An example concerning the Whitehead Theorem in shape theory

by

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Abstract. Let $F\colon (X, x)\to(Y, y)$ be a shape morphism with (X, x) and (Y, y) pointed movable metric continua of finite dimension. A theorem of M. Moszyńska states that if $F_*\colon \underline{\pi}_k(X, x)\to\underline{\pi}_k(Y, y)$ is an isomorphism for all k, then F is a shape equivalence. In this paper an example is given to show that if X and Y are not finite-dimensional, then the above result may not hold.

Let T be the category of pointed topological spaces and HT be the homotopy category of pointed topological spaces with $H\colon T\to HT$ the homotopy functor. Let $S\colon T\to ST$ be the shape functor to the shape category in the sense of S. Mardešić [5]. If (X, x) is a pointed topological space, then there is for each n an inverse system of groups associated with (X, x) called the *n-th homotopy pro-group* of (X, x) (see [6]) which we will denote by $\pi_n\{(X, x)\}$. A shape morphism $F\colon (X, x)\to(Y, y)$ induces a unique morphism $F_n\colon \pi_n\{(X, x)\}\to\pi_n\{(Y, y)\}$ in the category of pro-groups. There is also associated with (X, x) a group $\underline{\pi}_n(X, x)$ which is the projective limit of $\pi_n\{(X, x)\}$. This we will call the *n-th shape group* of (X, x). The morphisms F_n (and hence F) induce unique homomorphisms $F_*\colon \underline{\pi}_n(X, x)\to\underline{\pi}_n(Y, y)$ in the category of groups. These structures $\pi_n\{(X, x)\}$ and $\underline{\pi}_n(X, x)$ play the analogous role in shape theory that the homotopy groups $\pi_n(X, x)$ play in homotopy theory.

An important result in homotopy theory is a classical theorem of J. H. C. Whitehead.

THEOREM 1. *Let $f\colon (X, x)\to(Y, y)$ be a continuous map with $f_i\colon \pi_i(X, x)\to\pi_i(Y, y)$ an isomorphism for $i<n_0 = \max\{1+\dim X, \dim Y\}$ and an epimorphism for $i = n_0$ where (X, x) and (Y, y) are connected* CW*-complexes. Then f is a homotopy equivalence.*

In shape theory several analogous results of this theorem have been proved. The first such theorem was due to M. Moszyńska [10].

THEOREM 2. *Let $F\colon (X, x)\to(Y, y)$ be a shape morphism where X and Y are finite-dimensional metric continua and let $F_k\colon \pi_k\{(X, x)\}\to\pi_k\{(Y, y)\}$ be the induced*

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morphisms of homotopy pro-groups. If F_k is an isomorphism for $1 \leqslant k < n_0+1 = \max\{1+\dim X, \dim Y\}+1$ and an epimorphism for $k = n_0+1$, then F is a shape equivalence.

Mardešić [6] has generalized this theorem to show that one can replace the assumption that X and Y are metric continua by the assumption that X is a Hausdorff continuum and Y is a metric continuum. What is more significant is the following theorem in [6].

THEOREM 3. *Let $f\colon (X, x) \to (Y, y)$ be a continuous map with X and Y connected, finite-dimensional spaces. Suppose that $S(f)_k\colon \pi_k\{(X, x)\} \to \pi_k\{(Y, y)\}$ is an isomorphism of pro-groups for $1 \leqslant k < n_0+1 = \max\{1+\dim X, \dim Y\}+1$ and an epimorphism for $k = n_0+1$. Then $S(f)$ is a shape equivalence.*

Another theorem of Moszyńska [10] is a shape version of Theorem 1 using the shape groups rather than the homotopy pro-groups.

THEOREM 4. *Let (X, x) and (Y, y) be finite-dimensional movable pointed continua and let $F\colon (X, x) \to (Y, y)$ be a shape morphism such that $F_*\colon \underline{\pi}_k(X, x) \to \underline{\pi}_k(Y, y)$ is an isomorphism for $1 \leqslant k < n_0+1 = \max\{1+\dim X, \dim Y\}+1$ and an epimorphism for $k = n_0+1$. Then F is a shape equivalence.*

There appears to be a nontrivial gap in the proof of Theorem 4 in [10]. However, this gap has been filled in [4]. The purpose of this paper is to give an example of a continuous map $f\colon (X, x) \to (Y, y)$ where (X, x) and (Y, y) are movable pointed metric continua such that $S(f)_k\colon \pi_k\{(X, x)\} \to \pi_k\{(Y, y)\}$ is an isomorphism of homotopy pro-groups for all k and $S(f)_*\colon \underline{\pi}_k(X, x) \to \underline{\pi}_k(Y, y)$ is an isomorphism for all $k \geqslant 1$, but with $S(f)$ not a shape equivalence. This example shows that the assumption that X and Y are finite dimensional in each of Theorems 2, 3, and 4 cannot be eliminated. It also shows that adding the condition that (X, x) and (Y, y) be movable in Theorem 2 or Theorem 3 would still not allow one to eliminate the requirement that X and Y be finite-dimensional. In [3] Keesling gave an example of a pointed movable nonmetric continuum (X, x) with $\underline{\pi}_i(X, x) = H_i(X) = 0$ for $i \geqslant 1$, but with X not having the shape of a point. Thus the map $e\colon x \to (X, x)$ is an example of a map inducing isomorphisms $S(e)_*\colon \underline{\pi}_i(x) \to \underline{\pi}_i(X, x)$ and $e_*\colon H_i(x) \to H_i(X)$ for all $i \geqslant 1$, but with $S(e)$ not a shape equivalence. However, in addition to X not being metric, the map e did not induce an isomorphism of homotopy pro-groups in dimension one. Thus, the counterexample presented in this paper is more useful in defining the limits of Theorems 2, 3, and 4.

The example. In [2] D. S. Kahn has constructed for each odd prime p a sequence of compact connected polyhedra $\{Z_i\}_{i=0}^{\infty}$ and maps $h_i\colon Z_{i+1} \to Z_i$ for $i \geqslant 0$ such that:

(1) for $i < j$ the map $h_i \circ \ldots \circ h_j\colon Z_{j+1} \to Z_i$ is essential;

(2) $\dim Z_i = (2p+1)+(2p-2)i$; and

(3) each Z_i is $[(2p-1)+(2p-2)i]$-connected.

This sequence was also described by J. F. Adams ([1], Theorem 1.7). Let Z be the inverse limit of $\{Z_i; h_i\}$ and let $z \in Z$ be a fixed point with $z_i \in Z_i$ the projection of z in Z_i for each $i \geqslant 0$. We will use the sequence (Z_i, z_i) and bonding maps $h_i\colon (Z_{i+1}, z_{i+1}) \to (Z_i, z_i)$ in constructing the example proving the main theorem.

THEOREM. *There are pointed movable metric continua (X, x) and (Y, y) and a continuous map $f\colon (X, x) \to (Y, y)$ such that for all $n \geqslant 1$ $S(f)$ induces isomorphisms of $\underline{\pi}_n(X, x)$ to $\underline{\pi}_n(Y, y)$ and $\pi_n\{(X, x)\}$ to $\pi_n\{(Y, y)\}$ in the categories of groups and pro-groups, respectively, but with $S(f)$ not a shape equivalence.*

Proof. Let $\{(Z_i, z_i); h_i\}$ be the sequence of spaces and bonding maps described above. Let $(X_n, x_n) = \bigvee_{i=0}^{n} (Z_i, z_i)$. Then let the bonding maps $g_n\colon (X_{n+1}, x_{n+1}) \to (X_n, x_n)$ be defined by $g_n(x) = x$ for $x \in Z_i$, $0 \leqslant i \leqslant n$ and $g_n(x) = h_n(x)$ for $x \in Z_{n+1}$. Then g_n is well-defined for all n and letting (X, x) be the inverse limit of $\{(X_n, x_n); g_n\}$ we have that (X, x) is a movable pointed metric continuum.

Now we define (Y, y). Let $(Y_n, y_n) = \bigvee_{i=0}^{n} (Z_i, z_i) = (X_n, x_n)$. However, we define different bonding maps $h_n\colon (Y_{n+1}, y_{n+1}) \to (Y_n, y_n)$ than were used in defining the sequence associated with (X, x). Let $h_n(x) = x$ for $x \in Z_i$, $0 \leqslant i \leqslant n$ and $h_n(x) = y_n$ for $x \in Z_{n+1}$. Then h_n is well-defined. Letting (Y, y) be the inverse limit of $\{(Y_n, y_n); h_n\}$ we have that (Y, y) is a movable pointed metric continuum.

For each $n \geqslant 1$ we now define a map

$$f_n\colon \bigvee_{i=0}^{n+1} (Z_i, z_i) = (X_{n+1}, x_{n+1}) \to (Y_n, y_n) = \bigvee_{i=0}^{n} (Z_i, z_i)$$

by $f_n(z) = z$ if $z \in \bigvee_{i=0}^{n} (Z_i, z_i)$ and $f_n(z) = y_n$ if $z \in Z_{n+1}$. Then f_n is well-defined and the following diagram commutes.

$$\begin{array}{ccc} (X_n, x_n) & \xleftarrow{g_n} & (X_{n+1}, x_{n+1}) \\ \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} \\ (Y_{n-1}, y_{n-1}) & \xleftarrow[h_{n-1}]{} & (Y_n, y_n) \end{array}$$

Consequently the maps $\{f_n\}$ induce a continuous map $f\colon (X, x) \to (Y, y)$. We will now show that this map is the one required in the theorem. Note that if $S(f)$ induces isomorphisms in the pro-group category between $\pi_n\{(X, x)\}$ and $\pi_n\{(Y, y)\}$ for all $n \geqslant 1$, then $S(f)$ will automatically induce isomorphisms of $\underline{\pi}_n(X, x)$ to $\underline{\pi}_n(Y, y)$ for all n. Thus we only need to show that $S(f)$ induces isomorphisms of the homotopy pro-groups.

Fix an integer k and let m satisfy $k \leqslant (2p-1)+(2p-2)m$. Then let

$$r\colon (S^k, *) \to (X_n, x_n) = \bigvee_{i=0}^{n} (Z_i, z_i),$$

then by property (3) of the sequence of Z_i's, r is homotopic to a map $r'\colon (S^k, *) \to (X_n, x_n)$ such that the image of r' is contained in $\bigvee_{i=0}^{m} (Z_i, z_i) = (X_m, x_m)$. Thus the pro-group $\pi_k\{(X, x)\}$ which is equivalent to the pro-group $\{\pi_k(X_n, x_n);\ g_{n*};\ n \geqslant m\}$ is equivalent to the pro-group:

$$\pi_k(X_m, x_m) \xleftarrow{1_*} \pi_k(X_m, x_m) \xleftarrow{1_*} \ldots = \{\pi_k(X_m, x_m);\ 1_*\}$$

where $1\colon (X_m, x_m) \to (X_m, x_m)$ is the identity map. Similarly, the pro-group $\pi_k\{(Y, y)\}$ is isomorphic to $\{\pi_k(Y_n, y_n);\ h_{n*};\ n \geqslant m\}$ which is equivalent to $\{\pi_k(Y_m, y_m)\colon 1_*\}$ in the pro-group category where $k \leqslant (2p-1)+(2p-2)m$. Clearly, the pro-group morphism $S(f)_k\colon \pi_k\{(X, x)\} \to \pi_k\{(Y, y)\}$ is the same as that induced by the identity map $1\colon (X_m, x_m) \to (Y_m, y_m)$ and thus $S(f)_k$ is an isomorphism of pro-groups for all $k \geqslant 1$.

$$\begin{array}{ccccc}
\pi_k(X_m, x_m) & \xleftarrow{1_*} & \pi_k(X_m, x_m) & \xleftarrow{1_*} & \ldots \\
\downarrow{\scriptstyle 1_*} & & \downarrow{\scriptstyle 1_*} & & \\
\pi_k(Y_m, y_m) & \xleftarrow[1_*]{} & \pi_k(Y_m, y_m) & \xleftarrow[1_*]{} & \ldots
\end{array}$$

Now we will show that $S(f)$ is not a shape equivalence. Suppose that $S(f)$ is a shape equivalence. Then there is a shape morphism $Q\colon (Y, y) \to (X, x)$ such that $Q \circ S(f) = S(1_{(X,x)})$. Using the ANR-systems approach to shape theory [8], the shape morphism Q can be thought of as a function $q\colon N \to N$ such that for $n \geqslant m$, $q(n) \geqslant q(m)$ and a system of continuous maps $q_n\colon (Y_{q(n)}, y_{q(n)}) \to (X_n, x_n)$ such that the following diagram commutes up to homotopy

$$\begin{array}{ccc}
(Y_{q(n)}, y_{q(n)}) & \xleftarrow{h_{q(n)} \circ \ldots \circ h_{q(n+1)-1}} & (Y_{q(n+1)}, y_{q(n+1)}) \\
\downarrow{\scriptstyle q_n} & & \swarrow{\scriptstyle q_{n+1}} \\
(X_n, x_n) & \xleftarrow[g_n]{} & (X_{n+1}, x_{n+1})
\end{array}$$

and such that for each n, there is an $m \geqslant q(n)$ such that $g_n \circ \ldots \circ g_m\colon (X_{m+1}, x_{m+1}) \to (X_n, x_n)$ is homotopic to $q_n \circ p_{q(n)} \circ g_{q(n)+1} \circ \ldots \circ g_m\colon (X_{m+1}, x_{m+1}) \to (X_n, x_n)$.

$$\begin{array}{ccccc}
(X_n, x_n) & \xleftarrow{g_n} \ldots \xleftarrow{g_{q(n)}} & (X_{q(n)+1}, x_{q(n)+1}) & \leftarrow \ldots \leftarrow & (X_{m+1}, x_{m+1}) \\
\nwarrow{\scriptstyle q_n} & & \swarrow{\scriptstyle p_{q(n)}} & & \\
& (Y_{q(n)}, y_{q(n)}) & & &
\end{array}$$

However, $g_n \circ \ldots \circ g_m | (Z_{m+1}, z_{m+1})$ is an essential map onto (Z_n, z_n) by property (1) of the sequence $\{Z_1;\ h_i\}$. But the map $q_n \circ f_{q(n)} \circ g_{q(n)+1} \circ \ldots \circ g_m | (Z_{m+1}, z_{m+1})$ is a constant map since $f_{q(n)}$ takes $(Z_{q(n)}, z_{q(n)})$ to the point $y_{q(n)} \in Y_{q(n)}$. This is a contradiction. Thus $S(f)$ cannot be a shape equivalence.

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