On the Whitehead Theorem in shape theory

by

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Abstract. Let F: $(X, x) \to (Y, y)$ be a shape morphism with (X, x) and (Y, y) pointed movable metric continua of finite dimension. A theorem of M. Moszyńska asserts that if $F_*: \underline{\pi}_k(X, x) \to \underline{\pi}_k(Y, y)$ is an isomorphism for all k, then F is a shape equivalence. In this note a gap is pointed out in Moszyńska's proof and a correction given. Some related results are also presented.

Let T be the category of pointed topological spaces and HT be the homotopy category of pointed topological spaces with $H: T \to HT$ the homotopy functor. Let S: $T \to ST$ be the shape functor to the shape category in the sense of S. Mardešić [6]. If (X, x) is a pointed topological space, then there is for each n an inverse system of groups associated with (X, x) called the *n-th homotopy pro-group* of (X, x) (see [7]) which we will denote by $\pi_n\{(X, x)\}$. A shape morphism F: $(X, x) \to (Y, y)$ induces a unique morphism F_n: $\pi_n\{(X, x)\} \to \pi_n\{(Y, y)\}$ in the category of pro-groups. There is also associated with (X, x) a group $\underline{\pi}_n(X, x)$ for each positive integer n which is the projective limit of $\pi_n\{(X, x)\}$. This we will call the *n-th shape group* of (X, x). The morphisms F_n and hence F induce unique homomorphisms F_*: $\underline{\pi}_n(X, x) \to \underline{\pi}_n(Y, y)$ in the category of groups. These structures $\pi_n\{(X, x)\}$ and $\underline{\pi}_n(X, x)$ in shape theory are analogous to the homotopy groups $\pi_n(X, x)$ in homotopy theory. Similarly we have the n-dimensional homology pro-groups $H_n(\{X\}; G)$ with coefficient group G. The projective limit of these pro-groups are the n-dimensional Čech homology groups with coefficient group G, $\check{H}_n(X; G)$.

The classical theorem of J. H. C. Whitehead has played an important role in homotopy theory.

THEOREM 1. *Let f: $(X, x) \to (Y, y)$ be a continuous map with f_i: $\pi_i(X, x) \to \pi_i(Y, y)$ an isomorphism for $i < n_0 = \max\{1+\dim X, \dim Y\}$ and an epimorphism for $i = n_0$ where (X, x) and (Y, y) are connected* CW-*complexes. Then f is a homotopy equivalence.*

In shape theory there are several results analogous to this theorem which have been proved. The following theorem is due to Mardešić [7].

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THEOREM 2. *Let* $F\colon (X,x)\to(Y,y)$ *be a shape morphism where* X *and* Y *are finite-dimensional spaces and suppose that one of the two following conditions is satisfied.*

(i) F *is induced by a continuous map* f (*i.e.* $F=S(f)$), *or*

(ii) X *and* Y *are continua with* Y *metrizable.*

Then if $F_k\colon \pi_k\{(X,x)\}\to\pi_k\{(Y,y)\}$ *is an isomorphism of pro-groups for* $1\leqslant k<n_0+1=\max\{1+\dim X, \dim Y\}+1$ *and an epimorphism for* $k=n_0+1$, *then* F *is a shape equivalence.*

In case (ii) is satisfied the theorem is a slight generalization of a theorem of M. Moszyńska [11]. A version of the Whitehead Theorem involving the groups $\underline{\pi}_n(X,x)$ was also presented in [11].

THEOREM 3. *Let* (X,x) *and* (Y,y) *be finite-dimensional movable pointed continua and let* $F\colon (X,x)\to(Y,y)$ *be a shape morphism such that* $F_*\colon \underline{\pi}_k(X,x)\to\underline{\pi}_k(Y,y)$ *is an isomorphism for* $1\leqslant k<n_0+1=\max\{1+\dim X, \dim Y\}+1$ *and an epimorphism for* $k=n_0+1$. *Then* F *is a shape equivalence.*

However, there seems to be a gap in the proof of this theorem in [11]. It is the purpose of this paper to point out the gap in the proof and provide a correct proof of theorem. There is an advantage to the proof of Theorem 3 presented in this paper even apart from the gap in [11]. The proof presented here does not make use of the auxiliary notion of uniform movability introduced in [10]. What is crucial in the proof presented here is the use of a natural topology on the groups $\underline{\pi}_n(X,x)$ which is described in [4] and which was first introduced by M. Atiyah and G. B. Segal [1]. We are also able to prove a homological version of Theorem 3 using these notions based on results in [8].

1. The gap. Theorem 3 appears as Theorem 4.3 in [11]. There appears to be a gap in the proof of this theorem which appears in statement (6) on page 261 of [11]. The statement depends on Corollary 6.6 of [10] which we state here for examination.

COROLLARY (6.6 of [10]). *If* (X,x) *and* (Y,y) *are uniformly movable pointed compact Hausdorff spaces and* $F\colon (X,x)\to(Y,y)$ *is a shape morphism, then*

(1) $F_*\colon \underline{\pi}_n(X,x)\to\underline{\pi}_n(Y,y)$ *a group monomorphism implies that* $F_n\colon \pi_n\{(X,x)\}\to\pi_n\{(Y,y)\}$ *is a monomorphism in the category of pro-groups,*

(2) $F_*\colon \underline{\pi}_n(X,x)\to\underline{\pi}_n(Y,y)$ *a group epimorphism implies that* $F_n\colon \pi_n\{(X,x)\}\to\pi_n\{(Y,y)\}$ *is an epimorphism in pro-groups,*

(3) $F_*\colon \underline{\pi}_n(X,x)\to\underline{\pi}_n(Y,y)$ *a group bimorphism (a monomorphism and an epimorphism) implies that* $F_n\colon \pi_n\{(X,x)\}\to\pi_n\{(Y,y)\}$ *is a bimorphism in pro-groups.*

No proof is given for Corollary 6.6 in [10] only the statement that it follows from 6.1 and 4.3-4.5 in [10]. It appears on carefully examining 6.1 and 4.3-4.5 and their proofs that one is justified in making assertion (2). However, in place of (1) one appears to be able state only (1′) $F_*\colon \underline{\pi}_n(X,x)\to\underline{\pi}_n(Y,y)$ a group monomorphism implies that $F_n\colon \pi_n\{(X,x)\}\to\pi_n\{(Y,y)\}$ is a *monomorphism in the category of uniformly movable pro-groups.*

The difficulty is that a monomorphism in the category of uniformly movable pro-groups need *not* be a monomorphism in the category of pro-groups. The next example demonstrates this. In the proof of Theorem 3 Moszyńska needed F_n to be a monomorphism in the category of pro-groups.

It should be remarked that we do not have a counterexample to Corollary 6.6 (1). That appears to be difficult. Corollary 6.6 is true if X and Y are *metrizable* which is basically what we prove in § 2 of this paper.

1.1. EXAMPLE. Let $A=B=\prod_{i=1}^{\infty}Z$ where Z is the group of integers. Let $A_n=\prod_{i=1}^{\infty}Z=A$ for all $n\geqslant 1$ and let $f_n=\mathrm{id}_A$ for all $n\geqslant 1$, $f_n\colon A_{n+1}\to A_n$. Let $B_n=\prod_{i=1}^{n}Z$ for $n\geqslant 1$ and let $g_n\colon B_{n+1}\to B_n$ be projection onto the first n coordinates. Then B is the inverse limit of the inverse system $\{B_n; g_n\}$ and A is the inverse limit of $\{A_n; f_n\}$. Now let $\underline{h}\colon \{A_n\}\to\{B_n\}$ be defined by $h_n\colon A_n\to B_n$ be the projection homomorphism onto the first n coordinates. Then the following ladder of groups and homomorphisms commutes.

$$\begin{array}{ccccccc} A_1 & \xleftarrow{f_1} & A_2 & \xleftarrow{f_2} & A_3 & \xleftarrow{f_3} & \dots \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \\ B_1 & \xleftarrow[g_1]{} & B_2 & \xleftarrow[g_2]{} & B_3 & \xleftarrow[g_3]{} & \dots \end{array}$$

Clearly $h\colon A\to B$ which is the inverse limit of $\underline{h}$ is just the identity homomorphism. However, we will now show that $\underline{h}$ is *not* a monomorphism in the category of pro-groups.

Let $C_n=\prod_{i=n+1}^{\infty}Z$ for $n\geqslant 1$ and $i_n\colon C_{n+1}\to C_n$ be the inclusion homomorphism. Let $j_n\colon C_n\to A_n$ be the inclusion homomorphism also. Then the following ladder commutes.

$$\begin{array}{ccccccc} C_1 & \xleftarrow{i_1} & C_2 & \xleftarrow{i_2} & C_3 & \xleftarrow{i_3} & \dots \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_3} & & \\ A_1 & \xleftarrow[f_1]{} & A_2 & \xleftarrow[f_2]{} & A_3 & \xleftarrow[f_3]{} & \dots \end{array}$$

Clearly the pro-group homomorphism $\underline{j}$ is not equivalent to the zero-homomorphism $\underline{k}\colon \{C_n\}\to\{A_n\}$ with $k_n\colon C_n\to A_n$ the zero-homomorphism for all n. However, $\underline{h}\circ\underline{j}=\underline{h}\circ\underline{k}$ is the zero pro-group homomorphism for both $\underline{j}$ and $\underline{k}$. Thus $\underline{h}$ is not a monomorphism.

Note that the pro-group $\{C_n; i_n\}$ is not uniformly movable. Corollary 4.4 of [10] shows that $\underline{h}$ is a monomorphism in the category of uniformly movable pro-groups. Note also that $\underline{h}$ is an epimorphism in the category of pro-groups. This follows from Corollary 4.3 of [10].

2. A proof of Theorem 3. In this section we give a proof of Theorem 3. In the next section we give another version of a Whitehead theorem using the Čech homology groups based on the results in [8].

2.1. Definition. Let $\{G_\alpha; p_{\alpha\beta}; \alpha \leqslant \beta \in A\}$ be an object in the category of pro-groups. Then this object satisfies the *Mittag–Leffler condition* (ML) provided that for each $\alpha \in A$ there is a $\beta \geqslant \alpha$ such that for all $\gamma \geqslant \beta$

$$p_{\alpha\gamma}(G_\gamma) = p_{\alpha\beta}(G_\beta) \subset G_\alpha .$$

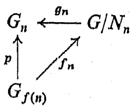
The condition ML simply states that for a fixed α, the images $p_{\alpha\gamma}(G_\gamma)$ are finally constant.

Now for any pro-group $\{G_\alpha; p_{\alpha\beta}; \alpha \leqslant \beta \in A\}$ we can topologize the inverse limit group G by giving it the inverse limit topology considering each G to have the discrete topology. Then G is a topological group. This topology is useful if the set A is countable and the pro-group satisfies ML. In this case let G be the inverse limit of $\{G_\alpha\}$ and consider the inverse system of groups $\{G/N; p_{NN'}; N' \subset N \in \eta\}$ where η is the set of all open and closed normal subgroups of G. The following proposition was observed in [1] (see also [4], Proposition 2).

2.2. Proposition. *Let $\{G_n; p_n; n \in P\}$ be a pro-group satisfying* ML *with P the positive integers. Let $\{G/N; p_{NN'}; N' \subset N \in \eta\}$ be as above. Then $\{G_n\}$ and $\{G/N\}$ are isomorphic as pro-groups. The isomorphism between $\{G_n\}$ and $\{G/N\}$ can be taken to induce the identity map on G.*

Proof. The sets $N_n = \ker p_n \subset G$ are open and closed normal subgroups of the inverse limit group G. We will show that $\{G/N_n\}$ is equivalent to the pro-group $\{G_n; p_n\}$. Once this is established, then since $\{N_n\}$ forms a basis for $e \in G$, $\{N_n\}$ is cofinal in η. Thus $\{G/N_n\}$ is equivalent to $\{G/N\}$ by [4], Proposition 1. Thus $\{G_n; p_n\}$ and $\{G/N\}$ will be equivalent also.

To show that $\{G/N_n\}$ is equivalent to $\{G_n; p_n\}$ note that there is a homomorphism $g_n\colon G/N_n \to G_n$ for all $n \geqslant 1$. Note also that by the condition ML, for each n, there is an $f(n) \geqslant n$ such that for all $k \geqslant f(n)$, $p_{nk}(G_k) = p_{nf(n)}(G_{f(n)})$. Note that this implies that $p_n(G) = p_{nf(n)}(G_{f(n)}) = g_n(G/N_n)$ as well. Let $f(n)$ be defined in this way such that $f(m) > f(n)$ for $m > n$. For each n now let $f_n\colon G_{f(n)} \to G/N_n$ be defined by $f_n = g_n^{-1} \circ p_{nf(n)}$.



It is routine to verify that $\underline{f} \circ \underline{g}$ and $\underline{g} \circ \underline{f}$ are the identities in the category of pro-groups where $\underline{f}$ is defined by the function $f\colon P \to P$ and the homomorphisms $\{f_n; n \in P\}$ given above. Thus $\{G/N_n\}$ and $\{G_n\}$ are isomorphic as pro-groups and thus $\{G_n\}$ and $\{G/N\}$ are isomorphic as pro-groups. All the pro-group isomorphisms in the proof will induce the identity homomorphism on G. The last statement in the proposition follows from this.

2.3. Remark. If the topologies described above would have been introduced in Example 1.1, then h would have been a continuous isomorphism, but *not* a homeomorphism. The topology on A is discrete in the example and that on B is not. If h were a homeomorphism, then by Proposition 2.2, $\underline{h}$ would had to be an isomorphism of pro-groups and hence a monomorphism of pro-groups.

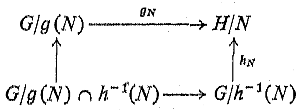
2.4. Corollary. *If $\{G_n; n \in P\}$ and $\{H_n; n \in P\}$ are pro-groups satisfying* ML *with G and H the inverse limit groups, then if $h\colon G \to H$ is an isomorphism which is a homeomorphism, then $\{G_n\}$ and $\{H_n\}$ are isomorphic by an isomorphism which induces h.*

2.5. Proposition. *If $\{G_n; n \in P\}$ and $\{H_n; n \in P\}$ are pro-groups satisfying* ML *with G and H the inverse limit groups, then if $h\colon G \to H$ is a continuous surjective homomorphism, then there is a unique pro-group homomorphism $\underline{h}\colon \{G_n\} \to \{H_n\}$ with h the inverse limit of $\underline{h}$ and $\underline{h}$ is an epimorphism in the pro-group category.*

Proof. The pro-group $\{G_n\}$ is equivalent to $\{G/N\}$ and $\{H_n\}$ is equivalent to $\{H/N\}$ by Proposition 2.3. Thus we need only to prove the proposition for $\{G/N\}$ and $\{H/N\}$. We define a pro-group homomorphism $\underline{h}\colon \{G/N\} \to \{H/N\}$ as follows. Let $N \subset H$ be an open and closed normal subgroup. Then $h^{-1}(N)$ is an open and closed normal subgroup of G. Then let

$$h_N\colon G/h^{-1}(N) \to H/N$$

be the naturally induced homomorphism. Then the correspondence $h^{-1}\colon \eta(H) \to \eta(G)$ and the homomorphisms $\{h_N\colon N \in \eta(H)\}$ define $\underline{h}$. We will show that $\underline{h}$ is an epimorphism, then show that $\underline{h}$ is unique. Using the fact that h is an epimorphism and that all bonding maps are, epimorphisms in $\{G/N\}$ and $\{H/N\}$ one has clearly that condition (e) of Theorem 6 in [8] is satisfied and thus $\underline{h}$ is an epimorphism. If $\underline{g}\colon \{G/N\} \to \{H/N\}$ also induced h as its inverse limit, then if $g\colon \eta(H) \to \eta(G)$ and $\{g_N\colon N \in \eta(H)\}$ represents $\underline{g}$, then for each $N \in \eta(H)$ the following diagram commutes because $\underline{g}$ induces h.



But this implies that $\underline{g}$ and $\underline{h}$ are equivalent ([4], § 2).

In order to apply this to prove Theorem 3 we need the following well-known result of Banach [2].

2.6. Proposition. *If G and H are separable and completely metrizable topological groups and if $h\colon G \to H$ is a surjective continuous homomorphism, then h is open.*

Proof of Theorem 3. Let $F\colon (X, x) \to (Y, y)$ be a shape morphism satisfying the hypotheses of Theorem 3. Then the pro-group homomorphisms $F_k\colon \pi_k\{(X, x)\}$

$\to \pi_k\{(Y, y)\}$ induce continuous group homomorphisms F_*: $\underline{\pi}_k(X, x) \to \underline{\pi}_k(Y, y)$ for all k where $\underline{\pi}_k(X, x)$ and $\underline{\pi}_k(Y, y)$ are given the limit topologies induced by $\pi_k\{(X, x)\}$ and $\pi_k\{(Y, y)\}$ described earlier in this section. By the movability of (X, x) and (Y, y), the pro-groups $\pi_k\{(X, x)\}$ and $\pi_k\{(Y, y)\}$ are movable and hence satisfy condition ML of 2.1. Since F_*: $\underline{\pi}_k(X, x) \to \underline{\pi}_k(Y, y)$ is onto for $1 \leqslant k \leqslant n_0+1$, F_k is the *unique* pro-group homomorphism inducing F_* by Proposition 2.5. Now $\pi_k\{(X, x)\}$ and $\pi_k\{(Y, y)\}$ are each clearly equivalent to a pro-group which is a countable inverse sequence of countable groups. Thus the groups $\underline{\pi}_k(X, x)$ and $\underline{\pi}_k(Y, y)$ are complete separable metric groups. Applying Proposition 2.6 to the continuous isomorphisms F_*: $\underline{\pi}_k(X, x) \to \underline{\pi}_k(Y, y)$ for $1 \leqslant k \leqslant n_0$ we have that F_* is also a homeomorphism and thus by Corollary 2.4 F_k: $\pi_k\{(X, x)\} \to \pi_k\{(Y, y)\}$ is a pro-group isomorphism for $1 \leqslant k \leqslant n_0$. By Proposition 2.5 F_{n_0+1}: $\pi_{n_0+1}\{(X, x)\} \to \pi_{n_0+1}\{(Y, y)\}$ is an epimorphism of pro-groups. Thus F satisfies the hypotheses of Theorem 2 and must be a shape equivalence. One could also apply Theorem 4.3 of [11] to complete the proof.

3. The Whitehead Theorem in homology. The technique of topologizing the shape groups $\underline{\pi}_n(X, x)$ described in § 2 can also be applied to topologizing the Čech homology groups. If X is a movable compactum, then the pro-group $H_n(\{X\}; G)$ satisfies the Mittag-Leffler condition for all n. If X is also metric and G is countable, then $H_n(\{X\}; G)$ is equivalent to a countable inverse system of countable groups. We could apply the techniques of § 2 to thus obtain the following lemma.

3.1. LEMMA. *If F: $X \to Y$ is a shape morphism with X and Y movable metric compacta and if G is a countable group, then if F_*: $H_n(X; G) \to H_n(Y; G)$ is an isomorphism for some n, then F induces an isomorphism of pro-groups from $H_n(\{X\}; G)$ to $H_n(\{Y\}; G)$. If F_*: $H_n(X; G) \to H_n(Y; G)$ is an epimorphism for some n, then F induces an epimorphism of pro-groups from $H_n(\{X\}; G)$ to $H_n(\{Y\}; G)$.*

We will use Lemma 3.1 to obtain a homological version of Theorem 3. In [8] Mardešić proved the following theorem and its corollary.

3.2. THEOREM. *Let F: $(X, x) \to (Y, y)$ be a shape map of 1-shape connected finite-dimensional topological spaces. We assume in addition that either*

(i) *X is compact Hausdorff and Y compact metric, or*

(ii) *F is induced by a continuous map f.*

If F_*: $H_k(\{X\}; G) \to H_k(\{Y\}; G)$ is an isomorphism of pro-groups for $2 \leqslant k < n_0 = \max\{1+\dim X, \dim Y\}$ and an epimorphism for $k = n_0$, then F is a shape equivalence.

3.3. COROLLARY. *Let (X, x) be a 1-shape connected finite-dimensional space. If $H_k(\{X\}; Z) = 0$ for $2 \leqslant k \leqslant \dim X$, then X has trivial shape.*

Using the techniques of proof for Theorem 3 given in § 2 together with Lemma 3.1 we have the following results.

3.4. THEOREM. *Let F: $(X, x) \to (Y, y)$ be a shape map of finite-dimensional movable pointed metric continua. If $\underline{\pi}_1(X, x) = 0 = \underline{\pi}_1(Y, y)$ and F_*: $H_k(X; Z) \to H_k(Y; Z)$ is an isomorphism for $2 \leqslant k < n_0 = \max\{1+\dim X, \dim Y\}$ and an epimorphism for $k = n_0$, then F is a shape equivalence.*

Proof. Since (X, x) and (Y, y) are movable pointed metric continua, $\underline{\pi}_1(X, x) = \underline{\pi}_1(Y, y) = 0$ implies that (X, x) and (Y, y) are 1-shape connected. By Lemma 3.1, F induces an isomorphism of pro-groups from $H_k(\{X\}; Z)$ to $H_k(\{Y\}; Z)$ for $2 \leqslant k < n_0$ and an epimorphism of pro-groups for $k = n_0$. Thus F is a shape equivalence by Theorem 3.2.

3.5. COROLLARY. *Let (X, x) be a finite-dimensional movable pointed continuum. Then if $\underline{\pi}_1(X, x) = 0$ and $H_k(X; Z) = 0$ for $2 \leqslant k \leqslant \dim X$, then (X, x) has trivial shape.*

3.6. EXAMPLE. In [3], J. Draper and the author give an example of two movable pointed metric continua (X, x) and (Y, y) and a continuous function f: $(X, x) \to (Y, y)$ such that $S(f)$ induces isomorphisms of the homotopy pro-groups and the shape groups for all $k \geqslant 1$ but with $S(f)$ not a shape equivalence. An examination of that example also shows that both (X, x) and (Y, y) are 1-shape connected and that $S(f)$ induces isomorphisms of $H_k(\{X\}; Z)$ to $H_k(\{Y\}; Z)$ and $H_k(X; Z)$ to $H_k(Y; Z)$ for all $k \geqslant 1$. Thus the finite-dimensionality of X and Y is necessary in Theorem 3.4.

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