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On group nil rings

by

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Abstract. The main result of this paper (Theorem A) is a generalization of a theorem of D. S. Passman [9] saying that under some assumptions the group ring $R[G]$ for commutative R contains no non-zero nil ideals. Our result is then applied in § 2 to find some new statements equivalent to the still open Koethe problem: if a ring R contains a one-sided nil ideal A, is A contained in a two-sided nil ideal of R? Finally, § 3 is devoted to an investigation, by means of Theorem A, of the $\mathcal{N}$-radical of certain group rings, where $\mathcal{N}$ is the absolutely nil property defined by S. A. Amitsur [2], [6].

Contents

§ 1. Semisimplicity of group rings. Let R be a ring with unity and M its multiplicatively closed subset contained in the centre of R. Assume that M contains no zero-divisors of R and consider the complete ring of right quotiens Q for R. Then ([8], Lemma 5, p. 160)

$$RM^{-1} = \{q \in Q |\ \underset{m\in M}{\exists}\ qm \in R\}$$

is a subring of the ring Q. If M is an empty set, put $RM^{-1} = R$. Every element of RM^{-1} can be written in the form rm^{-1}, where $r \in R$, $m \in M$. Since $(rm^{-1})^n = r^n m^{-n}$ for any integer n, R contains no nil ideals if and only if RM^{-1} contains no nil ideals. Moreover, for any group G, rings $(RM^{-1})[G]$ and $(R[G])M^{-1}$ are isomorphic to each other.

For a group G we denote by G_n the set of elements of G of order n. We shall say that the groups G and H are *torsion disjoint* if for any integer $n \geqslant 2$ at least one of the sets G_n, H_n is empty. It is not hard to check that groups G and H are torsion disjoint if and only if for any prime p one of the sets G_p, H_p is empty.

A nil ring R will be shortly denoted as a $\mathcal{K}$-ring. It is well known that $\mathcal{K}$ is a radical property [4]. The $\mathcal{K}$-radical of a ring R will be denoted by $\mathcal{K}(R)$.

PROPOSITION 1.1. *If the additive group R^+ of a ring R is torsion disjoint with a group G, then the additive group $\bar{R}^+$ of $\bar{R}$ is also torsion disjoint with G, where $\bar{R} = R/\mathcal{K}(R)$.*

Proof. If the set $\bar{R}_p^+$ is not empty for some prime p then the ideal $J = \{x \in R|\ px \in \mathcal{K}(R)\}$ of R is not contained in $\mathcal{K}(R)$ and therefore cannot be a nil ideal. Hence there exists such an element $x \in R$ that $x^k \neq 0$, $(px)^k = 0$ for some integer k. If l is the smallest integer such that $p^l x^k = 0$, then $p^{l-1}x^k \in R_p^+$; hence R_p^+ is not empty. Therefore by assumption G_p is empty, i.e., $\bar{R}^+$ is torsion disjoint with G.

PROPOSITION 1.2. *Let R be a commutative $\mathcal{K}$-semisimple ring with unity. If the additive group R^+ of R is torsion disjoint with a group G, then the group ring $R[G]$ is $\mathcal{K}$-semisimple.*

Proof. Let M be the multiplicatively closed set generated by elements of R of the form $n\cdot 1$, where n runs through orders of elements of G. Using the fact that R^+ is torsion disjoint with G, one can verify that M contains no zero-divisors of R. Since R and $R[G]$ are $\mathcal{K}$-semisimple if and only if RM^{-1} and $(R[G])M^{-1} \approx (RM^{-1})[G]$ are $\mathcal{K}$-semisimple, one can assume without loss of generality that elements from M are invertible in R. A commutative and $\mathcal{K}$-semisimple ring R can be represented as a subdirect sum of integral domains R_α. Therefore $R[G]$ is a subdirect sum of the rings $R_\alpha[G]$. Since a subdirect sum of semisimple rings is semisimple [4], is enough to show that each of the rings $R_\alpha[G]$ is $\mathcal{K}$-semisimple. Let N be the set of non-zero elements from the ring R_α. Since elements from M are invertible in R and R_α is a homomorphic image of R, the group G contains no elements the orders of which are divisible by the characteristic of the field $R_\alpha N^{-1}$. Now, applying D. S. Passman's result [9], we obtain

$$\mathcal{K}((R_\alpha[G])N^{-1}) \approx \mathcal{K}((R_\alpha N^{-1})[G]) = 0 .$$

Therefore $\mathcal{K}(R_\alpha[G]) = 0$.

A similar result for the Baer radical has been obtained by similar methods by J. Lambek [8]. Considering the above argument, one can also show that $R[G]$ contains no non-zero one-sided nil ideals.

Now we extend this result to non-commutative rings.

THEOREM A [12]. *Let R be a $\mathcal{K}$-semisimple ring. If the additive group R^+ of R is torsion disjoint with a group G, then the group ring $R[G]$ is $\mathcal{K}$-semisimple.*

Proof. Let us assume that $\mathcal{K}(R[G]) \neq 0$. Take a non-zero element $\alpha = \sum_{i=1}^{n} a_i g_i$, $a_i \in R$, $g_i \in G$ from $\mathcal{K}(R[G])$ of the least length (i.e., n is minimal) and consider the subring A of R generated by the elements a_i, $i = 1, \ldots, n$. The ring A is commutative because for any i the length of the element $\beta = a_i\alpha - \alpha a_i = \sum_{j=2}^{n}(a_i a_j - a_j a_i) g_i \in \mathcal{K}(R[G])$ is less than n, β must be zero and therefore $a_i a_j = a_j a_i$, $i, j = 1, \ldots, n$. We show now that the elements a_i, $i = 1, \ldots, n$ are nilpotent. It is sufficient to show that $\mathcal{K}(A)[G] = \mathcal{K}(A[G])$ since $\alpha \in \mathcal{K}(R[G]) \cap A[G] \subseteq \mathcal{K}(A[G])$. Since the fact that R^+ is torsion disjoint whith G implies that A^+ is also torsion disjoint with G, we find by applying Proposition 1.1 that the additive group $\bar{A}^+$ of $\bar{A}$ is torsion disjoint with G, where $\bar{A} = A/\mathcal{K}(A)$. Let us extend $\bar{A}$ to the ring A^* with unity, where $A^*/\bar{A}$ is the ring of integers. It is easy to check that A^* is $\mathcal{K}$-semisimple and $(A^*)^+$ is torsion disjoint with G; therefore by Proposition 1.2 $\mathcal{K}(A^*[G]) = 0$. Since $\bar{A}$ is an ideal of A^*, $\bar{A}[G]$ is an ideal of $A^*[G]$. It is well known [3] that $\mathcal{K}(\bar{A}[G])$ is an ideal of $A^*[G]$; hence $\mathcal{K}(A[G]) \subseteq \mathcal{K}(A^*[G]) = 0$. Therefore

$$\mathcal{K}\left(\frac{A[G]}{\mathcal{K}(A)[G]}\right) \approx \mathcal{K}\left(\frac{A}{\mathcal{K}(A)}[G]\right) = 0 ;$$

which implies $\mathcal{K}(A[G]) \subseteq \mathcal{K}(A)[G]$. The converse inclusion is obvious since A is commutative.

Consider now the set I of $a \in R$ such that $ag_1 + b_2 g_2 + \ldots + b_n g_n \in \mathcal{K}(R[G])$ for some $b_2, \ldots, b_n \in R$. If $\beta = ag_1 + b_2 g_2 + \ldots + b_n g_n \neq 0$, then β has the least length in $\mathcal{K}(R[G])$ and therefore a is nil. Such a set I is of course a non-zero ideal of R, since $a_1 \in I$. Therefore a $\mathcal{K}$-semisimple ring contains a non-zero nil ideal I, which is impossible.

§ 2. Remarks on the Koethe problem.

DEFINITION (J. Krempa [7]). Let G be a group. We shall say that a radical property S defined in the class of algebras over a field F is *G-invariant* if for any F-algebra A

$$S(A[G]) = S(A)[G] .$$

Let A be an F-algebra and σ such an F-automorphism of A that $\sigma^2 = \mathrm{id}$. It is easy to check that the set of all elements of the form $a + bx$, $a, b \in A$ with operations defined as follows:

$$(a_1 + b_1 x) + (a_2 + b_2 x) = (a_1 + a_2) + (b_1 + b_2)x ,$$

$$(a_1 + b_1 x)\cdot(a_2 + b_2 x) = \big(a_1 a_2 + b_1\sigma(b_2)\big) + \big(b_1\sigma(a_2) + a_1 b_2\big)x ,$$

$$\gamma(a_1 + b_1 x) = \gamma a_1 + \gamma b_1 x ,$$

$a_1, a_2, b_1, b_2 \in A$, $\gamma \in F$ is an F-algebra. This algebra will be denoted by $A^\sigma[C_2]$. If $\sigma = \mathrm{id}$, then this is the group algebra $A[C_2]$ where C_2 is the cyclic group of order 2.

PROPOSITION 2.1. *Let G be a group and H its subgroup of index 2. If G contains an element x of order two which does not belong to H, then for any ring R one can establish an isomorphism between $R[G]$ and $(R[H])^\sigma[C_2]$, where σ is the automorphism of $R[H]$ induced by the inner automorphism $h \to xhx^{-1}$, $h \in H$ of H.*

Proof. Since $G = H \cup Hx$, every element $\alpha \in R[G]$ can be written in a unique way in the form $a + bx$, where $a, b \in R[H]$. Consider the cyclic group of order 2 $C_2 = \{e, y\}$ and define the map φ: $R[G] \to (R[H])^\sigma[C_2]$ as follows:

$$\varphi(a + bx) = a + by .$$

Since

$$\begin{aligned}\varphi\big((a+bx)(a_1+b_1x)\big) &= \varphi(aa_1+bxb_1x^{-1}+bxa_1x^{-1}x+ab_1x)\\ &= \varphi(aa_1+b\sigma(b_1)+b\sigma(a_1)x+ab_1x)\\ &= aa_1+b\sigma(b_1)+\big(b\sigma(a_1)+ab_1\big)y\\ &= (a+by)(a_1+b_1y) = \varphi(a+bx).\varphi(a_1+b_1x)\,,\end{aligned}$$

φ is a homomorphism. Now it is easy to check that φ is also an isomorphism.

THEOREM B. *For any field F of characteristic $p \neq 2$ the following conditions are equivalent:*

(i) *for any nil F-algebra A the matrix algebra A_2 is nil;*

(ii) *the property $\mathscr{K}$ is G-invariant for any finite G such that the characteristic p of F does not divide the order of G if $p \neq 0$;*

(iii) *the property $\mathscr{K}$ is G_0-invariant in the class of F-algebras, where G_0 is the transformation group of a square, i.e., the group of 2×2-matrices of the form*

$$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix},\quad \begin{pmatrix}0 & \delta\\ \varepsilon & 0\end{pmatrix}$$

where $\lambda, \mu, \delta, \varepsilon$ are equal ± 1.

(iv) *for any nil F-algebra A and for any involutive automorphism σ of A the algebra $A^\sigma[C_2]$ is also nil.*

Proof. Since $\mathscr{K}(A)[G]$ can be isomorphicaly embedded into $(\mathscr{K}(A))_m$, where m is the order of G, then by (i) $\mathscr{K}(A)[G]$ is nil. Therefore $\mathscr{K}(A)[G]\subseteq\mathscr{K}(A[G])$. But, on the other hand, by the assumption on the characteristic in (ii) we get that the additive group of $A/\mathscr{K}(A)$ is torsion disjoint with G. Therefore from Theorem A we have

$$\mathscr{K}\left(\frac{A[G]}{\mathscr{K}(A)[G]}\right) \approx \mathscr{K}\left(\frac{A}{\mathscr{K}(A)}[G]\right) = 0\,,$$

i.e., $\mathscr{K}(A)[G]\supseteq\mathscr{K}(A[G])$.

The fact that (ii) implies (iii) is obvious.

We show that (iii) implies (i). Let A be a nil algebra. Since $\mathscr{K}$ is G_0-invariant, $A[G_0]$ is also nil. Let us consider the map $\varphi\colon A[G_0]\to A_2$ defined as follows:

$$\varphi\big(\sum_i a_i g_i\big) = \sum_i a_i\cdot g_i\,,$$

where $a_i\cdot g_i$ on the right-hand side are understood as products of a_i and matrices g_i. Such a map is then an F-homomorphism. Since the characteristic p of F is $\neq 2$, φ is an onto mapping. Therefore $\mathscr{K}(A_2) = A_2$, i.e., A_2 is nil.

The implication (i)⇒(iv) follows immediately from the fact that the map

$$f(a+bx) = \begin{pmatrix} a & b\\ \sigma(b) & \sigma(a)\end{pmatrix},$$

where $a, b\in A$, is a monomorphism of $A^\sigma[C_2]$ into the nil algebra A_2.

Now we show that (iv) implies (iii). Since the characteristic of F is $\neq 2$, we can apply Theorem A. Therefore, for any F-algebra A,

$$\mathscr{K}\left(\frac{A[G_0]}{\mathscr{K}(A)[G_0]}\right) \approx \mathscr{K}\left(\frac{A}{\mathscr{K}(A)}[G_0]\right) = 0\,,$$

which means that $\mathscr{K}(A[G_0])\subseteq\mathscr{K}(A)[G_0]$. To prove the converse inclusion it is enough to observe that if A is a nil algebra, then $A[G_0]$ is also nil. Since G_0 contains a normal subgroup $H = C_2\times C_2$ and an element of order 2 which does not belong to H, hence by Proposition 2.1, $A[G_0]$ and $(A[H])^\sigma[C_2]$ are isomorphic to each other. The set $I = \{a-ax|\ a\in A\}$ is a nil ideal of $A[C_2]$. Since $A[C_2]/I\approx A$ is nil, so is $A[C_2]$. But, on the other hand, $(A[C_2])[C_2]\approx A[C_2\times C_2] = A[H]$, which means that $A[G_0]$ is nil.

It is well known [6] that the condition (i) is equivalent to the open Koethe problem.

§ 3. Absolutely nil rings.

DEFINITION (S. A. Amitsur). We call a ring R can *absolutely nil ring* if for every $n>0$ the ring $R[x_1, \ldots, x_n]$ of polynomials in commutative indeterminates $x_1, \ldots, x_n$ is a nil ring.

One can easily observe that a ring R is absolutely nil if and only if the ring $R[x_1, x_2, \ldots]$ of polynomials in a denumerable set of commutative indeterminates $x_1, x_2, \ldots$ is a nil ring.

An absolutely nil ring R will be denoted shortly as an $\mathscr{N}$-ring. It is not hard to check that $\mathscr{N}$ is a radical property [7]. The $\mathscr{N}$-radical of a ring R will be denoted by $\mathscr{N}(R)$. If R is an $\mathscr{N}$-ring, then for any $m>0$ the polynomial ring $R[x_1, \ldots, x_m]$ is also an $\mathscr{N}$-ring.

PROPOSITION 3.1. *If R is an $\mathscr{N}$-ring, then for any $n>0$ the matrix ring R_n is also an $\mathscr{N}$-ring.*

Proof. Since $R[x_1, \ldots, x_{m+1}]$ is nil, therefore, as it is well known, $(R[x_1, \ldots, x_{m+1}])_n\approx(R_n[x_1, \ldots, x_m])[x_{m+1}]$ is a Jacobson radical. Now applying S. A. Amitsur's result [1] we find that $R_n[x_1, \ldots, x_m]$ is nil, which means that R_n is also an $\mathscr{N}$-ring.

DEFINITION. We shall call a group G an *$\mathscr{N}$-group* if for any $\mathscr{N}$-ring R the group ring $R[G]$ is an $\mathscr{N}$-ring.

Let R be an $\mathscr{N}$-ring which is not locally nilpotent (examples of such rings have been constructed by E. S. Golod [5]) and let W be a free group generated by at least two elements. Let P be a free semi-group with the same set of generators as W. Then $R[P]\subseteq R[W]$. Now applying A. Sierpińska's result [11], we obtain that $R[W]$ is nil if and only if R is locally nilpotent. Therefore W is not an $\mathscr{N}$-group.

The class of all $\mathscr{N}$-groups is homomorphicaly invariant and any subgroup of an $\mathscr{N}$-group is also an $\mathscr{N}$-group. It is easy to check that every abelian group is an $\mathscr{N}$-group and every locally $\mathscr{N}$-group is an $\mathscr{N}$-group.

PROPOSITION 3.2. *A discrete direct sum of $\mathcal{N}$-groups is also an $\mathcal{N}$-group.*

Proof. Let H_1, H_2 be $\mathcal{N}$-groups and let R be an $\mathcal{N}$-ring. Since $(R[H_1])[H_2] \approx R[H_1 \times H_2]$, $H_1 \times H_2$ is an $\mathcal{N}$-group. By simple induction arguments one can prove that the direct sum of finitely many $\mathcal{N}$-groups is also an $\mathcal{N}$-group. Now let G be a discrete direct sum of any family of $\mathcal{N}$-groups. Then G is a locally $\mathcal{N}$-group, i.e., an $\mathcal{N}$-group.

PROPOSITION 3.3. *If a group G contains an $\mathcal{N}$-subgroup H of finite index in G, then G is an $\mathcal{N}$-group.*

Proof. Let R be an $\mathcal{N}$-ring. Since $R[H]$ is an $\mathcal{N}$-ring, by Proposition 3.1 the matrix ring $(R[H])_n$ is also an $\mathcal{N}$-ring for any integer n. If we extend R to the ring R^* with unity element, then $R^*[G]$ is a right free $R^*[H]$-module of rank k, where k is the index of H in G. Now we can take a regular representation of the ring $R^*[G]$ into the ring of $R^*[H]$-endomorphism of the right $R^*[H]$-module $R^*[G]$. Thus $R^*[G]$ can be embedded into the matrix ring $(R^*[H])_k$, i.e., $R[G]$ can be embedded into the $\mathcal{N}$-ring $(R[H])_k$. Therefore $R[G]$ is an $\mathcal{N}$-ring, which means that G is an $\mathcal{N}$-group.

COROLLARY. *Any finite group is an $\mathcal{N}$-group.*

For a group G by $\Delta(G)$ we shall denote the set of those elements from G which have only finitely many conjugates [10].

PROPOSITION 3.4. *If $G = \Delta(G)$, then G is an $\mathcal{N}$-group.*

Proof. If H is a finitely generated subgroup of G, then the centre $Z(H)$ of H has a finite index in H ([10], Lemma 2.2). Thus $Z(H)$ as an abelian group is an $\mathcal{N}$-group. Therefore by Proposition 3.3 H is an $\mathcal{N}$-group. This means that G is a locally $\mathcal{N}$-group, i.e., G is an $\mathcal{N}$-group.

THEOREM C. *If the additive group R^+ of a ring R is torsion disjoint with an $\mathcal{N}$-group G, then*

$$\mathcal{N}(R[G]) = \mathcal{N}(R)[G].$$

Proof. The inclusion $\mathcal{N}(R)[G] \subseteq \mathcal{N}(R[G])$ is obvious since G is an $\mathcal{N}$-group. Conversely, by S. A. Amitsur's result [1], $\mathcal{K}(R[x_1, x_2, \ldots]) = \mathcal{N}(R)[x_1, x_2, \ldots]$; hence the ring $R/\mathcal{N}(R)[x_1, x_2, \ldots]$ is $\mathcal{K}$-semisimple. It is easy to check that the additive group $R[x_1, x_2, \ldots]^+$ of the ring $R[x_1, x_2, \ldots]$ is torsion disjoint with G. Now from Proposition 1.1 it follows that the additive group $R/\mathcal{N}(R)[x_1, x_2, \ldots]^+$ of the ring

$$\frac{R[x_1, x_2, \ldots]}{\mathcal{K}(R[x_1, x_2, \ldots])} \approx \frac{R[x_1, x_2, \ldots]}{\mathcal{N}(R)[x_1, x_2, \ldots]} \approx \frac{R}{\mathcal{N}(R)}[x_1, x_2, \ldots]$$

is also torsion disjoint with G. Applying Theorem A implies that the ring

$$\left(\frac{R}{\mathcal{N}(R)}[G]\right)[x_1, x_2, \ldots] \approx \left(\frac{R}{\mathcal{N}(R)}[x_1, x_2, \ldots]\right)[G]$$

is $\mathcal{K}$-semisimple. Therefore the ring

$$\frac{R[G]}{\mathcal{N}(R)[G]} \approx \frac{R}{\mathcal{N}(R)}[G]$$

is $\mathcal{N}$-semisimple, which means that $\mathcal{N}(R)[G] \supseteq \mathcal{N}(R[G])$.

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