An example concerning the Whitehead Theorem in shape theory

by

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Abstract. Let \( F: (X, x) \rightarrow (Y, y) \) be a shape morphism with \( (X, x) \) and \( (Y, y) \) pointed movable metric continua of finite dimension. A theorem of M. Morzyńska states that if \( F_k: \pi_k(X, x) \rightarrow \pi_k(Y, y) \) is an isomorphism for all \( k \), then \( F \) is a shape equivalence. In this paper an example is given to show that if \( X \) and \( Y \) are not finite-dimensional, then the above result may not hold.

Let \( T \) be the category of pointed topological spaces and \( HT \) be the homotopy category of pointed topological spaces with \( H: T \rightarrow HT \) the homotopy functor. Let \( S: T \rightarrow ST \) be the shape functor to the shape category in the sense of S. Mardešić [5]. If \( (X, x) \) is a pointed topological space, then there is for each \( n \) an inverse system of groups associated with \( (X, x) \) called the \( n \)-th homotopy pro-group of \( (X, x) \) (see [6]) which we will denote by \( \pi_n((X, x)) \). A shape morphism \( F: (X, x) \rightarrow (Y, y) \) induces a unique morphism \( F_*: \pi_n((X, x)) \rightarrow \pi_n((Y, y)) \) in the category of pro-groups. There is also associated with \( (X, x) \) a group \( \pi_n(X, x) \) which is the projective limit of \( \pi_n((X, x)) \). This we will call the \( n \)-th shape group of \( (X, x) \). The morphisms \( F_* \) (and hence \( F \)) induce unique homomorphisms \( F_*: \pi_n(X, x) \rightarrow \pi_n(Y, y) \) in the category of groups. These structures \( \pi_n((X, x)) \) and \( \pi_n(X, x) \) play the analogous role in shape theory that the homotopy groups \( \pi_n(X, x) \) play in homotopy theory.

An important result in homotopy theory is a classical theorem of J. H. C. Whitehead.

**Theorem 1.** Let \( f: (X, x) \rightarrow (Y, y) \) be a continuous map with \( f_*: \pi_n(X, x) \rightarrow \pi_n(Y, y) \) an isomorphism for \( 1 \leq n_0 = \max \{1 + \dim X, \dim Y\} \) and an epimorphism for \( i = n_0 \) where \( (X, x) \) and \( (Y, y) \) are connected CW-complexes. Then \( f \) is a homotopy equivalence.

In shape theory several analogous results of this theorem have been proved. The first such theorem was due to M. Morzyńska [10].

**Theorem 2.** Let \( F: (X, x) \rightarrow (Y, y) \) be a shape morphism where \( X \) and \( Y \) are finite-dimensional metric continua and let \( F_*: \pi_n((X, x)) \rightarrow \pi_n((Y, y)) \) be the induced morphism. Then if \( F_*: \pi_n(X, x) \rightarrow \pi_n(Y, y) \) is an isomorphism for all \( n \), then \( F \) is a shape equivalence.

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of homotopy pro-groups. If $F_k$ is an isomorphism for $1 \leq k < n_0 + 1 = \max\{1 + \dim X, \dim Y\} + 1$ and an epimorphism for $k = n_0 + 1$, then $F$ is a shape equivalence.

Mardeljević [6] has generalized this theorem to show that one can replace the assumption that $X$ and $Y$ are metric continua by the assumption that $X$ is a Hausdorff continuum and $Y$ is a metric continuum. What is more significant is the following theorem in [6].

**Theorem 3.** Let $f: (X, x) \to (Y, y)$ be a continuous map with $X$ and $Y$ connected and finite-dimensional spaces. Suppose that $S(f)_k: \pi_k((X, x)) \to \pi_k((Y, y))$ is an isomorphism for pro-groups for $1 \leq k < n_0 + 1 = \max\{1 + \dim X, \dim Y\} + 1$ and an epimorphism for $k = n_0 + 1$. Then $S(f)$ is a shape equivalence.

Another theorem of Moszyńska [10] is a shape version of Theorem 1 using the shape groups rather than the homotopy pro-groups.

**Theorem 4.** Let $(X, x)$ and $(Y, y)$ be finite-dimensional movable pointed continua and let $F: (X, x) \to (Y, y)$ be a shape morphism such that $F_k: \pi_k((X, x)) \to \pi_k((Y, y))$ is an isomorphism for $1 \leq k < n_0 + 1 = \max\{1 + \dim X, \dim Y\} + 1$ and an epimorphism for $k = n_0 + 1$. Then $F$ is a shape equivalence.

There appears to be a nontrivial gap in the proof of Theorem 4 in [10]. However, this gap has been filled in [4]. The purpose of this paper is to give an example of a continuous map $f: (X, x) \to (Y, y)$ where $(X, x)$ and $(Y, y)$ are movable pointed metric continua such that $S(f)_k: \pi_k((X, x)) \to \pi_k((Y, y))$ is an isomorphism of homotopy pro-groups for all $k$ and $S(f)_k: \pi_k((X, x)) \to \pi_k((Y, y))$ is an isomorphism for all $k \geq 1$, but with $S(f)$ not a shape equivalence. This example shows that the assumption that $X$ and $Y$ are finite dimensional in each of Theorems 2, 3, and 4 cannot be eliminated. It also shows that adding the condition that $(X, x)$ and $(Y, y)$ be movable in Theorem 2 or Theorem 3 would still not allow one to eliminate the requirement that $X$ and $Y$ be finite-dimensional. In [3] Keenling gave an example of a pointed movable nonmetric continuum $(X, x)$ with $\pi_2(X, x) = H_2(X) = 0$ for $i \geq 1$, but with $X$ not having the shape of a point. The map $x \to (X, x)$ is an example of a map inducing isomorphisms of the $S(f)_k: \pi_k((X, x)) \to \pi_k((Y, y))$ and $s_k: H_k(x) \to H_k(Y)$ for all $i \geq 1$, but with $S(f)$ not a shape equivalence. However, in addition to $X$ not being metric, the map $e$ did not induce an isomorphism of homotopy pro-groups in dimension one. Thus, the counterexample presented in this paper is more useful in defining the limits of Theorems 2, 3, and 4.

The example. In [2] D. S. Kahn has constructed for each odd prime $p$ a sequence of compact connected polyhedra $(Z_n)_n$ and maps $h_i: Z_{i+1} \to Z_i$ for $i \geq 0$ such that:

1. for $i < j$ the map $h_{i+1} \cdots h_j: Z_{i+1} \to Z_j$ is essential;
2. $\dim Z_i = (2p+1)(p-2)i$; and
3. each $Z_i$ is $(2p+1)(p-2)i$-connected.

This sequence was also described by I. F. Adams ([1], Theorem 1.7). Let $Z$ be the inverse limit of $(Z_n; h_i)$ and let $z \in Z$ be a fixed point with $z \in Z_0$, the projection of $z$ in $Z_n$ for each $n \geq 0$. We will use the sequence $(Z_i, z)$ and bonding maps $h_i: (Z_{i+1}, z_{i+1}) \to (Z_i, z_i)$ in constructing the example proving the main theorem.

**Theorem.** There are pointed movable metric continua $(X, x)$ and $(Y, y)$ and a continuous map $f: (X, x) \to (Y, y)$ such that for all $n \geq 1$ $S(f)$ induces isomorphisms of $\pi_k((X, x)) \to \pi_k((Y, y))$ in the categories of groups and pro-groups, respectively, but with $S(f)$ not a shape equivalence.

**Proof.** Let $(Z_i, z_i; h_i)$ be the sequence of spaces and bonding maps described above. Let $(X_0, x_0) = \bigvee_{i=0}^{n-1} (Z_i, z_i)$. Then let the bonding maps $g_i: (X_{i+1}, x_{i+1}) \to (X_i, x_i)$ be defined by $g_i(x) = x$ for $x \in Z_0$, $0 \leq i < n$ and $g_n(x) = h_n(x)$ for $x \in Z_n$. Then $g_i$ is well-defined for all $n$ and letting $(X, x)$ be the inverse limit of $(Z_i, x_i; h_i)$ we have that $(X, x)$ is a movable pointed metric continuum.

Now we define $(Y, y)$. Let $(Y_0, y_0) = \bigvee_{i=0}^{n-1} (Z_i, z_i)$. Then we define different bonding maps $h_i: (Y_{i+1}, y_{i+1}) \to (Y_i, y_i)$ than were used in defining the sequence associated with $(X, x)$. Let $h_i(x) = x$ for $x \in Z_i$, $0 \leq i < n$ and $h_n(x) = y_n$ for $x \in Z_n$. Then $h_i$ is well-defined. Letting $(Y, y)$ be the inverse limit of $(Y_i, y_i; h_i)$ we have that $(Y, y)$ is a movable pointed metric continuum.

For each $n \geq 1$ we now define a map

$$f_n^*: \bigvee_{i=0}^{n-1} (Z_i, z_i) = (X_{n+1}, x_{n+1}) \to (Y_i, y_i) = \bigvee_{i=0}^{n-1} (Z_i, z_i)$$

by $f_n^*(x) = y$ if $x \in Z_n$ and $f_0^*(x) = y_0$ if $x \in Z_0$. Then $f_n^*$ is well-defined and the following diagram commutes.

\[
\begin{array}{ccc}
(X_{n+1}, x_{n+1}) & \xrightarrow{f_{n+1}^*} & (Y_{n+1}, y_{n+1}) \\
\downarrow \quad h_{n+1} & & \downarrow h_n \\
(Y_n, y_n) & \xrightarrow{f_n^*} & (Y_n, y_n)
\end{array}
\]

Consequently the maps $\{f_n^*\}$ induce a continuous map $f: (X, x) \to (Y, y)$. We will now show that this map is the one required in the theorem. Note that if $S(f)$ induces isomorphisms in the pro-group category between $\pi_k((X, x))$ and $\pi_k((Y, y))$ for all $n \geq 1$, then $S(f)$ will automatically induce isomorphisms of $\pi_k((X, x))$ to $\pi_k((Y, y))$ for all $n$. Thus we only need to show that $S(f)$ induces isomorphisms of the homotopy pro-groups.

Fix an integer $k$ and let $m$ satisfy $k \in (2p-1)(2p-2)m$. Then let

$$r: (S^m, x) \to (X_0, x_0) = \bigvee_{i=0}^{n-1} (Z_i, z_i).$$
then by property (3) of the sequence of $Z_n$, $r$ is homotopic to a map $r': (S^n, *) \rightarrow (X_n, x_n)$ such that the image of $r'$ is contained in $\bigvee (Z_n, z_n) = (X_n, x_n)$. Thus the pro-group $\pi_k((X, x))$ which is equivalent to the pro-group $\{\pi_n(X_n, x_n); \ast \rightarrow n\}, \ast \equiv n \geq m$ is equivalent to the pro-group:

$$
\pi_k((X, x)) \cong \{\pi_n(X_n, x_n); \ast \rightarrow n\}, \ast \equiv n \geq m
$$

where $1: (X_n, x_n) \rightarrow (X_n, x_n)$ is the identity map. Similarly, the pro-group $\pi_k((Y, y))$ is isomorphic to $\{\pi_n(Y, y); \ast \rightarrow n\}, \ast \equiv n \geq m$ which is equivalent to $\{\pi_n(Y_n, y_n); \ast \rightarrow n\}, \ast \equiv n \geq m$ in the pro-category whose $k \leq (2p-1)+2(2p-2)m$. Clearly, the pro-group morphism $S(f)_k: \pi_k((X, x)) \rightarrow \pi_k((Y, y))$ is the same as that induced by the identity map $1: (X_n, x_n) \rightarrow (Y_n, y_n)$ and thus $S(f)_k$ is an isomorphism of pro-groups for all $k \geq 1$.

Now we will show that $S(f)$ is not a shape equivalence. Suppose that $S(f)$ is a shape equivalence. Then there is a shape morphism $Q: (Y, y) \rightarrow (X, x)$ such that $(Q \circ S(f)) = S(1_{(X, x)})$. Using the ANR-systems approach to shape theory [8], the shape morphism $Q$ can be thought of as a function $q: N \rightarrow Y$ such that for $n \geq m$, $q(n) \geq q(m)$ and a system of continuous maps $g_n: (Y_{(n+1), (n+1)} \rightarrow (X_{n+1}, x_{n+1})$ such that the following diagram commutes up to homotopy

$$
\begin{array}{ccc}
(Y_{(n+1), (n+1)}, (Y_{(n+1), (n+1)}) & \stackrel{q(n)+k}{\leftarrow} & (Y_{(n+1), (n+1)}) \\
(X_{n+1}, x_{n+1}) & \stackrel{g_n}{\leftarrow} & (X_{n+1}, x_{n+1})
\end{array}
$$

and such that for each $n$, there is an $m \geq q(n)$ such that $g_n = \ldots = g_m: (X_{n+1}, x_{n+1}) \rightarrow (X_{n+1}, x_{n+1})$ is homotopic to $g_n: (Z_{(n+1), (n+1)}, z_{(n+1)}) \rightarrow (Z_{(n+1), (n+1)}, z_{(n+1)})$ is an essential map onto $(Z_{(n+1), (n+1)}$ by property (1) of the sequence $[Z_{(n+1), h_n}]$. But the map $g_n: (Z_{(n+1), (n+1)}, z_{(n+1)}) \rightarrow (Z_{(n+1), (n+1)}, z_{(n+1)})$ is a constant map since $h_n$ takes $(Z_{(n+1), (n+1)}, z_{(n+1)})$ to the point $y_0$ of $Y_0$. This is a contradiction. Thus $S(f)$ cannot be a shape equivalence.