On the actions of SO(3) on lens spaces

by

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Abstract. In this paper, an effective SO(3)-action on a 5-dimensional lens space is studied via the lifting action on the universal covering space of a lens space. The results are the following:

Suppose SO(3) acts effectively on a 5-dimensional lens space. If there is a 3-dimensional orbit, then there is a non-trivial finite cyclic isotropy subgroup, and if the highest dimension of any orbit is two, then there is an isotropy subgroup S_p. Furthermore, if the fixed point set of the action is non-empty, then it is a circle.

The effective use of the universal covering space $\tilde{X}$ over X on the study of transformation groups (G, X) has been demonstrated in [1], [2], [3], [4], and others. The purpose of this short note is to add one more example to the above list.

The effective action of a 3-dimensional rotation group SO(3) on a 5-dimensional lens space $L_5(m; q_1, q_2)$, m odd and $(m, q_i) = 1$ for $i = 1, 2$, is studied via S^5, the universal covering space of $L_5(m; q_1, q_2)$. We will use $L_5(m)$ to denote $L_5(m; q_1, q_2)$.

All known results indicate that there is no finite cyclic isotropy subgroups of $(\mathrm{SO}(3), S^n)$. It is shown here that if SO(3) acts effectively on $L_5(m)$ then there exists a point $x \in L_5(m)$ such that the isotropy group $\mathrm{SO}(3)_x = \{g \in \mathrm{SO}(3) | \, gx = x\}$ is a non-trivial finite cyclic subgroup of SO(3) or S_p subgroup of SO(3). Furthermore, if the fixed point set $F(\mathrm{SO}(3), L_5(m)) \neq \emptyset$, then it must be a circle.

The group SO(3) has the following conjugacy classes of (closed) subgroups: the group S_p of all rotations about an axis determined by a point $p \in S^2$; the group N_p of all rotations which take the plane perpendicular to the axis determined by $p \in S^2$ into itself; the cyclic groups Z_n of order n; the dihedral group D_n of order $2n$; the group H_T, H_C, and H_I of all rotational symmetries of the tetrahedron, the cube, and the icosahedron, respectively.

Let S^{2n+1} be the $(2n+1)$-dimensional unit sphere in Euclidean $(2n+2)$-space defined in terms of $(n+1)$-complex coordinates $(Z_0, \ldots, Z_n)$ satisfying $Z_0\bar{Z}_0 + \ldots + Z_n\bar{Z}_n = 1$. Let $m \geqslant 2$ be a fixed integer, and $q_1, \ldots, q_n$ be n integers relatively prime to m. We define an action θ on S^{2n+1} onto itself by

$$\theta(t, (Z_0, \ldots, Z_n)) = (e^{2\pi i/m}Z_0, e^{2\pi i q_1/m}Z_1, \ldots, e^{2\pi i q_n/m}Z_n) .$$

Then t generates a fixed point free cyclic group $Z_m(t)$ of rotations of S^{2n+1} of order m. The orbit space $S^{2n+1}/Z_m(t) = L_{2n+1}(m; q_1, \ldots, q_n)$ is an orientable $(2n+1)$-dimensional manifold called a lens space. If $P\colon S^{2n+1}\to L_{2n+1}(m; q_1, \ldots, q_n)$ is the projection map and $g\in T_m(t)$, then $Pg(x) = P(x)$ for $x\in S^{2n+1}$. $Z_m(t)$ is the group of covering transformations.

Let us examine the known results of actions $(SO(3), S^n)$ from [5] and [6]. First assume SO(3) acts on S^5 with 3-dimensional orbits. Denote by X the set of points on 3-dimensional orbits and $Y = S^5 - X$. It is easy to see that the orbit space $S^5/SO(3) \simeq D^2$, a 2-dimensional disk, $Y/SO(3) \simeq \partial D^2 = S^1$, and $X/SO(3) \simeq D^2 - \partial D^2$ = interior of D^2. If there is an S^2-orbit then all orbits of Y are S^2-orbits and $G_x = e$ for $x\in X$. If there is an orbit homeomorphic to the real projective space P^2 then G_x must be a dihedral group D_n. There will be precisely two stationary points on Y and all other orbits on Y correspond to P^2-orbits. If all orbits of Y are stationary points, then for $x\in X$, G_x must be an icosahedral group H_I.

If an action of SO(3) on S^5 is such that the highest dimension of any orbit is two, then the orbit space is a closed 3-disk D^3. The interior points correspond to 2-sphere orbits and the boundary points correspond to stationary points. In other words, $G_x = S_p$ for all $x\in X$.

On the other hand, it has been shown [5] that if SO(3) acts on any sphere S^n such that the principal isotropy group is a finite cyclic group, then it must be the trivial group. In the same paper, if SO(3) acts differentiably on S^n with three dimensional principal orbits and if $\dim B < n-2$ (B is the set of points on orbits of dimension less than the highest dimension of any orbit), then the principal isotropy group is the identity.

Let (G, X) be an action. Define a map $f\colon (G, e)\to(X, x_0)$ by $f(g) = gx_0$. This map induces a homomorphism $f_*\colon \pi_1(G, e)\to\pi_1(X, x_0)$.

LEMMA 1. *Any effective action* $(SO(3), L_5(m))$, *m odd, can be lifted to an effective action of* $(SO(3), S^5)$.

Proof. The induced homomorphism $f_*\colon \pi_1(SO(3))\to\pi_1(L_5(m))$ is a trivial homomorphism and the lemma follows from [2; 4.3].

THEOREM 2. *If* SO(3) *acts effectively on* $L_5(m)$, *m odd, with a* 3-*dimensional orbit, then there exists a point* $x\in L_5(m)$ *such that* $SO(3)_x$ *is a non-trivial finite cyclic isotropy subgroup of* SO(3).

If SO(3) *acts effectively on* $L_5(m)$, *m odd, with a* 2-*dimensional principal orbit, then there exists a point* $x\in L_5(m)$ *such that* $SO(3)_x = S_p$.

Proof. We combine two proofs in one. By Lemma 1 we can lift SO(3) action to S^5 such that $g(b\gamma) = (gb)\gamma$ for $b\in S^5$ and $\gamma\in\pi_1(L_5(m)) = Z_m$. Now we can see easily that there is an induced action of $Z_m = \pi_1(L_5(m))$ on $S^5/SO(3) = D^j$, where $j = 2$ or 3. The following diagram commutes:

$$\begin{array}{ccc} S^5 & \xrightarrow{P_1} & D^j \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P'} \\ L_5(m) & \xrightarrow[P_1']{} & D^j \end{array}$$

Now a Z_m action on D^j has a fixed point which lies in the interior. To see this let us examine the case $j = 3$ (the D^2 case follows more easily). Since m is odd we can write $Z_m = Z_{q_1^\alpha}\times Z_{q_2}$ where $(q_1, q_2) = 1$. By the Smith theorem we may assume $F(Z_{q_1^\alpha}; D^3)$ is a line segment $I\subset D^3$ such that only two end points ∂I lie on $\partial D^3 = S^2$. Now we claim $F(Z_m; D^3) = I$. Otherwise a generator β of Z_{q_2} will permute I in D^3 with period q_2. We can see that if $x\in\beta(I)$, then the isotropy subgroup of Z_m at x is $Z_{q_1^\alpha}$. That is, $x\in F(Z_{q_1^\alpha}; D^3)$ and this is a contradiction to the Smith theorem. Therefore I is invariant under the Z_{q_2} action. Since q_2 is odd Z_{q_2} acts trivially on I and $F(Z_m; D^3) = I$. Thus we have a point $x\in F(Z_m; D^3)$ which lies in the interior of D^3.

At each $b\in S^5$ we define a map $\eta_b\colon SO(3)_{P(b)}\to\pi_1(L_5(m))$ by defining $\eta_b(g)$ to be the unique element in $\pi_1(L_5(m))$ such that $gb = b\eta_b(g)$. It is not too hard to see that η_b is a homomorphism for each $b\in S^5$. We have the following exact sequence from [2; 4.16]:

$$e\to SO(3)_b\to SO(3)_{P(b)}\xrightarrow{\eta_b}\pi_1(L_5(m))_{P_1(b)}\to 0\,.$$

Take $P_1(b)$ to be the point in $F(Z_m; D^j)$, $j = 2, 3$, which lies in the interior of D^j. It follows that SO(3)(b) in S^5 is a principal orbit and we know that $SO(3)_b$ is either e, a dihedral group D_n, icosahedral group H_I, or S_p. Now take all possible closed subgroups of SO(3) for $SO(3)_{P(b)}$. No combination allows $SO(3)_{P(b)}/SO(3)_b \simeq Z_m$, m odd, except $G_b = e$, the identity element, or $SO(3)_b = S_p$. In the first case we get $SO(3)_{P(b)} \simeq Z_m$ and in the second case we get $SO(3)_{P(b)} \simeq S_p$. This completes the proof.

THEOREM 3. *If* $F(SO(3); L_5(m)) \neq \emptyset$, *then it must be a circle* S^1.

Proof. It is known that $F(SO(3); S^5)$ must be S^0, S^1, or S^2. Since $g(b\gamma) = (gb)\gamma$ for $b\in S^5$, $\gamma\in\pi_1(L_5(m))$, S^0, S^1 and S^2 are invariant under the free Z_m action. Since m odd $Z_m\neq Z_2$ and Z_m cannot act freely on S^0 and S^2, so these two cases are eliminated. Now S^1 is invariant under Z_m and we have $F(SO(3); L_5(m)) \simeq S^1/Z_m \simeq S^1$. This completes the proof.

Finally we give the following theorem which follows by a lifting action and an application of [6].

THEOREM 4. *If* S^3, *the double covering group of* SO(3), *acts effectively on* $L_5(m)$, *then the action is unique. Furthermore, there exists a point at which the isotropy subgroup contains* Z_m, *m odd.*

Note. The results on general lens spaces will appear shortly.

References

[1] P. E. Conner and D. Montgomery, *Transformation groups on a* $K(\pi, 1)$, I., Michigan Math. J. 6 (1957), pp. 405–412.

[2] — and F. Raymond, *Actions of compact Lie groups on aspherical manifolds*, Topology of manifolds, edited by Cantrell and Edwards, (1969), pp. 227–264.

[3] J. Pak, *The fixed point theorems of circle and toroid groups on lens spaces*, Fund. Math. 66 (1970), pp. 195–197.
[4] J. C. Su, *Transformation groups on cohomology projective spaces*, Trans. Amer. Math. Soc. 106 (1963), pp. 305–318.
[5] D. Montgomery and H. Samelson, *On the action of* SO(3) *on* S^n, Pacific J. Math. 12 (1962), pp. 649–659.
[6] R. W. Richardson, Jr., *Actions of the rotation group on the 5-sphere*, Ann. of Math. 74 (1961), pp. 414–423.

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On group nil rings

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Abstract. The main result of this paper (Theorem A) is a generalization of a theorem of D. S. Passman [9] saying that under some assumptions the group ring $R[G]$ for commutative R contains no non-zero nil ideals. Our result is then applied in § 2 to find some new statements equivalent to the still open Koethe problem: if a ring R contains a one-sided nil ideal A, is A contained in a two-sided nil ideal of R? Finally, § 3 is devoted to an investigation, by means of Theorem A, of the $\mathcal{N}$-radical of certain group rings, where $\mathcal{N}$ is the absolutely nil property defined by S. A. Amitsur [2], [6].

Contents

§ 1. Semisimplicity of group rings. Let R be a ring with unity and M its multiplicatively closed subset contained in the centre of R. Assume that M contains no zero-divisors of R and consider the complete ring of right quotiens Q for R. Then ([8], Lemma 5, p. 160)

$$RM^{-1} = \{q \in Q | \underset{m \in M}{\exists} qm \in R\}$$

is a subring of the ring Q. If M is an empty set, put $RM^{-1} = R$. Every element of RM^{-1} can be written in the form rm^{-1}, where $r \in R$, $m \in M$. Since $(rm^{-1})^n = r^n m^{-n}$ for any integer n, R contains no nil ideals if and only if RM^{-1} contains no nil ideals. Moreover, for any group G, rings $(RM^{-1})[G]$ and $(R[G])M^{-1}$ are isomorphic to each other.

For a group G we denote by G_n the set of elements of G of order n. We shall say that the groups G and H are *torsion disjoint* if for any integer $n \geqslant 2$ at least one of the sets G_n, H_n is empty. It is not hard to check that groups G and H are torsion disjoint if and only if for any prime p one of the sets G_p, H_p is empty.

A nil ring R will be shortly denoted as a $\mathcal{K}$-ring. It is well known that $\mathcal{K}$ is a radical property [4]. The $\mathcal{K}$-radical of a ring R will be denoted by $\mathcal{K}(R)$.

Proposition 1.1. *If the additive group R^+ of a ring R is torsion disjoint with a group G, then the additive group $\bar{R}^+$ of $\bar{R}$ is also torsion disjoint with G, where $\bar{R} = R/\mathcal{K}(R)$.*