On the actions of $\mathrm{SO}(3)$ on lens spaces

by

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Abstract. In this paper, an effective $\mathrm{SO}(3)$-action on a 5-dimensional lens space is studied via the lifting action on the universal covering space of a lens space. The results are the following: Suppose $\mathrm{SO}(3)$ acts effectively on a 5-dimensional lens space. If there is a 3-dimensional orbit, then there is a non-trivial finite cyclic isotropy subgroup, and if the highest dimension of any orbit is two, then there is an isotropy subgroup $S_k$. Furthermore, if the fixed point set of the action is non-empty, then it is a circle.

The effective use of the universal covering space $\tilde{X}$ over $X$ on the study of transformation groups $(G, X)$ has been demonstrated in [1], [2], [3], [4], and others. The purpose of this short note is to add one more example to the above list.

The effective action of a 3-dimensional rotation group $\mathrm{SO}(3) \times \mathbb{S}^3$ on a 5-dimensional lens space $L_k(m; q_1, q_2)$, $m$ odd and $(m, q_i) = 1$ for $i = 1, 2$, is studied via $\mathbb{S}^3$, the universal covering space of $L_k(m; q_1, q_2)$. We will use $L_k(m)$ to denote $L_k(m; q_1, q_2)$.

All known results indicate that there is no finite cyclic isotropy subgroups of $(\mathrm{SO}(3), \mathbb{S}^3)$. It is shown here that if $\mathrm{SO}(3)$ acts effectively on $L_k(m)$ then there exists a point $x \in L_k(m)$ such that the isotropy group $\mathrm{SO}(3)_x = \{ g \in \mathrm{SO}(3) : gx = x \}$ is a non-trivial finite cyclic subgroup of $\mathrm{SO}(3)$ or $S_k$ subgroup of $\mathrm{SO}(3)$. Furthermore, if the fixed point set $F(\mathrm{SO}(3), L_k(m)) \neq \emptyset$, then it must be a circle.

The group $\mathrm{SO}(3)$ has the following conjugacy classes of (closed) subgroups: the group $S_k$ of all rotations about an axis determined by a point $p \in S^2$; the group $N_k$ of all rotations which take the plane perpendicular to the axis determined by $p \in S^2$ into itself; the cyclic groups $Z_n$ of order $n$; the dihedral group $D_n$ of order $2n$; the group $H_7$, $H_8$, and $H_9$ of all rotational symmetries of the tetrahedron, the cube, and the icosahedron, respectively.

Let $S^{2n+1}$ be the $(2n+1)$-dimensional unit sphere in Euclidean $(2n+2)$-space defined in terms of $(n+1)$-complex coordinates $(z_0, \ldots, Z_n)$ satisfying $Z_0 Z_0 + \ldots + Z_n Z_n = 1$. Let $m \geqslant 2$ be a fixed integer, and $q_1, \ldots, q_4$ be integers relatively prime to $m$. We define an action $\theta$ on $S^{2n+1}$ onto itself by

$$\theta(t, (z_0, \ldots, z_n)) = (e^{2\pi i q_0 t} z_0, e^{2\pi i q_1 t} z_1, \ldots, e^{2\pi i q_n t} z_n).$$

Then $t$ generates a fixed point free cyclic group $Z_n(t)$ of rotations of $S^{2n+1}$ of order $m$. The orbit space $S^{2n+1}/Z_n(t) = L_{2n+1}(m; q_1, \ldots, q_4)$ is an orientable $(2n+1)$-dimen-
sional manifold called a lens space. If \( P: S^{2n+1} \rightarrow T(\mathbb{Z}; \mathbb{Z}_p; \ldots; \mathbb{Z}_p) \) is the projection map and \( g \in T(\mathbb{Z}; \mathbb{Z}_p; \ldots; \mathbb{Z}_p) \), then \( P(x) = P(x) \) for \( x \in S^{2n+1}, Z_d(i) \) is the group of covering transformations.

Let us examine the known results of actions (SO(3), S^3) from [5] and [6]. First assume SO(3) acts on \( S^3 \) with 3-dimensional orbits. Denote by \( X \) the set of points on 3-dimensional orbits and \( Y = S^3 - X \). It is easy to see that the orbit space \( S^3/\text{SO}(3) \cong D^2 \), a 2-dimensional disk, \( Y/\text{SO}(3) \cong D^2 \), and \( X/\text{SO}(3) \cong D^3 \). If there is an \( S^2 \)-orbit then all orbits of \( Y \) are \( S^2 \)-orbits and \( G_x = e \) for \( x \in X \). If there is an orbit homeomorphic to the real projective space \( \mathbb{RP}^3 \) then \( G_x \) must be a dihedral group \( D_2 \). There will be precisely two stationary points on \( Y \) and all other orbits on \( Y \) correspond to \( P^3 \)-orbits. All orbits of \( Y \) are stationary points, then for \( x \in X \), \( G_x \) must be an icosahedral group \( H_1 \).

If an action of SO(3) on \( S^3 \) is such that the highest dimension of any orbit is two, then the orbit space is a closed 3-disk \( D^3 \). The interior points correspond to 2-sphere orbits and the boundary points correspond to stationary points. In other words, \( G_x = S_3 \), for all \( x \in X \).

On the other hand, it has been shown [5] that if SO(3) acts on any sphere \( S^3 \) such that the principal isotropy group is a finite cyclic group, then it must be the trivial group. In the same paper, if SO(3) acts differentiably on \( S^3 \) with three dimensional principal orbits and if \( \dim S = n - 1 \) (B is the set of points on orbits of dimension less than the highest dimension of any orbit), then the principal isotropy group is the identity.

Let \( (G, X) \) be an action. Define a map \( f: (G, e) \rightarrow (X, x_0) \) by \( f(g) = g.x_0 \). This map induces a homomorphism \( f_2: \pi_2(\text{SO}(3)) \rightarrow \pi_2(L_3(m)) \) of the homotopy groups.

**Lemma 1.** Any effective action (SO(3), \( L_3(m) \)), \( m \) odd, can be lifted to an effective action of (SO(3), \( S^3 \)).

**Proof.** The induced homomorphism \( f_2: \pi_2(\text{SO}(3)) \rightarrow \pi_2(L_3(m)) \) is a trivial homomorphism and the lemma follows from [2; 4.3].

**Theorem 2.** If SO(3) acts effectively on \( L_3(m) \), \( m \) odd, with a 3-dimensional orbit, then there exists a point \( x \in L_3(m) \) such that SO(3) is a non-trivial finite cyclic isotropy subgroup of SO(3).

**Proof.** We combine two proofs in one. By Lemma 1 we can lift SO(3) action to \( S^3 \) such that \( g(b^2) = (g(b)g) \) for \( b \in S^3 \) and \( \gamma \in \pi_2(L_3(m)) = Z_2 \). Now we can see easily that there is an induced action of \( Z_2 = \pi_1(L_3(m)) \) on \( S^3/\text{SO}(3) \cong D^3 \), where \( j = 2 \) or 3. The following diagram commutes:

\[
\begin{array}{ccc}
S^3 & \overset{\pi_2}{\longrightarrow} & D^3 \\
\downarrow & & \downarrow \\
L_3(m) & \overset{\pi_2}{\longrightarrow} & D^3
\end{array}
\]

Now a \( Z_2 \) action on \( D^3 \) has a fixed point which lies in the interior. To see this let us examine the case \( j = 3 \) (the \( D^3 \) case follows more easily). Since \( m \) is odd we can write \( Z_2 = \mathbb{Z}_2 \times Z_2 \) where \( (q_1, q_2) = 1 \). By the Smith theorem we may assume \( F(Z_2; D^3) \) is a line segment in \( D^3 \) such that only two end points \( \partial D^3 \) lie on \( \partial D^3 = S^2 \).

Now we claim \( F(Z_2; D^3) = 1 \). Otherwise a generator of \( F(Z_2; D^3) \) will permute \( l \in D^3 \) with period \( q_2 \). We can see that if \( x \in \beta(l) \), then the isotropy subgroup of \( Z_2 \) at \( x \) is \( Z_2^4 \). That is, \( x \in F(Z_2^4; D^3) \) and this is a contradiction to the Smith theorem. Therefore \( l \) is invariant under the \( Z_2 \) action.

Since \( q_2 \) is odd \( Z_2 \) acts trivially on \( l \) and \( F(\mathbb{Z}_2^4; D^3) = 1 \). Thus we have a point \( x \in F(\mathbb{Z}_2^4; D^3) \) which lies in the interior of \( D^3 \).

At each \( b \in S^3 \) we define a map \( \eta_b: (\text{SO}(3); \eta_b) \rightarrow \pi_2(L_3(m)) \) by defining \( \eta_b(g) \) to be the unique element in \( \pi_2(L_3(m)) \) such that \( gb = \eta_b(g) \). This is not too hard to see that \( \eta_b \) is a homomorphism for each \( b \in S^3 \). We have the following exact sequence from [2; 4.16]:

\[
n \circ \rightarrow \pi_2(\text{SO}(3); \eta_b) \rightarrow \pi_3(L_3(m)) \rightarrow \mathbb{Z}_2 \rightarrow 0.
\]

Take \( P_i \) to be the point in \( F(Z_2; D^3), j = 2, 3 \), which lies in the interior of \( D^3 \).

It follows that \( \text{SO}(3) \) acts trivially and we know that \( \text{SO}(3) \) is either \( e \), a dihedral group \( D_2 \), icosahedral group \( H_1 \), or \( S_3 \). Now take all possible closed subgroups of \( \text{SO}(3) \) for \( \text{SO}(3) \). No combination allows \( \text{SO}(3) \cong \text{SO}(3) \), odd, \( m \) odd, except \( \eta_b = e \), the identity element, or \( \text{SO}(3) \cong S_3 \). In the first case we get \( \text{SO}(3) \cong Z_2 \) and in the second case we get \( \text{SO}(3) \cong S_3 \). This completes the proof.

**Theorem 3.** If \( F(\text{SO}(3); L_3(m)) = \emptyset \), then it must be a circle \( S^1 \).

**Proof.** It is known that \( F(\text{SO}(3); S^3) \) must be \( S^3 \), \( S^2 \), or \( S^1 \). Since \( \text{SO}(3) \) acts effectively on \( L_3(m) \), \( m \) odd, \( S^2 \) and \( S^3 \) are invariant under the free \( \text{SO}(3) \) action. Since \( m \) odd \( Z_2 \neq \mathbb{Z}_2 \) and \( Z_2 \) cannot act freely on \( S^2 \) and \( S^3 \), so these two cases are eliminated. Now \( S^1 \) is invariant under \( \text{SO}(3) \) and we have \( F(\text{SO}(3); L_3(m)) \cong S^1/\mathbb{Z}_2 \simeq S^1 \). This completes the proof.

Finally we give the following theorem which follows by a lifting action and an application of [6].

**Theorem 4.** If \( S^1 \), the double covering group of \( \text{SO}(3) \), acts effectively on \( L_3(m) \), then the action is unique. Furthermore, there exists a point at which the isotropy subgroup contains \( Z_2 \), odd.

Note. The results on general lens spaces will appear shortly.

References


On group nil rings

by

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Abstract. The main result of this paper (Theorem A) is a generalization of a theorem of D. S. Passman [9] saying that under some assumptions the group ring $R[G]$ for commutative $R$ contains no non-zero nil ideals. Our result is then applied in §2 to find some new statements equivalent to the still open Koethe problem: if a ring $R$ contains a one-sided nil ideal $A$, is $A$ contained in a two-sided nil ideal of $R$? Finally, §3 is devoted to an investigation, by means of Theorem A, of the $N$-radical of certain group rings, where $N$ is the absolutely nil property defined by S. A. Amitsur [2], [4].

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§ 1. Semisimplicity of group rings

Let $R$ be a ring with unity and $M$ its multiplicatively closed subset contained in the centre of $R$. Assume that $M$ contains no zero-divisors of $R$ and consider the complete ring of right quotients $Q$ for $R$. Then

$RM^{-1} = \{ q \in Q \mid \exists qm \in R \}$

is a subring of the ring $Q$. If $M$ is an empty set, put $RM^{-1} = R$. Every element of $RM^{-1}$ can be written in the form $rm^{-1}$, where $r \in R$, $m \in M$. Since $(rm^{-1})^n = r^nm^{-n}$ for any integer $n$, $R$ contains no nil ideals if and only if $RM^{-1}$ contains no nil ideals. Moreover, for any group $G$, rings $(RM^{-1})[G]$ and $(R[G])M^{-1}$ are isomorphic to each other.

For a group $G$ we denote by $G_n$ the set of elements of $G$ of order $n$. We shall say that the groups $G$ and $H$ are torsion disjoint if for any integer $n \geq 2$ at least one of the sets $G_n$, $H_n$ is empty. It is not hard to check that groups $G$ and $H$ are torsion disjoint if and only if for any prime $p$ one of the sets $G_p$, $H_p$ is empty.

A nil ring $R$ will be shortly denoted as a $\mathcal{A}$-ring. It is well known that $\mathcal{A}$ is a radical property [4]. The $\mathcal{A}$-radical of a ring $R$ will be denoted by $\mathcal{A}(R)$.

Proposition 1.1. If the additive group $R^+$ of a ring $R$ is torsion disjoint with a group $G$, then the additive group $\overline{R}^+$ of $\overline{R}$ is also torsion disjoint with $G$, where $\overline{R} = R/\mathcal{A}(R)$.

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