

$$[\![f]\!]^*_n \text{ is an epimorphism in } \begin{cases} \mathscr{C} & \text{for } n = 1, \\ \mathscr{G} & \text{for } n = 2, \ldots, n_0+1. \end{cases}$$

[1] Applying Theorem 3.6 of [7] instead of Theorem 2 of [2], we prove $f: X \to Y$ to be a homotopy equivalence but not necessarily pointed (as in Theorem 3.7 of [7]).

By 6.1, $f \in \mathrm{UM}$, and thus (Y, y_0) is uniformly movable and therefore $\pi_n(Y, y_0) \in \mathrm{Ob}_{\mathscr{G}^*_0}$. Thus, by Proposition 3.3, we get

(3) $[f_n]$ is a monomorphism in $(\hat{\mathscr{C}}^*, \hat{\mathscr{C}}^*_0)$ for $n = 1, \ldots, n_0$ and

$$[f_n] \text{ is an epimorphism in } \begin{cases} \hat{\mathscr{C}}^* & \text{for } n = 1, \\ \hat{\mathscr{G}}^* & \text{for } n = 2, \ldots, n_0+1. \end{cases}$$

Hence, by Theorem 5.2, f is a homotopy equivalence, and thus $[\![f]\!]$ is a shape equivalence. ■

In applications of Theorem 6.2 the following condition sufficient for a shape map to be uniformly movable may be useful.

6.3. Proposition. *Let* (X, x_0) *and* (Y, y_0) *be metric pointed compacta. If the shape map* $[\![f]\!]: (X, x_0) \to (Y, y_0)$ *has a representative* $f: (X, x_0) \to (Y, y_0)$ *with movable* (C_f, X, x_0) *then* $[\![f]\!]$ *is uniformly movable.*

Proof. As proved by S. Spież [8], every movable inverse sequence is uniformly movable; thus (C_f, X, x_0) is uniformly movable. By 4.2, $f \in \mathrm{UM}$, whence $[\![f]\!]$ is uniformly movable. ■

Remark. The shape category in the sense of Mardešić restricted to metric compacta is known to be isomorphic to the fundamental category in the sense of Borsuk. Thus, using 6.5 of [6], we can express Theorem 6.2 equivalently in terms of fundamental sequences.

The following problems remain open:

Problem 1. Does the movability of a map f of ANR sequences imply the uniform movability of f?

Problem 2. Do the notions of UM and WUM differ essentially?

Problem 3. Does the uniform movability of the inclusion $i: (X, x_0) \to (Z, x_0)$ imply the uniform movability of (Z, X, x_0)?

References

[1] S. Mardešić, *Retracts in shape theory*, Glasnik Mat. 6 (26) (1971), pp. 153–163.
[2] — *On the Whitehead theorem in the shape theory* I, Fund. Math. 91 (1976), pp. 51–64.
[3] — and J. Segal, *Shapes of compacta and ANR systems*, Fund. Math. 72 (1971), pp. 41–59.
[4] M. Moszyńska, *On shape and fundamental deformation retracts* II, Fund. Math. 77 (1972), pp. 235–240.
[5] — *Uniformly movable compact spaces and their algebraic properties*, Fund. Math. 77 (1972), pp. 125–144.
[6] — *Various approaches to the fundamental groups*, Fund. Math. 78 (1973), pp. 107–118.
[7] — *The Whitehead Theorem in the theory of shapes*, Fund. Math. 80 (1973), pp. 221–263.
[8] S. Spież, *Movability and uniform movability*, Bull. Acad. Polon. Sci. 22 (1974), pp. 43–45.

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