Homotopy properties of pathwise connected continua

by

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Abstract. Some continuity invariant properties and some related homotopy properties of pathwise connected continua are investigated. For each pathwise connected continuum, a certain subgroup of the Bruschlinsky group is distinguished which is known to carry an invariant of continuous onto mappings, and a Vietoris-type mapping theorem for this subgroup is proved. Also, six classes of continua are considered, each of the classes being invariant under continuous mappings, and some relations between the classes are established.

In [7], the first author introduced, for each pathwise connected space X, a group $A(X)$ that seems to be useful in certain problems involving transformations and continuous images of some spaces. In the present paper, we study pathwise connected continua X with $A(X) = 0$, and we prove, among other things, that each such a continuum is a continuous image of a one-dimensional continuum of the same type (see Theorem 4.2 below). Our proof depends essentially on a strong result (see Theorem 4.1) of R. D. Anderson who has kindly provided us with an outline of the proof of his result; it is published here for the first time. We also show that an analogue of the Vietoris mapping theorem holds for the group $A(X)$ (see Theorem 3.3).

Throughout this paper, by a *continuum* we understand a connected compact metric space, and a *mapping* always means a continuous function. Let S denote the unit circle, i.e. the set of all the complex numbers z with $|z| = 1$. Given a space X, we denote by $\pi^1(X)$ the Abelian group of all the homotopy classes $[\varphi]$ of mappings $\varphi\colon X\to S$ with the group operation defined by $[\varphi]+[\psi] = [\varphi\cdot\psi]$, where $(\varphi\cdot\psi)(x) = \varphi(x)\cdot\psi(x)$ for $x\in X$. Thus $\pi^1(X)$ is the so-called Bruschlinsky group of X. If $f\colon X\to Y$ is a mapping, we have an induced homomorphism $f^*\colon \pi^1(Y)\to\pi^1(X)$ defined by $f^*([\varphi]) = [\varphi\circ f]$, where $(\varphi\circ f)(x) = \varphi(f(x))$ for $x\in X$. Given a pathwise connected space X, we denote by $\pi_1(X)$ the fundamental group of X; and if $f\colon X\to Y$ is a mapping, we have an induced homomorphism $f_*\colon \pi_1(X)\to\pi_1(Y)$ defined by $f_*([\alpha]) = [f\circ\alpha]$ for mappings $\alpha\colon S\to X$. If $f\colon X\to Y$ and $g\colon X\to Y$ are mappings, we use the notation $f\simeq g$ to mean that f and g are homotopic, and we write $f\simeq 0$ to state the fact that f is homotopic to a constant mapping.

1. The group $A(X)$. Suppose X is a pathwise connected space. Let $A(X)$ be the subset of $\pi^1(X)$ consisting of those homotopy classes $[\varphi]$ of mappings $\varphi\colon X\to S$ for which it is true that

$$\varphi_*(\pi_1(X)) = 0\,,$$

or, in other words, $\varphi \circ \alpha \simeq 0$ for each mapping $\alpha\colon S \to X$. If $[\varphi] \in A(X)$ and $[\psi] \in A(X)$, then $[\varphi]-[\psi] = [\varkappa]$, where $\varkappa(x) = \varphi(x)\cdot(\psi(x))^{-1}$ for $x \in X$. We notice that here the homotopies $\varphi \circ \alpha \simeq 0$ and $\psi \circ \alpha \simeq 0$ imply the homotopy $\varkappa \circ \alpha \simeq 0$ since

$$(\varkappa \circ \alpha)(z) = \frac{\varphi(\alpha(z))}{\psi(\alpha(z))} = \frac{(\varphi \circ \alpha)(z)}{(\psi \circ \alpha)(z)} \qquad (z \in S),$$

and so $A(X)$ is a subgroup of the Bruschlinsky group $\pi^1(X)$. Moreover, if $f\colon X \to Y$ is a mapping and $[\varphi] \in A(Y)$, then $f_*(\pi_1(X)) \subset \pi_1(Y)$ and $(\varphi \circ f)_* = \varphi_* \circ f_*$, whence

$$(\varphi \circ f)_*(\pi_1(X)) \subset \varphi_*(\pi_1(Y)) = 0\,,$$

which means that $f^*([\varphi]) = [\varphi \circ f] \in A(X)$. As a result, we get an induced homomorphism $f^A\colon A(Y) \to A(X)$ defined by $f^A([\varphi]) = f^*([\varphi])$ for $[\varphi] \in A(Y)$. The following proposition is an immediate consequence of the definition of the group $A(X)$.

1.1. *If $X \subset Y$ are pathwise connected spaces and $[\varphi] \in A(Y)$, then $[\varphi|X] \in A(X)$.*

1.2. *If X is a locally connected continuum and $\varphi\colon X \to S$ is a mapping such that $\varphi|C \simeq 0$ for each simple closed curve $C \subset X$, then $\varphi \simeq 0$.* (See [9], pp. 427 and 430.)

1.3. *If X is a pathwise connected space and $\varphi\colon X \to S$ is a mapping, then $[\varphi] \in A(X)$ if and only if $\varphi|C \simeq 0$ for each simple closed curve $C \subset X$.*

Proof. Assume $[\varphi] \in A(X)$ and let $h\colon S \to C$ be a homeomorphism. Then $\varphi \circ h \simeq 0$, whence also $\varphi|C = \varphi \circ h \circ h^{-1} \simeq 0$. On the other hand, if $\varphi|C \simeq 0$ for each simple closed curve $C \subset X$, then, given a mapping $\alpha\colon S \to X$, the set $\alpha(S) \subset X$ is a locally connected continuum, and 1.2 yields $\varphi|\alpha(S) \simeq 0$. Consequently, $\varphi \circ \alpha \simeq 0$ and therefore $[\varphi] \in A(X)$.

1.4. *If X is a locally connected continuum, then $A(X) = 0$.*

Proof. Apply 1.2 and 1.3.

1.5. THEOREM. *If X, Y are pathwise connected continua and $f\colon X \to Y$ is a mapping such that $f(X) = Y$, then f^A is a monomorphism.* (See [7], Theorem 1.)

2. Monotone mappings. We say that a mapping $f\colon X \to Y$ is *monotone* provided $f^{-1}(y)$ is a connected set for each point $y \in Y$.

2.1. THEOREM (L. Vietoris). *If X, Y are compact metric spaces and $f\colon X \to Y$ is a monotone mapping such that $f(X) = Y$ and $\pi^1(f^{-1}(y)) = 0$ for $y \in Y$, then $f^*\colon \pi^1(Y) \to \pi^1(X)$ is an isomorphism.*

Proof. There is a commutative diagram

$$\begin{array}{ccc} H^1(Y) & \xrightarrow{f^{**}} & H^1(X) \\ {\scriptstyle i_Y}\uparrow & & \uparrow{\scriptstyle i_X} \\ \pi^1(Y) & \xrightarrow[f^*]{} & \pi^1(X) \end{array}$$

where $H^1(X)$, $H^1(Y)$ are the Čech cohomology groups with integer coefficients and i_X, i_Y are isomorphisms (compare [6], 226). Also, because of the existence of such isomorphisms and according to the condition imposed upon $f^{-1}(y)$, we have $H^1(f^{-1}(y)) = 0$ for $y \in Y$. Since the sets $f^{-1}(y)$ are continua, their reduced zero-dimensional cohomology groups are trivial too. It now follows from the Vietoris mapping theorem (see [10], pp. 334 and 344) that the induced homomorphism f^{**} is an isomorphism, and so is $f^* = (i_X)^{-1} \circ f^{**} \circ i_Y$.

2.2. THEOREM. *If X, Y are compact metric spaces, $\varphi\colon X \to S$ is a mapping and $f\colon X \to Y$ is a monotone mapping such that $f(X) = Y$ and $\varphi|f^{-1}(y) \simeq 0$ for $y \in Y$, then there exists a mapping $\psi\colon Y \to S$ such that $\psi \circ f \simeq \varphi$.*

Proof. For each point $y \in Y$, there exists (see [9], p. 364) an open set $G_y \subset X$ such that $f^{-1}(y) \subset G_y$ and $\varphi|G_y \simeq 0$. Thus there also exists a closed neighborhood V_y of y in Y such that $V_y \subset Y \setminus f(X \setminus G_y)$, whence

$$(1) \qquad f^{-1}(y) \subset U_y = f^{-1}(V_y) \subset G_y \qquad (y \in Y)$$

and $\varphi|U_y \simeq 0$. It follows (see [9], pp. 407 and 427) that there is a real-valued continuous function $\lambda_y\colon U_y \to R$ such that

$$(2) \qquad \varphi(x) = e^{2\pi i \lambda_y(x)} \qquad (x \in U_y)\,.$$

Let $\boldsymbol{D}$ be the decomposition of the space X into the sets $f^{-1}(y) \cap \lambda_y^{-1}\lambda_y(x)$, where $y = f(x)$ and $x \in X$. We show that $\boldsymbol{D}$ is an upper semi-continuous decomposition. Indeed, let

$$D = f^{-1}(y) \cap \lambda_y^{-1}\lambda_y(x)$$

be an arbitrarily chosen element of $\boldsymbol{D}$. By (1), the closed set U_y contains D in its interior, and U_y is the union of those elements of $\boldsymbol{D}$ which intersect U_y. If $v = f(u)$, where $u \in U_y$, the set $f^{-1}(v)$ is a continuum contained in U_y, by (1). Denote

$$D' = f^{-1}(v) \cap \lambda_v^{-1}\lambda_v(u)$$

and observe that, according to (2), the mappings $\lambda_v|f^{-1}(v)$ and $\lambda_y|f^{-1}(v)$ satisfy the condition

$$e^{2\pi i\lambda_v(p)} = \varphi(p) = e^{2\pi i \lambda_y(p)} \qquad (p \in f^{-1}(v))\,,$$

since $f^{-1}(v) \subset U_v \cap U_y$. We conclude that there exists an integer k such that $\lambda_y(p) = k + \lambda_y(p)$ for $p \in f^{-1}(v)$ (see [9], p. 406). Consequently, we have

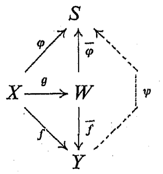
$$D' = f^{-1}(v) \cap \lambda_y^{-1}\lambda_y(u) = f^{-1}f(u) \cap \lambda_y^{-1}\lambda_y(u)\,,$$

which means that the elements of $\boldsymbol{D}$ contained in U_y coincide with the inverse images of points under the function $g_y\colon U_y \to Y \times R$ defined by $g_y(u) = (f(u), \lambda_y(u))$ for $u \in U_y$. But g_y is a continuous function, and therefore these inverse images form an upper semi-continuous decomposition of U_y. Hence $\boldsymbol{D}$ is upper semi-continuous at D, and D being an arbitrary element of $\boldsymbol{D}$, we have proved that $\boldsymbol{D}$ is an upper semi-continuous decomposition of X.

Let $Q = X/D$ be the decomposition space with the quotient topology, and let $g\colon X\to Q$ be the canonical projection which is a mapping of X onto Q. Then, for each point $q \in Q$, the set $g^{-1}(q)$ is an element of D; in particular, the set $fg^{-1}(q)$ is degenerate. Moreover, the set $\lambda_y g^{-1}(q)$, where $y = f(x)$ and $x \in g^{-1}(q)$, also is degenerate whence, by (1) and (2), so is the set $\varphi g^{-1}(q)$. Putting

$$\{\bar{\varphi}(q)\} = \varphi g^{-1}(q), \quad \{\bar{f}(q)\} = fg^{-1}(q) \quad (q \in Q),$$

we obtain mappings $\bar{\varphi}\colon Q\to S$, $\bar{f}\colon Q\to Y$, whose continuity follows from the definition of the mapping g and the quotient topology. We now have the commutative diagram



to which we shall add a mapping $\psi\colon Y\to S$ such that the new completed diagram be commutative up to a homotopy.

In fact, for each point $y \in Y$, the set $\bar{f}^{-1}(y)$ consists of exactly those points $q \in W$ which satisfy the inclusion $g^{-1}(q) \subset f^{-1}(y)$. Since $Q = X/D$ and the elements of D are subsets of the inverse images of points under f, it follows that $\bar{f}^{-1}(y)$ is homeomorphic to the quotient space $W_y = f^{-1}(y)/D_y$, where D_y is the upper semi-continuous decomposition of $f^{-1}(y)$ consisting of the elements of D which are contained in $f^{-1}(y)$. Each element of D_y is of the type $f^{-1}(y) \cap \lambda_y^{-1}\lambda_y(x)$, where $x \in f^{-1}(y)$, i.e. it is the inverse image of a point under the mapping $\lambda_y|f^{-1}(y)$. Thus W_y is homeomorphic to the image $\lambda_y(f^{-1}(y)) \subset R$ which is either a closed interval or a degenerate set, the set $f^{-1}(y)$ being a continuum. We conclude that $\bar{f}^{-1}(y)$ is either an arc or a point, whence $\pi^1(\bar{f}^{-1}(y)) = 0$ for $y \in Y$. By 2.1, the mapping $\bar{f}$ induces an isomorphism $\bar{f}^*$ between the groups $\pi^1(Y)$ and $\pi^1(W)$. In particular, there exists a mapping $\psi\colon Y\to S$ such that

$$[\bar{\varphi}] = \bar{f}^*([\psi]) = [\psi \circ \bar{f}],$$

or, in other words, $\bar{\varphi} \simeq \psi \circ \bar{f}$, which yields

$$\psi \circ f = \psi \circ \bar{f} \circ g \simeq \bar{\varphi} \circ g = \varphi.$$

2.3. *If X, Y are pathwise connected continua, $\varphi\colon X\to S$ is a mapping such that $[\varphi] \in A(X)$ and $f\colon X\to Y$ is a monotone mapping such that $f(X) = Y$, $f^{-1}(y)$ is pathwise connected and $A(f^{-1}(y)) = 0$ for $y \in Y$, then there exists a mapping $\psi\colon Y\to S$ such that $\psi \circ f \simeq \varphi$.*

Proof. By 1.1, we have $[\varphi|f^{-1}(y)] \in A(f^{-1}(y))$, whence $[\varphi|f^{-1}(y)] = 0$, that is $\varphi|f^{-1}(y) \simeq 0$ for $y \in Y$. Applying 2.2, we get 2.3.

2.4. COROLLARY. *Under the conditions of 2.3, it is true that*

$$A(X) \subset f^*(\pi^1(Y)).$$

3. Path-raising mappings. Let I denote the unit closed interval of the real line, i.e. $t \in I$ means $0 \leqslant t \leqslant 1$. We say that a mapping $p\colon I\to X$ is a *path* joining x_0 and x_1 provided $p(0) = x_0$ and $p(1) = x_1$. A mapping $f\colon X\to Y$ is said to be *path-raising* provided $f(X) = Y$ and, for each path $q\colon I\to Y$ joining $f(x_0)$ and $f(x_1)$, where $x_0, x_1 \in X$, there exists a path $p\colon I\to X$ joining x_0 and x_1 such that $fp(I) \subset q(I)$. Clearly, if f is path-raising, then f is monotone.

3.1. *If X, Y are metric spaces and $f\colon X\to Y$ is a mapping such that $f(X) = Y$, then the following conditions are equivalent to each other:*

(i) f is path-raising,

(ii) $f^{-1}(C)$ is pathwise connected for each pathwise connected set $C \subset Y$,

(iii) $f^{-1}(y)$ is pathwise connected for each point $y \in Y$, and each arc contained in Y is the image under f of a pathwise connected subset of X.

Proof. It is rather obvious that (ii) implies (i) and (i) implies (iii). To see that (iii) implies (ii), let us consider a pathwise connected subset C of Y, and let $x_0, x_1 \in f^{-1}(C)$. If $f(x_0) = f(x_1) = y$, there exists a path joining x_0 and x_1 in $f^{-1}(y)$. If $f(x_0) = y_0 \neq y_1 = f(x_1)$, we can take an arc $A \subset C$ which joins y_0 and y_1. Then there is, by (iii), a pathwise connected set $B \subset X$ such that $A = f(B)$. Let $b_0, b_1 \in B$ be points such that $f(b_0) = y_0$ and $f(b_1) = y_1$. Since $f^{-1}(y_0)$ and $f^{-1}(y_1)$ are pathwise connected, by (iii), there exists a path joining x_0 and b_0 in $f^{-1}(y_0)$; also, another path joins x_1 and b_1 in $f^{-1}(y_1)$. A third path joins b_0 and b_1 in $B \subset f^{-1}(A) \subset f^{-1}(C)$, and combining these three paths together, we obtain a path that joins x_0 and x_1 in $f^{-1}(C)$. Consequently, (ii) holds.

3.2. *If X, Y are metric spaces, $\beta\colon S\to Y$ and $\psi\colon Y\to S$ are mappings and $f\colon X\to Y$ is a path-raising mapping, then there exists a mapping $\alpha\colon S\to X$ such that $\psi \circ f \circ \alpha \simeq \psi \circ \beta$.*

Proof. The continuity of $\psi \circ \beta$ implies the existence of a decomposition $S = A_1 \cup \ldots \cup A_n$ of the circle S into non-over-lapping arcs A_j such that

$$\operatorname{diam} \psi\beta(A_j) < 2 = \operatorname{diam} S \quad (j = 1, \ldots, n), \tag{3}$$

and the end-points of A_j are a_j, a_{j+1} $(j = 1, \ldots, n)$, where $a_1 = a_{n+1}$. Let us select points $x_j \in X$ such that $\beta(a_j) = f(x_j)$ for $j = 1, \ldots, n$ and put $x_{n+1} = x_1$. Then $\beta|A_j$ is a path joining $f(x_j)$ and $f(x_{j+1})$. Since f is a path-raising mapping, there exists a path $p_j\colon A_j\to X$ joining x_j and x_{j+1} such that

$$fp_j(A_j) \subset \beta(A_j) \quad (j = 1, \ldots, n). \tag{4}$$

The paths p_j agree on junction-points and thus we actually have a mapping $\alpha\colon S\to X$ defined by $\alpha|A_j = p_j$ for $j = 1, \ldots, n$. Moreover, by (4), we get

$$(\psi \circ f \circ \alpha)(A_j) = \psi f p_j(A_j) \subset \psi\beta(A_j),$$

which, by (3), implies that

$$|\psi(f(\alpha(z)))-\psi(\beta(z))|\leqslant \operatorname{diam}\psi\beta(A_j)<2 \quad (z\in A_j),$$

and $\psi\circ f\circ\alpha\simeq\psi\circ\beta$ follows.

3.3. THEOREM. *If X, Y are pathwise connected continua and $f\colon X\to Y$ is a path-raising mapping such that $f(X)=Y$ and $A(f^{-1}(y))=0$ for $y\in Y$, then*

$$f^A\colon A(Y)\to A(X)$$

is an isomorphism.

Proof. By 1.5, f^A is a monomorphism. To see that f^A is an epimorphism, let $\varphi\colon X\to S$ be a mapping such that $[\varphi]\in A(X)$. By 3.1, $f^{-1}(y)$ is pathwise connected for $y\in Y$, and thus there exists, by 2.3, a mapping $\psi\colon Y\to S$ such that $\psi\circ f\simeq\varphi$. Hence $f^*([\psi])=[\varphi]$ and it suffices to show that $[\psi]\in A(Y)$. To this end, if $\beta\colon S\to Y$ is a mapping, we can apply 3.2. We obtain a mapping $\alpha\colon S\to X$ such that

$$\psi\circ\beta\simeq\psi\circ f\circ\alpha\simeq\varphi\circ\alpha\simeq 0,$$

since $[\varphi]\in A(X)$. Consequently, we have $[\psi]\in A(Y)$.

Remark. The condition that f is path-raising cannot be omitted in 3.3 (see 5.3 and 5.4).

4. Six classes of continua. Let $I^{\aleph_0}$ denote the Hilbert cube and let M^1 denote the Menger universal curve (compare [3]). Thus, in particular, M^1 is a locally connected, hence also pathwise connected, one-dimensional continuum that contains a topological copy of any other such continuum. We say that a mapping $f\colon X\to Y$ is *open* provided $f(G)$ is an open subset of Y for each open set $G\subset X$.

4.1. THEOREM (R. D. Anderson). *There exists an open mapping $\omega\colon M^1\to I^{\aleph_0}$ such that $\omega(M^1)=I^{\aleph_0}$ and $\omega^{-1}(C)$ is homeomorphic to M^1 for each locally connected continuum $C\subset I^{\aleph_0}$* (1).

Outline of proof (R. D. Anderson). The result follows by some modifications of proofs published in [1] and [2] together with the characterization of the Menger universal curve established in [3]. Specifically, a sequence $\{F_i\}$ of one-dimensional finite covers in R^3 may be set up with F_{i+1} a refinement of F_i, and $\{F_i\}$ incidence and containment isomorphic to a sequence $\{H_i\}$ of finite covers of $I^{\aleph_0}$ with mesh (H_i) converging to zero. Also, one can identify the set $\boldsymbol{F}$ of intersections $\bigcap_{i=1}^{\infty} f_i^*$, where f_i^* is the star in F_i of element $f_i\in F_i$, and $f_{i+1}\subset f_i$ for $i=1,2,\ldots$ By suitable conditions on the construction, $\boldsymbol{F}$ can be required to be a continuous collection (compare [9], p. 68) of one-dimensional continua whose decomposition space is $I^{\aleph_0}$ under a map ω induced by the sequence of isomorphisms of $\{F_i\}$ to $\{H_i\}$.

(1) Theorem 4.1 had been proved and announced by R. D. Anderson in the late 1950's but its proof was never published. An outline of the proof has been given recently in a letter to the second author of this paper and is included here with the permission of its author.

Now, using the characterization of M^1 as a locally connected one-dimensional continuum with no local cut-points and with no open non-empty subset embeddable in the plane, one can modify the constructions of the earlier papers to guarantee that, for each locally connected continuum $C\subset I^{\aleph_0}$, the set $\omega^{-1}(C)$ is a topological copy of M^1. Specifically, one must require that the elements of F_{i+1} (and the vertex elements of such elements) must be built with a local doubling up or duplication process (to insure no local cut-points), one must require that the elements of F_{i+1} locally contain one-skeletons of 4-simplexes to get the non-planar embedding, and one must require connectivity of elements of F_{i+1} in each "cell" (which they intersect) of each element of F_i containing them to get local connectivity, such requirement giving a kind of uniform local connectivity of inverses to give the desired result, i.e. paths in $I^{\aleph_0}$ induce local paths in the inverses of such paths.

4.2. THEOREM. *If Y is a pathwise connected continuum, then the following conditions are equivalent to each other:*

(i) $A(Y)=0$,

(ii) *Y is a continuous image of a pathwise connected continuum X with $A(X)=0$,*

(iii) *Y is a continuous image of a pathwise connected one-dimensional continuum X with $A(X)=0$.*

Proof. Obviously, (iii) implies (ii) and it follows from 1.5 that (ii) implies (i). To see that (i) implies (iii), let us consider Y as embedded in $I^{\aleph_0}$ and put $X=\omega^{-1}(Y)$, where ω is the mapping given by 4.1. Thus $X\subset M^1$ and ω is a path-raising mapping, by 3.1. Hence ω is monotone and X is a one-dimensional continuum. Again by 3.1, the continuum X is pathwise connected and, moreover, $f=\omega|X$ is a path-raising mapping such that $f(X)=Y$ and $f^{-1}(y)=\omega^{-1}(y)$ is homeomorphic to M^1 for $y\in Y$. Since M^1 is a locally connected continuum, we conclude from 1.4 that

$$A(f^{-1}(y))=A(\omega^{-1}(y))=A(M^1)=0 \quad (y\in Y),$$

and, by 3.3, the groups $A(X)$, $A(Y)$ are isomorphic. Consequently, one has $A(X)=0$, according to (i).

4.3. *If Y is a pathwise connected continuum, then Y is a continuous image of a pathwise connected one-dimensional continuum.*

Proof. Take the continuum $\omega^{-1}(Y)$, as in the proof of 4.2.

The properties of pathwise connected continua described in 4.2 and 4.3 suggest a classification of these continua closely related to the theory of dendroids. By a *dendroid* we understand a pathwise connected one-dimensional continuum X such that $\pi^1(X)=0$. The following notion is a generalization of an earlier concept (see [4], p. 193) that had been particularly useful in an investigation of dendroids (see [4] and [5]). A metric space X is said to be *uniformly pathwise connected* provided there exists a family of paths $p_{xy}\colon I\to X$, where $x,y\in X$, such that the path p_{xy} joins x and y $(x,y\in X)$ and, for each number $\varepsilon>0$, there exists a positive integer k with this property: for each pair of points $x,y\in X$, there is a decomposition

$$I=I_{xy,1}\cup\ldots\cup I_{xy,k}$$

of I into k subintervals $I_{xy,i}$ satisfying the inequalities

$$\operatorname{diam} p_{xy}(I_{xy,i}) \leqslant \varepsilon \quad (i = 1, \ldots, k).$$

4.4. THEOREM. *A continuum X is uniformly pathwise connected if and only if X is a continuous image of the cone over the Cantor set.* (See [8], Theorem 3.5.)

We distinguish the six classes of continua as follows.

(I) Locally connected continua.
(II) Continuous images of the cone over the Cantor set.
(III) Continuous images of dendroids.
(IV) Continuous images of pathwise connected continua X with $\pi^1(X) = 0$.
(V) Continuous images of pathwise connected continua X with $A(X) = 0$.
(VI) Pathwise connected continua.

Then (I)$\subset$(II)$\subset$(III)$\subset$(IV)$\subset$(V)$\subset$(VI) and each of these classes is invariant under continuous transformations. Also, observe that Classes (I) and (VI) are defined by means of intrinsic properties while Classes (II) and (V) can be characterized by such properties in a certain manner, according to 4.4 and 4.2. respectively.

PROBLEM. Give intrinsic characterizations of Classes (III) and (IV).

None of the above six classes of continua is equal to another one of them. In fact, the cone over the Cantor set belongs to (II)$\setminus$(I), and there are well-known examples of dendroids which fail to be uniformly pathwise connected (see [4], p. 201, for instance); they all belong to (III)$\setminus$(II). In 5.2 and 5.1 below, we construct continua that belong to (IV)$\setminus$(III) and (V)$\setminus$(IV), respectively. The so-called "Warsaw circle" is an element of (VI)$\setminus$(V) (see [7], Corollary 2).

5. Examples. The first and the last of our examples are one-dimensional continua. The other two are two-dimensional.

5.1. EXAMPLE. *There exists a pathwise connected one-dimensional continuum K with $A(K) = 0$ such that K is not a continuous image of any continuum X with $\pi^1(X) = 0$.*

Proof. Let us define arcs $A'_n \subset S \times I$ by the formula

$$A'_n = \{(e^{2\pi i t}, t^{-1}) \colon 2^n \leqslant t \leqslant 2^{n+1} - 1\} \quad (n = 1, 2, \ldots), \tag{5}$$

and put $S'_0 = S \times \{0\}$. The union $K' = S'_0 \cup (A'_1 \cup A'_2 \cup \ldots)$ is a closed subset of $S \times I$ and so is the set $E = \{(1, 0)\} \cup \{(1, 2^{-n}) \colon n = 1, 2, \ldots\}$. Let $\mathbf{E}$ be the decomposition of K' into E and the one-point sets $\{y\}$, where $y \in K' \setminus E$. We define K by setting $K = K'/\mathbf{E}$.

Since the set E contains an end-point of each arc A'_n $(n = 1, 2, \ldots)$ and E meets the limit circle S'_0, it is apparent that the decomposition space K is a pathwise connected continuum. Also, K is one-dimensional. Let $g\colon K' \to K$ denote the canonical projection and let

$$A_n = g(A'_n), \quad \{e\} = g(E), \quad S_0 = g(S'_0) \quad (n = 1, 2, \ldots),$$

whence $K = S_0 \cup (A_1 \cup A_2 \cup \ldots)$. To see that $A(K) = 0$, suppose $[\varphi] \in \pi^1(K)$ is the homotopy class of a mapping $\varphi\colon K \to S$ such that $[\varphi] \neq 0$. Then $\varphi \not\simeq 0$, and there exists a continuum $B \subset K$ such that $\varphi|B \not\simeq 0$ but $\varphi|Z \simeq 0$ for each closed proper subset Z of B (see [9], p. 425). We claim that $B \subset S_0$. If not, there would exist a positive integer m such that

$$B_m = (A_m \setminus \{e\}) \cap B = A_m \cap (B \setminus S_0) \neq \emptyset,$$

and $B \setminus B_m$ would be a proper subset of B. The set $A_m \setminus \{e\}$ is open in K, its boundary is $\{e\}$, and $\varphi|A_m \simeq 0$. Thus the set $B \setminus B_m$ is closed in B, whence $\varphi|(B \setminus B_m) \simeq 0$, and also B cannot be contained in A_m, whence $e \in B$, B being a continuum. Moreover, the set $B_m \cup \{e\}$ is a subarc of A_m and the identity mapping of B is homotopic to the retraction $r\colon B \to B \setminus B_m$ that sends B_m to e. Consequently, we would have

$$\varphi|B = (\varphi|B) \circ \mathrm{id}_B \simeq (\varphi|(B \setminus B_m)) \circ r \simeq 0,$$

which is a contradiction. Hence $B \subset S_0$. But, on the other hand, S_0 is a simple closed curve and, therefore, $\varphi|Z \simeq 0$ for each closed proper subset Z of S_0, too. We conclude that $B = S_0$, and any homeomorphism $\alpha\colon S \to S_0$ satisfies the conditions

$$\varphi \circ \alpha \circ \alpha^{-1} = \varphi|S_0 = \varphi|B \not\simeq 0,$$

whence $\varphi \circ \alpha \not\simeq 0$. As a result, we have $[\varphi] \notin A(K)$, which means that $A(K) = 0$.

It remains to be shown that K is not a continuous image of any continuum X with $\pi^1(X) = 0$. Suppose, on the contrary, that there exists such a continuum X and a mapping $f\colon X \to K$ of X onto K. The point e is an end-point of the arc A_n, and let a_n denote the other end-point of A_n $(n = 1, 2, \ldots)$. We need to know that, for each integer $n = 1, 2, \ldots$, the set $f^{-1}(A_n)$ is connected between $f^{-1}(a_n)$ and $f^{-1}(e)$. If not, there would exist a decomposition of $f^{-1}(A_n)$ into two disjoint closed sets F_1 and F_2 containing $f^{-1}(a_n)$ and $f^{-1}(e)$, respectively. Then the disjoint closed sets F_1 and $F_2 \cup f^{-1}(K \setminus A_n)$ would form a decomposition of X, which is impossible since X is connected. Thus $f^{-1}(A_n)$ is connected between $f^{-1}(a_n)$ and $f^{-1}(e)$, whence there exists a continuum $C_n \subset f^{-1}(A_n)$ joining $f^{-1}(a_n)$ and $f^{-1}(e)$ (see [9], p. 170). It follows that $f(C_n)$ is a subcontinuum of the arc A_n, joining its end-points, so that

$$A_n = f(C_n) \quad (n = 1, 2, \ldots). \tag{6}$$

Let ψ denote the projection $\psi\colon S \times I \to S$ of $S \times I$ onto S, i.e. $\psi(z, t) = z$ for $(z, t) \in S \times I$. Notice that the set E is the only non-degenerate element of $\mathbf{E}$, and $\psi(E) = \{1\}$. Consequently, for each point $q \in K$, the set $\psi g^{-1}(q)$ is degenerate, and the formula

$$\{\overline{\psi}(q)\} = \psi g^{-1}(q) \quad (q \in K) \tag{7}$$

defines a mapping $\overline{\psi}\colon K \to S$. But since $\pi^1(X) = 0$, we have $\overline{\psi} \circ f \simeq 0$, and there exists (see [9], p. 427) a mapping $\lambda\colon X \to R$ such that

$$\overline{\psi}(f(x)) = e^{2\pi i \lambda(x)} \quad (x \in X). \tag{8}$$

Let I_n denote the closed interval $I_n = \{t \in R: 2^n \leqslant t \leqslant 2^{n+1}-1\}$ and let $h_n: I_n \to A'_n$ be the homeomorphism given by the formula

$$h_n(t) = (e^{2\pi it}, t^{-1}) \quad (t \in I_n),$$

according to (5). Hence

$$(9) \qquad \psi(y) = e^{2\pi i h_n^{-1}(y)} \quad (y \in A'_n).$$

We also have the homeomorphism $g|A'_n$ of the arc A'_n onto the arc A_n, and let us put $g_n = (g|A'_n)^{-1}$ for $n = 1, 2, \ldots$ This yields

$$\{\psi(g_n(q))\} = \psi g^{-1}(q) = \{\overline{\psi}(q)\} \quad (q \in A_n),$$

by (7), whence

$$e^{2\pi i\lambda(x)} = \overline{\psi}(f(x)) = \psi(g_n(f(x))) = e^{2\pi i h_n^{-1}(g_n(f(x)))} \quad (x \in C_n),$$

by (6), (8) and (9). The set C_n is a continuum, and we infer (see [9], p. 406) that there exists an integer k_n such that

$$\lambda(x) = k_n + h_n^{-1}(g_n(f(x))) \quad (x \in C_n),$$

which implies that

$$\operatorname{diam}\lambda(C_n) = \operatorname{diam} h_n^{-1} g_n f(C_n) = \operatorname{diam} I_n = 2^n - 1 \quad (n = 1, 2, \ldots),$$

by (6). Since $C_n \subset X$, the last formula contradicts the boundedness of $\lambda(X)$ that follows from the compactness of X; and the proof of 5.1 is now complete.

Remark. The connectedness of X has been used only to guarantee the existence of continua $C_n \subset X$ such that (6) holds. What we have actually proved is the following property of the continuum K: it is not a continuous image of any compact metric space X such that $\pi^1(X) = 0$ and there exist continua $C_n \subset X$ satisfying (6). We shall refer to this property of K in our next construction.

5.2. Example. *There exists a pathwise connected two-dimensional continuum L with $\pi^1(L) = 0$ such that L is not a continuous image of any one-dimensional continuum X with $\pi^1(X) = 0$.*

Proof. Let D denote the unit disk whose boundary is S, i.e. the set of all the complex numbers z with $|z| \leqslant 1$. Furthermore, let $D'_0 = D \times \{0\}$ and $L' = D'_0 \cup \cup (A'_1 \cup A'_2 \cup \ldots)$, where the arcs A'_n are defined by (5). Then L' is a compact metric space and the set E, defined as in the proof of 5.1, is a closed subset of L'. Let $\boldsymbol{F}$ be the decomposition of L' into E and the one-point sets $\{y\}$, where $y \in L' \setminus E$. We define L by setting $L = L'/\boldsymbol{F}$.

As in 5.1, the decomposition space L is a pathwise connected continuum. Let $h: L' \to L$ denote the canonical projection and let

$$A_n = h(A'_n), \quad \{e\} = h(E), \quad D_0 = h(D'_0) \quad (n = 1, 2, \ldots),$$

whence $L = D_0 \cup (A_1 \cup A_2 \cup \ldots)$. To see that $\pi^1(L) = 0$, suppose on the contrary that there exists a non-zero element $[\varphi] \in \pi^1(L)$. Then there is a continuum $B \subset L$ such that $\varphi|B \not\simeq 0$ but $\varphi|Z \simeq 0$ for each closed proper subset Z of B (see [9], p. 425). An argument identical to that from 5.1 now shows that $B \subset D_0$. But $h|D'_0$ is a homeomorphism, and D_0 is a topological disk in L. Thus $\varphi|D_0 \simeq 0$, whence also $\varphi|B \simeq 0$, which is a contradiction.

We have $K' \subset L'$ and $g = h|K'$, where K' and g are taken from the proof of 5.1. Hence L contains a topological copy $K_L = h(K')$ of K, composed of the same arcs A_n $(n = 1, 2, \ldots)$ and their limit circle $h(S'_0)$. Moreover, the point e is also an end-point of A_n and the only boundary point of the open subset $A_n \setminus \{e\}$ of L. Therefore, if X is a continuum and $f: X \to L$ is a mapping of X onto L, we conclude, exactly as in the proof of 5.1, that there exist continua $C_n \subset X$ satisfying (6). Let X_L be the closure of the union $C_1 \cup C_2 \cup \ldots$ in X. By (6), we obtain

$$f(X_L) = \operatorname{cl} f(C_1 \cup C_2 \cup \ldots) = \operatorname{cl}(A_1 \cup A_2 \cup \ldots) = K_L,$$

and X_L is a compact metric space contained in X. If X were one-dimensional with $\pi^1(X) = 0$, we would have $\pi^1(X_L) = 0$ too (see [9], p. 354). This would violate the property of K, hence also of K_L, which was mentioned before, in the remark preceding 5.2.

5.3. Example. *There exists a pathwise connected two-dimensional continuum M and an open mapping $\mu: M \to S$ such that $\mu(M) = S$, $\mu^{-1}(z)$ is an arc for each point $z \in S$, and $\mu|C \simeq 0$ for each locally connected continuum $C \subset M$* ([2]).

Consequently, $A(\mu^{-1}(z)) = \pi^1(\mu^{-1}(z)) = 0$ for $z \in S$ and $A(S) = 0$, by 1.4. *On the other hand, $\mu \not\simeq 0$* (compare [9], pp. 427 and 433), *i.e. $[\mu] \neq 0$, and $[\mu] \in A(M)$, by* 1.3, *so that $A(M) \neq 0$, and μ^A is not an isomorphism.*

Proof. We define M to be the subset $M = W_1 \cup W_2$ of the plane, where the sets W_1 and W_2 are given by the formulae

$$W_1 = \{(x, y): \tfrac{5}{4} \leqslant (x-\tfrac{1}{2})^2 + (y-2)^2 \leqslant \tfrac{37}{4}\} \setminus \{(x, y): 0 < x < 1, -2 < y < 1\},$$

$$W_2 = \left\{(x, y): 0 < x \leqslant 1, -x + \sin\frac{\pi}{x} \leqslant y \leqslant x + \sin\frac{\pi}{x}\right\}.$$

Clearly, M is a pathwise connected continuum with a non-empty interior.

Let $\overline{pq}$ denote the straight-line closed interval with the end-points p and q. For each real number t, let $p_t = (t, 0)$ and $q_t = (t+2, 1)$. Let V be the part of the plane defined by the formula

$$(10) \qquad V = \bigcup_{t \geqslant 0} \overline{p_t q_t},$$

and let $V = V_1 \cup V_2$ be the decomposition of V into the sets $V_1 = \{(x, y) \in V: x \leqslant 3\}$ and $V_2 = \{(x, y) \in V: x \geqslant 3\}$. We also define points r_n, s_n $(n = 0, 1, \ldots)$

([2]) Example 5.3 is essentially due to W. S. Mahavier.

of M by $r_0 = (0, -1)$, $s_0 = (0, 1)$, $r_1 = (1, -1)$, $s_1 = (1, 1)$, and

$$r_n = \left(\frac{2}{2n-1}, -\frac{2}{2n-1} + \sin\frac{(2n-1)\pi}{2}\right), \qquad (n = 2, 3, \ldots)$$
$$s_n = \left(\frac{2}{2n-1}, \frac{2}{2n-1} + \sin\frac{(2n-1)\pi}{2}\right).$$

Both V_1 and W_1 are compact disks, so that there exists a homeomorphism $h_1\colon V_1 \to W_1$ satisfying the conditions $h_1(\overline{p_0q_0}) = \overline{r_0s_0}$ and $h_1(\overline{p_3q_1}) = \overline{r_1s_1}$. On the other hand, the sets V_2 and W_2 are homeomorphic to a disk with one point removed from its boundary, and the intervals $\overline{p_3q_1}$ and $\overline{r_1s_1}$ lie on the boundaries of V_2 and W_2, respectively. It can then be seen that there exists a homeomorphism $h_2\colon V_2 \to W_2$ such that $h_1|\overline{p_3q_1} = h_2|\overline{p_3q_1}$ and

$$h_2(\overline{p_{n+2}q_n}) = \overline{r_ns_n} \quad (n = 2, 3, \ldots).$$

Combining the homeomorphism h_1 and h_2, we get a one-to-one mapping $g\colon V \to M$ of V onto M, defined by

$$g(p) = \begin{cases} h_1(p) & (p \in V_1), \\ h_2(p) & (p \in V_2), \end{cases}$$

and let us consider the collection $\boldsymbol{G}$ of all the arcs $g(\overline{p_tq_t})$, where $t \geqslant 0$. According to (10), $\boldsymbol{G}$ is a decomposition of M, and the elements of $\boldsymbol{G}$ are mutually disjoint. Moreover, it can be verified that $\boldsymbol{G}$ is a continuous decomposition of M, whence the canonical projection $\mu\colon M \to M/\boldsymbol{G}$ is an open mapping of M onto the decomposition space $M/\boldsymbol{G}$ (compare [9], p. 68). There is also a natural correspondence between the arcs $g(\overline{p_tq_t})$ and the numbers $t \geqslant 0$ under which t converging to $+\infty$ correspond to arcs converging to $g(\overline{p_0q_0}) = \overline{r_0s_0}$. We conclude that $M/\boldsymbol{G}$ is topologically the circle S, and we can write $M/\boldsymbol{G} = S$.

For each point $z \in S$, $\mu^{-1}(z)$ is an element of $\boldsymbol{G}$. If $C \subset M$ is a locally connected continuum, C cannot contain points of W_2 arbitrarily close to $\overline{r_0s_0}$. Hence there must exist a point $z_0 \in S$ such that C does not meet $\mu^{-1}(z_0)$, and thus z_0 does not belong to $\mu(C)$. But each proper subset of S is contractible to a point in S, so that $\mu|C \simeq 0$.

5.4. Example. *There exists a pathwise connected one-dimensional continuum N and an open mapping $\nu\colon N \to S$ such that $\nu(N) = S$, $\nu^{-1}(z)$ is homeomorphic to the Menger universal curve for each point $z \in S$, and $\nu|C \simeq 0$ for each locally connected continuum $C \subset N$.*

Consequently, *$A(\nu^{-1}(z)) = 0$ for $z \in S$, by* 1.4, *and ν^A is not an isomorphism, as in* 5.3.

Proof. Let us consider the continuum M of 5.3, assuming that M is embedded in the Hilbert cube $I^{\aleph_0}$, and let us take the open mapping ω described in 4.1. We define $N = \omega^{-1}(M)$. Since, by 3.1, ω is a path-raising mapping, and M is pathwise connected, the continuum N is pathwise connected. Also, $\omega|N$ is an open mapping of N onto M, and, by 5.3, the composite $\nu = \mu \circ (\omega|N)$ is an open mapping of N onto S. For $z \in S$, the set $\mu^{-1}(z)$ is an arc, whence the set $\nu^{-1}(z) = \omega^{-1}(\mu^{-1}(z))$ is homeomorphic to the Menger universal curve, by 4.1. Finally, if $C \subset N$ is a locally connected continuum, so is $\omega(C) \subset M$, and $\mu|\omega(C) \simeq 0$, by 5.3. It follows that $\nu|C \simeq 0$.

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