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Some problem in elementary arithmetics *

by

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Abstract. Three different questions concerning peano arithmetic P are considered. (1) How large can the set of theories of the submodels or end extensions of some fixed non-standard model of P be? (2) What are the properties of the partial ordering of embeddibility between complete extensions of P? (3) How is the isomorphism type of a model of P related to the isomorphism types of its reducts?

This paper is concerned with (complete) extensions of elementary arithmetic P. The bulk of the paper is contained in §§ 2, 3, 4 and each one of these sections is concerned with a separate idea.

Let M be a non-standard model of P, and let M' be a submodel or end extension of M. What can $\mathrm{Th}(M')$ be, and how well does this family of theories characterize M? These questions are considered in § 2.

In § 3 a partial ordering of complete extensions of P is introduced. ($T_1 \leqslant T_2$ if each model of T_1 is embeddable in a model of T_2.) This ordering is shown to be a tree, and several of its other properties are considered.

It is well known that for elementary arithmetic the similarity type of the language used is relatively unimportant. In § 4 a study is made of the relationship between the isomorphism type of a model of P and the isomorphism type of certain of its reducts.

The paper is completed by § 1, which contains the required preliminaries, and § 5, which contains a collection of open problems.

§ 1. Preliminaries.

1A. P denotes the theory of *elementary Peano arithmetic*. When technicalities arise we may assume that the *basic* language L for P is suitably formalized with variables, logical symbols (e.g. $\neg$, $\wedge$, $\exists$, $=$), and the traditional symbols $<$, 0, $'$,

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$+$, $\cdot$. But in discussion we freely use abbreviation, and even regard any symbol defined from L as being present when it is convenient. $\forall_n$ and $\exists_n$ will denote the usual classes of prenex formulas of L, while B_n denotes the class of propositional combinations of $\forall_n$'s. Correspondingly, we class a defined symbol according to the class of its definition in L, while a Δ_n-symbol has both $\forall_n$ and $\exists_n$ definitions. By the theorem of Matijasevič (whose formal analog holds in P) we may disregard bounded quantifiers in determining the $\forall_n$ and $\exists_n$ classes. In particular, any informal recursive predicate may be present as a Δ_0 — (or recursive) — symbol. $\bar{n}$ denotes the *numeral* (term in L) for $n \in \omega$. Formulas are denoted by φ, ψ, α, etc. $y = \mu x \varphi(x)$ means $\varphi(y) \wedge \forall x(x<y \rightarrow \neg\varphi(y))$. Theories ($P$, its extensions, and reducts) are identified with their sets of theorems ($T \vdash \varphi$ indicates a theorem). Thus if A is a set of formulas, $P \cup A$ typically denotes the smallest theory containing P and A. And of course, P consists of the formulas of L obtained by logical deduction from defining recursion axioms for the symbols of L and the schema of induction.

1B. We assume all of the standard results of general model theory. M, M_1, M_2, etc. always denote models of P while N denotes the standard model with universe ω. $M \models \varphi$ means that M satisfies φ and $\mathrm{Th}(M)$ denotes the theory of formulas of L satisfied by M (a complete extension of P). $M_1 \subset M_2$ means that M_1 is embeddable as a *submodel* of M_2 (where both are models of P and the embedding preserves the basic symbols of L; note that the interpretations of the recursive symbols are also preserved). Where T_1 and T_2 are complete theories extending P, we say that T_1 *admits embedding* in T_2 provided every existential sentence of T_1 is also in T_2. Of course, this is equivalent to the condition that each model of T_1 is embeddable in some model of T_2. $M_1 \prec M_2$ indicates elementary embedding. $a \in M$ means that a belongs to the universe of M. N is a submodel of every M, which we call the *standard part* of M. $M-N$ is called the *non-standard part* of M. Note that for $M \neq N$, the standard part cannot be defined in M, even with parameters. As usual, by an *n-type* we mean a set of formulas, with at most n free variables, which is maximally consistent with P. If T is a complete extension of P, then T has a *prime* model M_T which is an elementary submodel of each model of T. Each element of M_T is a defined Skolem constant (e.g. some $\mu x \varphi(x)$). This implies that each type realized in M_T is *principal* (determined by a single formula). Given a non-principal 1-type t without parameters and consistent with a complete theory T, there is a unique smallest elementary extension of M_T in which t is realized. We say such a model is *single type generated*.

1C. Where $A \subset M$, we say that A is an *initial segment* of M (and M an *end-extension* of A) if every element of $M-A$ is greater than every element of A. M_1 is a *cofinal* substructure of M_2 if every element of M_2 is less than some element of M_1. Every countable $M \neq N$ has $<$-order type $\omega+(\omega^*+\omega)\eta$ (η: rationals; substructure initial segments correspond to certain cuts in η). The following Gaifman result [3] tells us that to determine the class of complete extensions of P satisfied by submodels of a given M, we need only to look at the initial segments:

PROPOSITION 1.1. *Given $M_2 \subset M_1$, let M_3 be the initial segment of M_1 which is cofinal with M_2. Then $M_2 \prec M_3$.*

We note another Gaifman result which extends the MacDowell–Specker theorem:

PROPOSITION 1.2. *For any M there is a family F of continuum many elementary end-extensions of M which are pairwise non-isomorphic. If M is countable, then each member of F is also.*

1D. In §§ 2 and 3 we will use some arithmetization of syntax. If φ is a formula of L, $\bar{\varphi}$ denotes the numeral for the code number of φ. If A is a set of formulas, the set of code numbers of these formulas is called the *code set for A*. We use a "dot" notation similar to that of Feferman (cf. [5], reference 6) to indicate recursive function symbols which are the code analogs of syntactic operations (thus $\dot{\neg}\bar{\varphi}$ is a term with value $\overline{\neg\varphi}$, $\bar{\varphi}\dot{\rightarrow}\bar{\psi}$ has value $\overline{\varphi\rightarrow\psi}$, $\bar{\varphi}(\dot{\bar{x}})$ has value as the code of the result of substituting in φ the $(x+1)$-st numeral for free occurrences of an understood variable). Also, corners ($\ulcorner$, $\urcorner$), placed around the description of a syntactic property, will indicate the code analog. We let $\mathrm{prf}_P(y, x)$ be a recursive proof predicate for: y $\ulcorner$is a proof in P of$\urcorner$ x. We also assume that this definition is natural so that the usual intensional derivability conditions hold (cf. [6]), we will use various consequences of these. For each $n \in \omega$ we will have a truth definition $\mathrm{tr}_n(x)$ whose properties are noted below.

PROPOSITION 1.3. (i) (*Gödel lemma of self reference*) *given $\varphi(x)$ with only x free, obtain ψ such that $P \vdash \psi \leftrightarrow \varphi(\bar{\psi})$*; (ii) *$P \vdash \varphi$ iff for some $m \in \omega$ $P \vdash \mathrm{prf}_P(\bar{m}, \bar{\varphi})$*; (iii) *$P \vdash \mathrm{prf}_P(y, x) \rightarrow x<y$*; (iv) *for each n, tr_n is a Δ_{n+1} symbol such that for any B_n-formula $\varphi(x)$ with only x free, $P \vdash \varphi(x) \leftrightarrow \mathrm{tr}_n(\bar{\varphi}(\dot{\bar{x}}))$*; and (v) *for each formula ψ, and each $m \in \omega$, $P \vdash \exists y \exists z(y<\bar{m} \wedge \mathrm{tr}_n(z) \wedge \mathrm{prf}_P(y, z \dot{\rightarrow} \bar{\psi})) \rightarrow \psi$.*

Remarks. We also use generalizations of (i). The $\exists_n$ subclass of B_n has an $\exists_n$ truth definition. This leads to tr_n being Δ_{n+1}, which is the best possible because of (i). If T extending P is recursively enumerable, then (v) holds with P replaced by T, for suitably chosen prf_T (follows from (ii), (iii), (iv), and the derivability conditions).

1E. P may be regarded as a set theory in which the axiom of infinity is negated: certain numbers may be regarded as sets (e.g. the canonical sequence codes for increasing sequences), and we take ε to be the recursive symbol for set membership. We let power (x) be the recursive symbol whose value is the set of all subsets of x. A *standard set* in M is a subset $A \subset \omega$ such that for some $a \in M$, $A = \{n \in \omega |\ M \models \bar{n}\varepsilon a\}$. Here a is called a *representative* of A. (Note that A might not be definable in M, although it is a segment of a definable set). The family of all standard sets in M is called the *standard system* of M (denoted $\mathrm{SSy}(M)$). The following theorem of Friedman [2] will be important for some of our results:

PROPOSITION 1.4. *Let M_1 and M_2 be countable. Then $M_1 \subset M_2$ iff* (i): $\mathrm{Th}(M_1)$ *admits embedding in* $\mathrm{Th}(M_2)$, *and* (ii) $\mathrm{SSy}(M_1) \subset \mathrm{SSy}(M_2)$. *Furthermore, M_1 is isomorphic to an initial segment of M_2 iff* (i) *and* (iii) $\mathrm{SSy}(M_1) = \mathrm{SSy}(M_2)$.

1F: In § 4 we consider reducts of arithmetic. If K is a set (finite here) of symbols definable in P, then L_K is the language (sublanguage of L) determined by the symbols in K. $\mathrm{red}_K P$ denotes $P \cap L_K$. Similarly for a model M, $\mathrm{red}_K M$ is the structure with the universe of M, but only the functions and predicates of K (determined by their interpretations in M). We say that M_1 and M_2 are isomorphic under K ($M_1 \simeq_K M_2$) exactly if $\mathrm{red}_K M_1 \simeq \mathrm{red}_K M_2$. An *$n$-type under K* means an n-type in L_K consistent with $\mathrm{red}_K P$.

PROPOSITION 1.5. *Both* $\mathrm{red}_{\{+\}} P$ *and* $\mathrm{red}_{\{\cdot\}} P$ *are complete.*

The first result is due to Presberger and the second to Mal'sev (cf. [1] references 60 and 50). We also note a consequence of the Keisler–Shelah theorem (cf. [8]):

PROPOSITION 1.6. *For M_1 and M_2 models (countable for our purposes), there are ultrapowers $M_1^* = M_1^I/U$ and $M_2^* = M_2^I/U$ such that: if* $\mathrm{Th}(\mathrm{red}_K M_1) = \mathrm{Th}(\mathrm{red}_K M_2)$, *then M_1^* and M_2^* are isomorphic under K.*

§ 2 The complete theories of submodels of arithmetic. In this section we will find that each non-standard model of P has very many submodels and these submodels satisfy very many distinct complete extensions of P. We also obtain similar results about end-extensions. We first note some useful results about standard systems.

LEMMA 2.1. *Assume every model here is non-standard.* (i) *If* $A \in \mathrm{SSy}(M)$ *then A has an arbitrarily small representative in the non-standard part of M*; (ii) *if* $M_1 \subset M_2$ *then* $\mathrm{SSy}(M_1) \subset \mathrm{SSy}(M_2)$ *and if M_1 and M_2 have isomorphic initial non-standard segments then* $\mathrm{SSy}(M_1) = \mathrm{SSy}(M_2)$; (iii) *if $\varphi(x)$ is a formula which may include parameters from M, then* $A = \{n \in \omega \mid M \models \varphi(\bar{x})\}$ *is in the standard system of M*; (iv) $\mathrm{SSy}(M)$ *is closed under usual recursion*; (v) *if M_T is the prime model of a complete extension T of P and* $A \in \mathrm{SSy}(M)$, *then* $A = \{n \in \omega \mid T \vdash \varphi(\bar{n})\}$ *for some formula φ of L with one free variable and no parameters.*

Proof. (i) Let a be a representative for A and let $b \in M - N$ be arbitrary. Choose c such that power (power(c))$<b$. Let $d = a \cap \mathrm{power}(c)$. Then d is a representative of A and $d<b$. (ii) Both parts follow from (i) and the absoluteness of the recursive symbol ε under submodel embedding. (iii) Any set definable and bounded in M behaves like a "finite" set in M. (iv) Clearly $\mathrm{SSy}(M)$ is closed under stronger operations which are peculiar to M, but relative recursion depends only on standard part. (v) Let a be a representative of some $A \in \mathrm{SSy}(M_T)$. Since M_T is prime, a is some Skolem constant $\mu z \psi(z)$. So $A = \{n \in \omega \mid T \vdash \bar{n} \in \mu z \psi(z)\}$. ■

Our first observation concerning the diversity of submodels, of a given non-standard model, is an easy consequence of theorems of Gaifman and Friedman:

PROPOSITION 2.2. *Each countable $M \neq N$ has a family F of initial segments which realize $2^{\aleph_0}$ distinct isomorphism types but such that for each $M_1 \in F$,* $\mathrm{Th}(M_1) = \mathrm{Th}(M)$.

Proof. By Proposition 1.2, there is a family F of $2^{\aleph_0}$ elementary end-extensions of M (each countable) which are pairwise non-isomorphic. If $M_1 \in F$, then $\mathrm{SSy}(M_1) = \mathrm{SSy}(M)$ by Lemma 2.1 (ii). But $\mathrm{Th}(M_1) = \mathrm{Th}(M)$ so the other condition of Proposition 1.4 (ii) is satisfied and thus M_1 is embeddible as an initial segment of M. ■

Note also that this argument indicates that M has a proper initial segment isomorphic to itself (in fact at least $\aleph_0$ such segments).

LEMMA 2.3. *Let M be given and let T be a complete extension of P with* $T \neq \mathrm{Th}(N)$. *Then the prime model M_T of T is a submodel of M iff* (i) *T admits embedding in* $\mathrm{Th}(M)$ *and* (ii) *for each $n \in \omega$ the code set of $T \cap B_n$ is in* $\mathrm{SSy}(M)$.

Proof. We will suppose that M is countable (otherwise take a countable elementary submodel of M preserving the code sets of the $T \cap B_n$). We use Proposition 1.4 (i) (the Friedman theorem).

Necessity. We need only to show that the code set for each $T \cap B_n$ is in $\mathrm{SSy}(M_T)$. But this code set is just $\{m \in \omega \mid T \vdash \mathrm{tr}_n(m)\}$ by Proposition 1.3 (i) and such a set $\in \mathrm{SSy}(M_T)$ by Lemma 2.1 (iii).

Sufficiency. We need to show that $\mathrm{SSy}(M_T) \subset \mathrm{SSy}(M)$. By Lemma 2.1 (v), if $A \in \mathrm{SSy}(M_T)$ then $A = \{m \in \omega \mid T \vdash \varphi(\bar{m})\}$ where φ has no parameters. Let $\varphi \in B_n$. Then the set $\{\bar{\varphi}(\bar{\bar{m}}) \mid m \in \omega\}$ is a recursive subset of the code set for B_n. So $C = \{k \in \omega \mid k = \bar{\varphi}(\bar{\bar{m}})$ where $T \vdash \varphi(\bar{m})\}$ is recursive in the code set of $T \cap B_n$ and thus $C \in \mathrm{SSy}(M)$ by Lemma 2.1 (iv). Similarly $A \in \mathrm{SSy}(M)$ by recursively decoding B. ■

The characterization of the above lemma will enable us to show that each non-standard model M has submodels satisfying very many distinct complete extensions of P. Briefly, the idea will be to "define" these complete extensions in M. The next lemma shows how we may represent in M, complete extensions with certain syntactic properties.

LEMMA 2.4. *Let $M \neq N$ be given, along with a set of sentences A whose code set is in* $\mathrm{SSy}(M)$, *and a sentence β such that $P \cup A$ not $\vdash \neg\beta$. Then there is some C such that:* (i) *C is a complete extension of $P \cup A \cup \{\beta\}$, and* (ii) *the code set for C is in* $\mathrm{SSy}(M)$. (*However, we will usually ask only that C satisfy* (i) *up to some B_n.*)

Proof. For each element $b \in M$ we will form the propositional closure of the theorems of $P \cup A \cup \{\beta\}$ with proof codes $<b$. For this purpose regard sentences as atomic propositions and theorems as propositional combinations of sub-sentences. Let a be a representative in M for the code set of A, and define

$$\mathrm{prop}(b, z) \leftrightarrow (z \ulcorner\text{is a propositional consequence of}\urcorner \{w \mid \exists x \exists y (y<b \wedge x \varepsilon a \wedge \mathrm{prf}_P(y, x \mathbin{\dot\wedge} \beta \dot\to w))\}).$$

Let $\varphi(v, b)$ be

$$\exists s \big(v \varepsilon s \wedge \forall u \big(u \varepsilon s \leftrightarrow (u \leqslant v \wedge u \ulcorner\text{is a sentence}\urcorner \wedge \neg \mathrm{prop}(b, \textstyle\bigwedge\!\!\bigwedge s_{<u} \dot\to \dot\neg u))\big)\big),$$

where $s_{<u}$ is a notation for the subset of s of elements $<$ value u. Now we must select a suitable parameter b so that $\varphi(b, v)$ defines the code set of a desired completion of $P \cup A \cup \{\beta\}$. Consider $\psi(b)$ given by

$$\forall v\big((v \ulcorner\text{is a sentence}\urcorner \wedge v<b)\to(\varphi(v,b)\vee\varphi(\dot{\neg} v,b))\big).$$

The propositional closure of $\text{prop}(\bar{n},\ )$ along with the actual consistency of $P \cup A \cup \{\beta\}$ ensure that $M \models \psi(\bar{n})$ for each $n \in \omega$. But $M \neq N$ forbids us from defining the standard part of M, so we may fix some $b \in M-N$ such that $M \models \psi(b)$. For this b, let $C = \{\gamma|\ M \models \varphi(\bar{\gamma}, b)\}$. Then the code set of C is in $\text{SSy}(M)$ by Lemma 2.1(iii).

Checking that C satisfies condition (i) of the lemma is straightforward: note that "proofs" in M of length $<b$ include all standard proofs (and possibly more), so C will be a consistent extension of $P \cup A \cup \{\beta\}$. Completeness is automatic because $M \models \psi(b)$. ■

LEMMA 2.5. *For each $n \in \omega$ there is a $\forall_{n+1}$ sentence α_n such that for any set of sentences $A \subset B_n$ which is closed under conjunction, if $P \cup A$ is consistent, then neither $P \cup A \vdash \alpha_n$ nor $P \cup A \vdash \neg\alpha_n$.*

(The corresponding result holds if P is replaced everywhere by a consistent recursively enumerable extension of P.)

Proof. By the Gödel lemma of self-reference, obtain a $\forall_{n+1}$ sentence α_n equivalent to

$$\forall w \forall x\big((\text{tr}_n(x)\wedge \text{prf}_P(w, x\dot{\to}\bar{\alpha}_n))\to\exists y\exists z(y<w\wedge \text{tr}_n(z)\wedge \text{prf}_P(y, z\dot{\to}\dot{\neg}\bar{\alpha}_n))\big).$$

(For $n = 0$, we delete the tr_0 parts and just obtain the Rosser sentence). Suppose $P \cup A \vdash \alpha_n$ and such a proof has code m. Then noting Proposition 1.3(ii) and (iv), we get

$$P \cup A \vdash \exists x(\text{tr}_n(x)\wedge \text{prf}_P(\bar{m}, x\dot{\to}\bar{\alpha}_n))\wedge\alpha_n .$$

So

$$P \cup A \vdash \exists y\exists z(y<\bar{m}\wedge \text{tr}_n(x)\wedge \text{prf}_P(y, z\dot{\to}\dot{\neg}\bar{\alpha}_n)).$$

But then $P \cup A \vdash \neg\alpha_n$ by Proposition 1.3(v). Then proof is similar if we assume $P \cup A \vdash \neg\alpha_n$. ■

We identify the *binary tree of length* ω (denoted Λ) with finite words on $\{0, 1\}$ If s is such a word, then $s0$ is the left successor of s, $s1$ is its right successor, and I_s is the set of initial subwords of s. Next we consider Λ labeled with sentences of L:

COROLLARY 2.6. *There is a labeled tree $\Lambda_n = \{\alpha_n^s|\ s \in \Lambda\}$ where:* (i) *α_n^0 is α_n as in Lemma* 2.5, (ii) *each α_n^{s1} is $\neg\alpha_n^{s0}$, and* (iii) *each α_n^{s0} satisfies the properties of α_n given in Lemma* 2.5, *but with $P \cup A \cup \{\alpha_n^r|\ r \in I_s\}$ replacing $P \cup A$. Furthermore, the labeling function f: $\Lambda \to$ code set of Λ_n (defined $f(s) = \overline{\alpha_n^s}$) is recursive.*

Producing the α_n^s is a simple modification of Lemma 2.5. We call Λ_n the *B_n-independence tree.*

DEFINITION 2A. T is called a *complete B_n extension of P* (C^nEP) provided $T \subset B_n$ sentences, T is complete for sentences in B_n, and T extends $P \cap B_n$. Let $\text{Th}^n(M)$ denote the C^nEP satisfied by M, and let $\text{Th}^n S(M)$ denote $\{\text{Th}^n(M_1)|\ M_1$ ranges over submodels of $M\}$.

LEMMA 2.7. *Let $M \neq N$ be given. For $n \geq 1$, let T be a C^nEP whose code set is in $\text{SSy}(M)$. Then there is a family F of $\aleph_0$ distinct C^{n+1}EP's which all extend T and all of whose code sets are in $\text{SSy}(M)$.*

Proof. Let $X \subset \Lambda$ be a set of $\aleph_0$ words which are pairwise incomparable in Λ. For $s \in X$, let β^s be $\bigwedge\!\!\bigwedge\{\alpha_n^r|\ r \in I_s\}$. Corollary 2.6(iii) tells us that $P \cup T$ not$\vdash \neg\beta^s$. So we may apply Lemma 2.4, taking T for A, β^s for β, and obtaining C from which we form T_s as $C \cap B_{n+1}$. Then let $F = \{T_s|\ s \in X\}$. Distinct words r and s in X will yield distinct C^{n+1}EP's noting Corollary 2.6(ii). ■

THEOREM 1. *If $M \neq N$, then there are $2^{\aleph_0}$ distinct complete extensions of P satisfied by submodels of M.*

Proof. Let T_1 be a C^1EP such that T_1 admits embedding in $\text{Th}(M)$, and the code set of T_1 is in $\text{SSy}(M)$. There exists such, namely $\text{Th}^1(M)$ by Lemma 2.3. Using Lemma 2.7 we form a tree Γ of length ω such that: (i) the nodes at height n are C^nEP's whose code sets are in $\text{SSy}(M)$; (ii) the base node is T_1; and (iii) each node T' has $\aleph_0$ distinct immediate successors in Γ all of which extend T'. Clearly the union of each branch of Γ yields a distinct complete extension T of P. And T satisfies the conditions of Lemma 2.3, so $M_T \subset M$. ■

COROLLARY 2.8. *For $M \neq N$, the initial segments of M satisfy $2^{\aleph_0}$ distinct complete extensions of P.*

Remarks. This result follows immediately from Theorem 1 and Proposition 1.1. We can obtain a stronger result: if M has a countable non-standard initial segment, then there is a family F of $2^{\aleph_0}$ initial segments of M where distinct members of F satisfy distinct complete extensions of P and each member of F is the *prime* model of its respective theory. The idea is to build the theories leading to F so that the standard system of M is coded in each theory (use Lemma 3.1 below).

We next note that elementary equivalence of models is not "local":

THEOREM 2. *Given a countable $M_1 \neq N$, there exists an M_2 such that M_1 and M_2 have identical sets of initial segments, but M_1 and M_2 are not elementarily equivalent. (In fact M_2 may be chosen to refute any sentence which is not a logical consequence of $P \cup \text{Th}^1(M_1)$.)*

Proof. Beginning with $\text{Th}^1(M_1)$, simply choose a branch of the tree Γ (of the proof of Theorem 1) which determines a complete extension T of P different from $\text{Th}(M_1)$. It follows from Lemmas 2.3 and 2.7 that $M_T \subset M_1$. Let M_2 be the initial segment of M_1 determined by M_T. So $\text{Th}(M_2) = T$ by Proposition 1.1 and $\text{SSy}(M_1) = \text{SSy}(M_2)$ by Lemma 2.1(ii). Using this and noting that $\text{Th}^1(M_1) = \text{Th}^1(M_2)$, Proposition 1.4 gives us that M_1 is itself isomorphic to an initial segment of M_2. The conclusion is immediate. ■

Notice that each C^1EP has $2^{\aleph_0}$ distinct C^1EP's $(n>1)$ extending it because each B_n independence tree has continuum many branches. We next check that these extensions cannot all be realized by submodels of a given model with a countable standard system. (Noting Lemma 2.3, they could all be realized by submodels of a suitable model of power continuum.) The full strength of Theorem 1 requires sentences of arbitrary complexity:

COROLLARY 2.9. *If M has a countable non-standard initial segment, then* $|\mathrm{Th}^n S(M)| = \aleph_0$ *for* $n>1$.

Proof. Note that by the construction of the proof of Theorem 1, $|\mathrm{Th}^n S(M)| \geqslant \aleph_0$ for $n>1$. But by Lemma 2.3, the family of code sets of $\mathrm{Th}^n S(M)$ is a subset of $\mathrm{SSy}(M)$. And $\mathrm{SSy}(M)$ is countable noting the hypothesis on M and Lemma 2.1(i). ■

We next consider end-extensions of a given model M:

COROLLARY 2.10. *If M is countable, then there are countable end-extensions of M which satisfy $2^{\aleph_0}$ distinct complete extensions of P.*

Proof. Note from the proofs of Theorem 1 and Corollary 2.8 that there exists a family F of initial segments of M which satisfy $\mathrm{Th}^1(M)$ *and* $2^{\aleph_0}$ distinct extensions of P. By Lemma 2.1(ii) and Proposition 1.4(ii), M can be embedded as an initial segment of each member of F. ■

The end-extensions of the above result all satisfy $\mathrm{Th}^1(M)$. In the next section we will see that this is sometimes but not always necessary. The following result characterizes what theories can be satisfied by end-extensions of M:

THEOREM 3. *Let $M_1 \neq N$ be countable and let T be a complete extension of P. Then there is an end-extension M_2 of M_1 such that* $\mathrm{Th}(M_2) = T$ *iff* (i) $\mathrm{Th}(M_1)$ *admits embedding in T, and* (ii) *for each $n \in \omega$, the code set of $T \cap B_n$ is in the standard system of M.*

Proof. Necessity. (i) is obvious; and for (ii), the code set of each $T \cap B_n$ is in $\mathrm{SSy}(M_2)$ by Lemma 2.3, and $\mathrm{SSy}(M_1) = \mathrm{SSy}(M_2)$ by Lemma 2.1(ii).

Sufficiency. By induction, we produce a sequence of restricted n-types, each of which is consistent with T *and* is "defined" in M_1: Let $\{A_1, A_2, \ldots\}$ be an enumeration of $\mathrm{SSy}(M_1)$ and let $\{v_1, v_2, \ldots\}$ be a sequence of variables. At the kth step we use the method of Lemma 2.4 to define in M_1 a k-type t_k such that: t_k is restricted to formulas in B_k, t_k extends $T \cap B_k$ *and* each t_j $(j<k)$, and $\bar{n}\, \varepsilon\, v_k$ is in t_k iff $n \in A_k$. Now let M_2 be the smallest elementary extension of M_T in which a sequence of elements $\{a_1, a_2, \ldots\}$ realize the n-types in $\bigcup \{t_k\}_{k \in w}$. Clearly $\mathrm{SSy}(M_1) \subset \mathrm{SSy}(M_2)$. On the other hand, if $A \in \mathrm{SSy}(M_2)$, then A is some $\{n \in \omega |\ M \models \varphi(\bar{n})\}$ where the only parameters in φ come from $\{a_1, a_2, \ldots\}$. But then $A \in \mathrm{SSy}(M_1)$ by noting that $\{n \in \omega |\ M \models \varphi(\bar{n})\}$ can be retrieved from some t_k which was defined in M_1. So, M_1 can be embedded as an initial segment of M_2 by Proposition 1.4(ii). ■

By using Theorem 3 and Lemma 3.1 (below), it is easy to produce a complete extension T of P such that a given countable $M \neq N$ is a submodel of every model of T but is embeddible as an initial segment of none.

The description of the $<$-order of a countable $M \neq N$ given in § 1C might tempt us to ask about the topological properties of the set of those cuts in the η-factor which yield submodels (note that the set of such cuts is order isomorphic to the reals). However, the following minor technical result suggests that we should not expect such a set to have simple topological properties:

THEOREM 4. *For a countable $M \neq N$, let X be the set of reals which correspond to submodels of M (in the sense described above under a fixed isomorphism). Then X is nowhere dense (usual topology), and furthermore, if* $\mathrm{Th}^1(M) \neq \mathrm{Th}^1(N)$ *then X is not closed.*

Proof. To see that X is nowhere dense: notice that there are cuts in the η-factor between any $a \in M-N$ and $2a$, but any submodel containing a must also contain $2a$. For the second result, let a be the sequence code for the first solution of a diophantine system which is solvable in M, but not N. Let A be the intersection of the set of initial segments of M which contain a and are themselves models. A cannot be a model for otherwise it would have proper initial segments which were also models containing a (by Proposition 2.2). ■

§ 3 Existential sentences and recursive numbers. How do the relative positions of (initial segment) submodels in the $<$-order of M depend on the complete theories satisfied by the submodels? Lemma 2.3 implies that the relative positioning of two submodels is restricted only by the $\exists_1$ parts of their theories. However, this restriction is rigid due to the presence of recursively defined elements whose definitions are absolute under submodel embeddings. In this section we compare the $\exists_1$ parts of the complete extensions of P. This inquiry leads to information concerning the distribution of recursively defined numbers in the non-standard part of models.

We first note two useful lemmas concerning how a subset of ω may be realized in, or omitted from, the standard system of the prime model of a complete extension of P. These are the natural questions, because given any $A \subset \omega$ and T, there is always a single type generated model M of T with $A \in \mathrm{SSy}(M)$.

LEMMA 3.1. *Given S a* $\mathrm{C}^n\mathrm{EP}$ *and $A \subset \omega$, there is a* $\mathrm{C}^{n+1}\mathrm{EP}$, *$T$ extending S such that A is a standard set of every model of $P \cup T$.*

Proof. Select a branch b of the B_n independence tree (cf. Corollary 2.6) as follows: if $\alpha_n^s \in b$ where s is a word of length k, then α_n^{s0} or $\alpha_n^{s1} \in b$ depending as $k \in A$ or $k \notin A$. Let T be a $\mathrm{C}^{n+1}\mathrm{EP}$ extending S and containing b. Then the code set for b (and hence A) is recursive in the code set for T. The conclusion follows by Lemmas 2.3 and 2.1(iv). ■

LEMMA 3.2. *Given $A \subset \omega$ is not recursive, there is a complete extension T of P such that A is not a standard set of the prime model of T (and T may be chosen to contain any fixed consistent r.e. extension of P).*

Proof. Enumerate the formulas with one free variable: $\varphi_0(x), \varphi_1(x), \ldots$ Let $T_0 = P$. Assume T_n is defined and is a consistent recursively enumerable extension of P. Now if $A_n = \{m|\ T_n \vdash \varphi_n(\overline{m})\}$ is the complement of $B_n = \{m|\ T_n \vdash \neg\varphi_n(\overline{m})\}$ then these sets are recursive and so neither could be A. In this case let $T_{n+1} = T_n$. If $A_n \cup B_n \neq \omega$, fix $m \notin A_n \cup B_n$. Then if $m \in A$, set $T_{n+1} = T_n \cup \{\neg\varphi_n(\overline{m})\}$; otherwise set $T_{n+1} = T_n \cup \{\varphi_n(\overline{m})\}$. Let T be any complete extension of $\bigcup \{T_n|\ n \in \omega\}$. Then $A \notin \mathrm{SSy}(M_T)$ by Lemma 2.1(v). ■

A typical application of Lemma 3.1 in the next theorem shows that all the models of a certain theory can be "universal" for certain families of models.

DEFINITION 3A. A family F of models is called *embedding compatible* provided $\bigcup \{\exists_1\text{-sentences} \cap \mathrm{Th}(M)|\ M \in F\} \cup P$ is consistent.

THEOREM 5. *Let F be an embedding compatible countable family of countable models. Then there are extensions T of P such that every model in F is a submodel of every model of T.*

Proof. Let S be a C^1EP containing $\bigcup \{\exists_1\text{-sentences} \cap \mathrm{Th}(M)|\ M \in F\}$, and let A be a subset of ω from which all of the standard sets of all of the models in F may be recovered recursively. For S and A, choose T as indicated by Lemma 3.1, and assume T is extended to include P. The conclusion follows by Lemma 2.1(iv) and Proposition 1.4(i). ■

Remarks. Such a universal model can be obtained for any embedding compatible family if we go to high enough cardinality. But the results of the last section (e.g. Corollary 2.9) imply that Theorem 5 would not hold, in general, for uncountable F. We now turn to the natural question raised by Theorem 5: which families of models (or C^1EP's) are embedding compatible?

THEOREM 6. *Let E denote the ordered system of all* C^1EP's *under the order "admits embedding in".* (i) *E is a tree with $2^{\aleph_0}$ branches, each of which has a maximal element*; (ii) $\mathrm{Th}^1(N)$ *is the only member of E which is comparable with every other member*; *and* (iii) *there are branches of E which are not well-ordered.*

Remarks. These results only suggest the structure of E. Special techniques might be needed in analyzing the structure of E because one is dealing with complete rather than recursive extensions of P. We suggest that the branches of E split very frequently and that each branch is order isomorphic to a closed segment of the real line. However, it is possible that not all branches are isomorphic (branches have different properties: for some, the maximal element has a Δ_2 definition). (ii) is significant because an analogous result is false for incomplete extensions of P (Kreisel has shown that any M is embedding compatible with some model of $P \cup \{\neg\mathrm{Consistent}_P\}$; cf. [6]). The proof of the theorem will be given following a lemma.

LEMMA 3.3. *Let γ be an existential sentence which is independent in P. Then there are existential sentences α and β such that $P \vdash \neg\alpha \vee \neg\beta$ and $P \cup \{\gamma\} \vdash \alpha \vee \beta$ but $P \cup \{\gamma\}$ not $\vdash \neg\alpha$ and $P \cup \{\gamma\}$ not $\vdash \neg\beta$. (Thus there are sentences which are Δ_1 in $P \cup \{\gamma\}$ but not decided in $P \cup \{\gamma\}$.)*

Proof. Let prf_S be the proof predicate $\mathrm{prf}_P(y, \bar{\gamma}\dot{\to}\bar{\gamma}x)$ for $P \cup \{\gamma\}$. We use a "dual" form of the Gödel lemma of self-reference to obtain:

$$\alpha \leftrightarrow \exists y\Big(\big(\mathrm{prf}_P(y, \bar{\gamma}) \vee \mathrm{prf}_S(y, \overline{\neg\alpha})\big) \wedge \forall z\big(z<y \to \neg\mathrm{prf}_S(z, \overline{\neg\beta})\big)\Big),$$

$$\beta \leftrightarrow \exists z\Big(\mathrm{prf}_S(z, \overline{\neg\beta}) \wedge \forall y\big(y \leqslant z \to (\neg\mathrm{prf}_P(y, \bar{\gamma}) \wedge \neg\mathrm{prf}_S(y, \overline{\neg\alpha}))\big)\Big).$$

It is routine to verify that α and β have the desired properties, using the fact that every true existential sentence is provable in P and the particular formal analog of this: $P \vdash \gamma \to \exists y\, \mathrm{prf}_P(y, \bar{\gamma})$. ■

Proof of Theorem 6. (i) Suppose T_1, T_2, and T_3 are C^1EP's such that each of T_1 and T_2 admits embedding in T_3. Let M_1, M_2, and M_3 be models of these C^1EP's respectively, where we may assume by Proposition 1.1 that each of M_1 and M_2 is an initial segment of M_3. So one of M_1 and M_2 must be an initial segment of the other, and correspondingly one of T_1 and T_2 admits embedding in the other. Thus E is a tree. Next we form, inductively, a binary, length ω tree Ω which is labeled with $\exists_1$-sentences such that the conjunction of the predecessors of any node is independent in P: fix the root node as an independent γ_0; if a node is labeled with ψ, label the right and left successors with α and β as given in Lemma 3.3 where we take γ as $\bigwedge \{\psi$ and its predecessors$\}$. Notice that C^1EP's which extend distinct branches of Ω must be incomparable in E. The maximal member of a branch B of E will be the C^1EP determined by $\bigcup \{\exists_1\text{-sentences} \cap T|\ T \in B\}$ (and the negations of all other $\exists_1$-sentences: verify consistency by a compactness argument).

(ii) If T is a C^1EP different from $\mathrm{Th}^1(N)$, then T contains an $\exists_1$-sentence γ independent in P. Apply Lemma 3.3 obtaining α and β. As T is a C^1EP it must contain either α or else β. Assuming T contains α, any C^1EP containing β will be incomparable with T.

(iii) Will follow from Theorem 8 (below) which shows branches for which every non-trivial initial segment is not well-ordered. ■

DEFINITION 3B. We call an element $a \in M$ a *recursive number* if there are formulas $\alpha \in \forall_1$ and $\beta \in \exists_1$ such that $P \vdash \alpha(x) \leftrightarrow \beta(x)$ and $M \models a = \mu x \beta(x)$.

We note the following simple facts about recursive numbers: (i) their Δ_0 definitions are preserved under submodel embeddings; (ii) they are closed under provably total recursive functions (and so with (i), form a "rigid" substructure, although a larger class might have this property); (iii) all standard numbers are recursive numbers, so we will mean non-standard recursive numbers where it makes sense; and (iv) a model M contains non-standard recursive numbers iff $\mathrm{Th}^1(M) \neq \mathrm{Th}^1(N)$. It is worth noting that if $\mathrm{Th}^1(M) \neq \mathrm{Th}^1(N)$, then the initial segment of M determined by the recursive numbers cannot be a model of P by (i), (ii) and Proposition 2.2.

DEFINITION 3C. Let TE denote the code set for the set of $\exists_1$-sentences which are true (in N). We say a theory T *requires* RTE (representation of true existentials) in case TE is a standard set in every non-standard model of $T \cup P$.

LEMMA 3.4. *For $M \neq N$, the recursive numbers are bounded below in $M-N$ iff* TE *is a standard set in M.*

Proof. Necessity. Let $a \in M-N$ be such a lower bound.

$$A = \{n \in \omega \mid M \models \exists y(y<a \wedge \mathrm{prf}_P(y, \bar{n}))\} \in \mathrm{SSy}(M)$$

by Lemma 2.1 (iii). Also A is the code set for the theorems of P, for otherwise there would be non-standard recursive numbers $<a$. So TE may be recovered from A recursively (assuming ω-consistency of P).

Sufficiency. The code set for $\mathrm{Th}^1(N)$ is recursive in TE and thus in $\mathrm{SSy}(M)$. So by Lemma 2.4 there is a complete extension $T \neq \mathrm{Th}(N)$ of P which extends $\mathrm{Th}^1(N)$ and whose code set is in $\mathrm{SSy}(M)$. Then by Lemma 2.3, $M_T \subset M$. Any element $a \in M_T - N$ must be a lower bound on non-standard recursive numbers in M because $\mathrm{Th}^1(M_T) = \mathrm{Th}^1(N)$. ∎

DEFINITION 3D. The *recursive index* of a number n is defined informally as the first number m which: (i) codes formulas α and β as in Definition 3B, (ii) also codes the proof of $\alpha(x) \leftrightarrow \beta(x)$, and (iii) satisfies $n = \mu x \beta(x)$. We also formalize this as a defined function symbol, $\mathrm{index}(x)$ where condition (iii) may be expressed properly using the truth definition tr_1 and noting Proposition 1.3 (iv).

We note that: (i) $\mathrm{index}(x)$ is a provably injective and total function (hint: "formulas" α and β may have non-standard codes); and (ii) $M \models \mathrm{index}(a) = \bar{n}$ for some $n \in \omega$ iff a is a recursive number.

LEMMA 3.5. *If $M \models a+\bar{n}<b$ for all $n \in \omega$, then there are non-recursive numbers $c \in M$ between a and b.*

Proof. If all of the numbers between a and b are recursive, then the set

$$A = \{y \mid \exists x(a<x<b \wedge y<\mathrm{index}(x))\}$$

is contained in N (by (ii) above) and definable in M. But there are infinitely many numbers between a and b, so the injectivity of $\mathrm{index}(x)$ makes $A = N$. But this violates the undefinability of N in M. ∎

The analogous result holds for the class of numbers with Δ_n definitions.

THEOREM 7. (i) *The recursive numbers are bounded above in every $M \neq N$;* (ii) *there are extensions T of P such that for every model M of T, the recursive numbers are bounded below in $M-N$;* (iii) *there are extensions T of P with some models $M \neq N$ such that the recursive numbers are not bounded below in $M-N$;* and (iv) *every maximal segment of non-standard recursive numbers has order type $\omega^* + \omega$.*

Proof. (i) follows immediately from Proposition 2.2 because each $M \neq N$ has a proper initial non-standard segment containing all of the recursive numbers. Both (ii) and (iii) follow from the characterization of Lemma 3.4 along with Lemmas 3.1 and 3.2 which allow us to obtain both T which require RTE and T which do not require RTE. For (iv), Lemma 3.5 tells us that a segment of recursive numbers cannot have elements which are infinitely distant, but also, recursive numbers are closed under $+1$ and -1. ∎

LEMMA 3.6. *Let T be a* $\mathrm{C^1EP}$ *which does not require* RTE *and let φ be an $\exists_1$-sentence in $T - \mathrm{Th}(N)$. Then there is an $\exists_1$-sentence $\psi \in T - \mathrm{Th}(N)$ such that $P \cup (T \cap \forall_1)$ not $\vdash \psi \to \varphi$.*

Proof. By Lemma 3.4, let $M \neq N$ be a model of $T \cup P$ in which the recursive numbers have no lower bound in $M-N$ (so clearly there is some φ as hypothesized). We define in M a function similar to the recursive index (notation: here z will be the sequence code $\langle z_0, z_1, z_2 \rangle$ while z_0 will be a recursive index and $\dot{\exists} z_0$ will $\exists y \beta(y)$ where β is the $\exists_1$-formula of definition 3B; let a be a representative in M for the code set for $\mathrm{Th}(M) \cap \forall_1$):

$$\varphi\,\mathrm{rank}(x) = \mu z\big(z_0 = \mathrm{index}(x) \wedge z_1 \varepsilon a \wedge \mathrm{prf}_P(z_2, z_1 \wedge \dot{\exists} z_0 \to \bar{\varphi})\big).$$

Now $\varphi\,\mathrm{rank}(x)$ assumes a value in N exactly if x is a recursive number *and* $P \cup (T \cap \forall_1) \vdash \exists y \beta(y) \to \varphi$ where $\beta(y)$ is the $\exists_1$-definition of x. Thus for $n \in \omega$, $\mathrm{rank}(n) \in M-N$ because P not $\vdash \gamma \to \varphi$ for any $\forall_1$-sentence γ unless $P \vdash \neg\gamma$. Now let

$$A = \{y \mid \forall x(x<y \to x<\varphi\,\mathrm{rank}(x))\}.$$

So clearly $N \subset A$. But if no ψ exists as asserted in the conclusion then $N = A$ (because there are arbitrarily small recursive numbers in $M-N$; consider the graph of $\varphi\,\mathrm{rank}(x)$). ∎

THEOREM 8. *There are complete extensions of P for which every model M has the properties: M has non-standard recursive numbers, and if A is an initial segment of M containing such a number, there are submodels included in A which satisfy $\aleph_0$ distinct* $\mathrm{C^1EP}$*'s.*

Proof. A suitable complete extension T needs only extend a $\mathrm{C^1EP}$ which does not require RTE, and such exists by Lemma 3.2. Given that M is a model of T with an initial segment submodel M_1 containing a non-standard recursive number, we will exhibit $M_2 \subset M_1$ such that $\mathrm{Th}^1(M_1) \neq \mathrm{Th}^1(M_2) \neq \mathrm{Th}^1(N)$. Clearly there is an $\exists_1$-sentence $\varphi \in \mathrm{Th}^1(M_1) - \mathrm{Th}(N)$. By Lemma 3.6. obtain ψ (using $T \cap B_1$ and φ). Now apply Lemma 2.4 taking $T \cap \forall_1$ for A and $\psi \wedge \neg\varphi$ for β thus obtaining a complete extension C of $P \cup (T \cap \forall_1) \cup \{\psi \wedge \neg\varphi\}$ such that the code set of C is in $\mathrm{SSy}(M)$. By Lemma 2.3 and Proposition 1.1 we may assume C has a model M_2 which is an initial segment of M. Clearly $M_2 \models \psi \wedge \neg\varphi$ forces the other conditions. ∎

Exactly as in Corollary 2.9, $\aleph_0$ is an upper bound in Theorem 8 for countable models. Are the only possibilities for $|\mathrm{Th}^1 S(M)|$ for countable models either $\aleph_0$ or 1? A similar question holds for the number of $\mathrm{C^1EP}$'s satisfied by end-extensions of a countable model (note that Theorem 6 (i) implies that this can be exactly 1). Either new techniques or a refinement of those of Lemma 3.6 might be needed for these questions. Corresponding to each of the $\mathrm{C^1EP}$'s in Theorem 8, there is a very large "hole" in the recursive numbers of M (i.e. the proof of Theorem 1 implies there are $2^{\aleph_0}$ cuts in M yielding submodels which contain all of the recursive numbers required by a given C^1EP, but prior to any recursive numbers rejected by the same C^1EP). If we regard a "large hole" as a gap in M which cannot be bridged by certain recursive functions, then the only "large holes" correspond to C^1EP's with code sets in $\mathrm{SSy}(M)$ (otherwise we should be able to define other C^1EP's in M).

§ 4. **Reducts of arithmetic.** In this section we consider certain "interesting" reducts K of P. We discover a fixed relationship between the isomorphism types of a model under different reducts of this sort. However, we also show that such reduct isomorphism type has almost no bearing on the elementary type of the full model. We refer to each of $\{+\}$, $\{\cdot\}$, and $\{|\}$ (divisibility) as familiar examples.

Definition 4A. We say that a reduct K is *rich* provided that for each $n \in \omega$ there is a formula $\varphi_n(v)$ of L_K such that whenever M is a model and $A \subset \omega$ we have: $A \in \mathrm{SSy}(M)$ iff there is some $a \in M$ such that $M \models \varphi_n(a)$ exactly when $n \in A$. We let Φ_A denote $\{\varphi_n(v)|\ n \in A\} \cup \{\neg\varphi_n(v)|\ n \notin A\}$, so the last condition is just that *a realizes* Φ_A. If in addition, the φ_n are effectively given and are all in some Δ_m, we call K *uniformly rich*.

Lemma 4.1. *Each of the reducts* $\{+\}$, $\{|\}$, *and* $\{\cdot\}$ *are uniformly rich.*

Proof. In the language $L_{\{+\}}$ we can define $\bar{n}|v$ with an $\exists_1$ formula for each $n \in \omega$, separately. Let p_n denote the nth prime, and set $\varphi_n(v)$ as $\overline{p_n}|v$. If $A \in \mathrm{SSy}(M)$ we can clearly define an a which realizes Φ_A. Conversely, if $a \in M$ realizes Φ_A then $A = \{n \in \omega|\ M \models \overline{p_n}|a\}$ is a standard set by Lemma 2.1(iii). To see that $\{|\}$ is rich, notice that for each $n \in \omega$ there is an $\exists_2$-formula $\varphi_n(v)$ of $L_{\{|\}}$ which defines $\exists p$ (p is a prime $\wedge p^{\overline{n+1}}|v \wedge p^{\overline{n+2}}|v$). Showing that each resulting Φ_A has the desired properties is similar to the case for $\{+\}$. The definability of | from $\cdot$ implies that $\{\cdot\}$ is uniformly rich. ■

Lemma 4.2. *If K is a rich reduct (e.g. $\{+\}$, $\{\cdot\}$, or $\{|\}$) and T is a consistent extension of P, there are $2^{\aleph_0}$ 1-types under K which are consistent with T (and thus $2^{\aleph_0}$ isomorphism types under K amongst countable models of T).*

Proof. Since any $A \in \omega$ can be in $\mathrm{SSy}(M)$ for some model M of T, Φ_A is consistent with T. And 1-types under K extending distinct Φ_A's are distinct. ■

Corollary 4.3. *The $2^{\aleph_0}$ countable isomorphism types under $\{+\}$ cannot all be embedded in a countable model.*

Proof. Notice that if the $\{+\}$-isomorphism type of a model M_1 is embedded in M_2, the embedding must be the identity on the standard part. Thus if Φ_A is as given in the proof of Lemma 4.1, and a realizes Φ_A in M_1, it will realize Φ_A in M_2. So if Φ_A are realized for $2^{\aleph_0}$ distinct A in $\{+\}$-isomorphism types embedded in M_2, Lemma 4.2 implies that $|\mathrm{SSy}(M_2)| \geqslant 2$ which implies $|M_2| \geqslant 2^{\aleph_0}$ (by Lemma 2.1 (i)). ■

We do not consider whether an analogous result might hold for $\{\cdot\}$ or $\{|\}$, except to note that the argument used for $\{+\}$ would not seem to work without requiring some additional preservation properties for the embeddings (e.g. preserves primes).

When does an isomorphism of models under one rich reduct imply the existence of an isomorphism (perhaps different) under another? Clearly definability implies that an isomorphism under $\{\cdot\}$ is also an isomorphism under $\{|\}$. Furthermore, since multiplication preserves sums of prime exponents, an isomorphism under $\{\cdot\}$ implies the existence of some isomorphism under $\{+\}$. We had guessed that the corresponding implications would fail for the other combinations from $\{+\}$, $\{\cdot\}$, and $\{|\}$. Thus the strong result of Theorem 9 (below) was unexpected.

Definition 4B. We say that a reduct K is *strongly complete* in P if there is a Δ_1 formula $\gamma(x)$ of P such that for each formula $\varphi(x_1, \ldots, x_j)$ of L_K,

$$P \vdash \gamma(\bar{\varphi}(\dot{\bar{x}}_1, \ldots, \dot{\bar{x}}_j)) \leftrightarrow \varphi(x_1, \ldots, x_j).$$

Lemma 4.4. *Each of the reducts $\{+\}$, $\{\cdot\}$, and $\{|\}$, is strongly complete in P.*

Proof. By Proposition 1.5 each of $\mathrm{red}_{\{+\}}P$ and $\mathrm{red}_{\{\cdot\}}P$ is complete, and in each case the proof of this was by elimination of quantifiers. The formalization of this recursive procedure is easily (but tediously) duplicated in P, thus yielding in each case a truth definition $\gamma(x)$ as required in Definition 4B. The definability of | from $\cdot$ extends the result to $\{|\}$. ■

Lemma 4.5. *Let M_1 and M_2 be countable non-standard models and let K be a strongly complete reduct of P. If $\mathrm{SSy}(M_1) = \mathrm{SSy}(M_2)$ then M_1 and M_2 are isomorphic under K. If K is rich, then conversely, an isomorphism under K implies* $\mathrm{SSy}(M_1) = \mathrm{SSy}(M_2)$.

Proof. The converse follows directly from the definition of K being rich. Assume $\mathrm{SSy}(M_1) = \mathrm{SSy}(M_2)$. We build the K-isomorphism by a back-and-forth argument. Let f be a mapping from a finite set $\{a_1, \ldots, a_j\} \subset M_1$ onto a set $\{c_1, \ldots, c_j\} \subset M_2$ (where $f(a_i) = c_i$). We say f is *K-good* provided $M_1 \models \varphi(a_1, \ldots, a_j)$ iff $M_2 \models \varphi(c_1, \ldots, c_j)$ for every formula φ of L_K. Clearly the empty mapping is K-good because K is complete in P. Let $b_0, b_1, \ldots$ and $d_0, d_1, \ldots$ be fixed enumerations of M_1 and M_2, respectively. Suppose at the $2n$th stage that we have a K-good mapping f as indicated above, and that b_n is not among the a_i's. For γ as in Definition 4B, consider the standard set (of M_1) $E = \{q \in \omega|\ M_1 \models \gamma(\bar{q}(\dot{\bar{a}}_1, \ldots, \dot{\bar{a}}_j, \dot{\bar{b}}_n))\}$. Since $\mathrm{SSy}(M_1) = \mathrm{SSy}(M_2)$, let $e \in M_2$ be a representative of E. We claim: for each fixed $m \in \omega$,

$$M_2 \models \exists x \forall y < \bar{m} \big(y \varepsilon e \leftrightarrow \gamma(y(\dot{\bar{c}}_1, \ldots, \dot{\bar{c}}_j, \dot{\bar{x}}))\big).$$

To see this, consider the set A of formulas of K with code numbers $< m$ and having, at most, the first $j+1$ variables free. Now let $\varphi(x_1, \ldots, x_j, x)$ be the conjunction of those formulas of A whose codes are in E, along with the negations of formulas of A whose codes are not in E. Clearly, $M_1 \models \exists x \varphi(a_1, \ldots, a_j, x)$ because γ is a truth definition, and so $M_2 \models \exists x\varphi(c_1, \ldots, c_j, x)$ because f is K-good. And if $M_2 \models \varphi(c_1, \ldots, c_j, d)$, then the property of γ gives us

$$M_2 \models \forall y<\dot{\overline{m}}\big(y\varepsilon e \leftrightarrow \gamma(y(\dot{c}_1, \ldots, \dot{c}_j, \dot{d}))\big),$$

and so the claim is verified. But since we cannot define N in M_2 there is some $g \in M_2 - N$ and $d \in M_2$ such that

$$M_2 \models \forall y<g\big(y\varepsilon e \leftrightarrow \gamma(y(\dot{c}_1, \ldots, \dot{c}_j, \dot{d}))\big).$$

Now extend f to f' by adding $f'(b_n) = d$. For large enough $y<g$, $\bigwedge\bigwedge_{i\leq j} x_j \neq x$ has code $<y$ so $d \neq$ any c_i. The choice of $g \in M_2 - N$ and the property of γ ensure that f' is K-good. At the $(2n+1)$-st stage of the construction, the argument is symmetric extending the range of f' to $d_n \in M_2$. ■

Lemma 4.5 is easily generalized to other applications, once we notice that the argument does not require K to be strongly complete, but only that there be a truth definition γ and that the two models are L_K-elementarily equivalent. Thus for example, two non-ω-models of second order number theory with countable bases have isomorphic bases exactly if they are first-order equivalent and the same subsets of ω are represented in each, using the full second-order language.

THEOREM 9. *Let M_1 and M_2 be countable models, and let K_1 and K_2 be rich, strongly complete reducts (e.g. $\{+\}$, $\{\cdot\}$, or $\{|\}$). If M_1 and M_2 are isomorphic under K_1, then they are isomorphic under K_2.*

Proof. We may dispose of the case in which one of the $M_i = N$. This is because, if K is rich then N is not isomorphic to any non-standard model under K (note the definition of rich and also that N is the only model whose standard sets are all finite). If both M_1 and M_2 are non-standard, the result follows directly from Lemma 4.5.

COROLLARY 4.6. *If $M \neq N$ is countable, and K is a strongly complete reduct, then both non-standard initial segment submodels, and countable end-extensions, of M are isomorphic to M under K.*

Proof. Both non-standard initial segments and end-extensions have the same standard systems as M by Lemma 2.1(ii). So the result follows by Lemma 4.5. ■

DEFINITION 4C. Let Γ_1 and Γ_2 be isomorphism types for reducts K_1 and K_2, respectively. We say that Γ_1 and Γ_2 are *compatible* if some model M has Γ_1 and Γ_2 as its isomorphism types under K_1 and K_2, respectively. And we call the isomorphism type of M under $K_1 \cup K_2$ an *amalgamation* of Γ_1 and Γ_2.

Pairs of reducts become interesting when we can define the full language L from them, in which case an amalgamation is just a model. We consider $\{+\}$ and $\{\cdot\}$ as an example. Each has $2^{\aleph_0}$ countable isomorphism types (Lemma 4.2) and compatibility is a one-one correspondence between these two families (Theorem 9). But furthermore:

COROLLARY 4.7. *Let Γ_1 and Γ_2 be compatible, non-standard, countable isomorphism types under $\{+\}$ and $\{\cdot\}$, respectively. The family of amalgamations of Γ_1 and Γ_2 satisfy $2^{\aleph_0}$ distinct complete extensions of P.*

Proof. Let M be a countable amalgamation of Γ_1 and Γ_2. The non-standard initial segment submodels of M satisfy $2^{\aleph_0}$ distinct complete extensions of P, by Corollary 2.8. But each of these is an amalgamation of Γ_1 and Γ_2 by Corollary 4.6. ■

If Γ is a non-standard, countable isomorphism type for some rich reduct K, Lemma 3.1 implies that there are complete extensions T of P such that no model of T has K-isomorphism type Γ. However this does not effect questions of independence:

THEOREM 10. *Let K_1 and K_2 be strongly complete reducts from which full arithmetic, L, can be defined (e.g. $\{+\}$ and $\{\cdot\}$). Let Γ_1 and Γ_2 be compatible, non-standard, countable isomorphism types under K_1 and K_2, respectively. Let φ be any sentence independent in P. Then there are amalgamations M_1 and M_2, K_1 and K_2 such that $M_1 \models \varphi$ and $M_2 \models \neg\varphi$.*

Proof. Let M_1 be an amalgamation of Γ_1 and Γ_2. Assume $M_1 \models \varphi$ since $M_1 \neq N$ we may apply Lemma 2.4 with A = empty set, $B = \neg\varphi$, and thus obtain a complete extension C of P ($C \vdash \neg\varphi$) whose code set is in $\mathrm{SSy}(M_1)$. Now let $\{A_1, A_2, \ldots\}$ be an enumeration of $\mathrm{SSy}(M_1)$ and let $\{v_1, v_2, \ldots\}$ be a sequence of variables. At the kth step use Lemma 2.4 to define in M_1 a k-type t_k (in the full language L) which extends C and also each t_j $(j<k)$, and such that $\bar{n}\varepsilon v_k$ is in t_k iff $n \in A_k$. Now let M_2 be the smallest elementary extension of the prime model M_c of C in which the types t_k are realized. So $\mathrm{SSy}(M_1) = \mathrm{SSy}(M_2)$ (exactly as in the proof of Theorem 3). Thus M_2 is an amalgamation of Γ_1 and Γ_2 by Lemma 4.5. And $M_2 \models \neg\varphi$ because $C \vdash \neg\varphi$. ■

The argument of Lemma 4.5 does not seem to be simply adaptable to uncountable models. Thus analogues of Theorem 9 and Corollary 4.6 for uncountable cases might fail. However, the authors believe that such failures would *not* be related significantly to the elementary types of models: the above results indicate that elementary type has only a small effect on structure in the countable case, and we would expect even less effect when we go up in power. We ignore the uncountable situation except for the following technical result which also applies to theories in general:

THEOREM 11. *Let $\{K_i\}_{i\in I}$ be any (finite) set of reducts such that each K_i is complete in P. Let T_1 and T_2 be any complete extensions of P and let $\aleph$ be any infinite cardinal. Then there are models M_1 and M_2 of T_1 and T_2, respectively, such that each has cardinality $\aleph$ and such that M_1 and M_2 are isomorphic under each K_i separately.*

Proof. Let M_1^* and M_2^* be Keisler–Shelah ultrapowers (Proposition 1.6) of models of T_1 and T_2, respectively, chosen so that $|M_1^*| = |M_2^*| \geq \aleph$. Since each K_i is complete in P, there is an isomorphism f_i of M_1^* and M_2^* under K_i for each i. Let $A_0 \subset M_1^*$ have cardinality $\aleph$. At the $2n$th stage let C_n be the closure under Skolem functions (in M_2^*) of $\bigcup_{i \in I} f_i[A_n]$. At the $(2n+1)$-st stage let A_{n+1} be the closure under Skolem functions (in M_1^*) of $\bigcup_{i \in I} f_i^{-1}[C_n]$. Let $M_1 = \bigcup_{n \in \omega} A_n$ and $M_2 = \bigcup_{n \in \omega} C_n$. Clearly M_1 and M_2 satisfy the conclusions of the theorem. ∎

If K_1 and K_2 are rich, strongly complete reducts (e.g. $\{+\}$ and $\{\cdot\}$) and Γ_1 is a countable K_1-isomorphism type, is there any reasonable way to determine its compatible K_2 mate (unique by Theorem 9)? In particular, if Γ_1 has a simple description (e.g. arithmetical) must Γ_2 also be simple? The answer will generally be "yes". For this we will use certain recursion theoretic notions as follows: If Γ is a countable isomorphism type, we will say Γ is Δ_2, *arithmetical*, etc., if there is a Δ_2, arithmetical, etc. coding of Γ in N. If M is a countable model we will correspondingly designate the *full diagram* of M as — if $\{\varphi|\ M \models \varphi$ with parameters$\}$ can be — coded in N. If S is a countable family of countable sets, then a *universal set* A for S is a set from which each member of S may be recursively recovered.

Lemma 4.8. *Let A be a universal set for the standard system S of some countable non-standard model. Then there is a model M such that $\mathrm{SSy}(M) = S$ and the full diagram of M is Δ_2 in A.*

Proof. Let $c_1, c_2, \ldots$ be a recursive listing of all of the Skolem set constants of P; let $A_1, A_2, \ldots$ be a listing of S given by A; and let $\sigma_1, \sigma_2, \ldots$ be a recursive listing of the sentences of L. Let $T_0 = P$. We proceed by induction, assuming that T_m produced at earlier stages are consistent. At the $(3n+1)$-st stage let

$$T_{3n+1} = \begin{cases} T_{3n} & \text{if } T_{3n} \vdash \neg\sigma_n, \\ T_{3n} \cup \{\sigma_n\} & \text{otherwise.} \end{cases}$$

At the $(3n+2)$-nd stage let

$$T_{3n+2} = T_{3n+1} \cup \{\overline{m}\,\varepsilon\, c_i|\ m \in A_n\} \cup \{\overline{m} \text{ not } \varepsilon\, c_i|\ m \notin A_n\}$$

where i is smallest integer for which this is consistent (such c_i exists by Lemma 3.1). At the $(3n+3)$-rd stage let

$$T_{3n+3} = T_{3n+2} \cup \{\overline{m}\,\varepsilon\, c_n|\ m \in A_i\} \cup \{\overline{m} \text{ not } \varepsilon\, c_n|\ m \notin A_i\}$$

where i is the smallest integer for which this is consistent. We claim that such an A_i must exist: The sets

$$C_1 = \{m \in \omega|\ T_{3n+2} \vdash \overline{m}\,\varepsilon\, c_n\} \quad \text{and} \quad C_2 = \{m \in \omega|\ T_{3n+2} \vdash \overline{m} \text{ not } \varepsilon\, c_n\}$$

are disjoint sets which are each r.e. in the A_j's previously used in the construction. Because S is the standard system of a non-standard model it must include a separating set $A_i \supset C_1$ and disjoint from C_2 (in a model with standard system S we formally define a set of elements which are put in C_1 before they are put in C_2; then A_i

is the standard initial segment of this set). Let T be $\bigcup T_m$. T is a complete extension of P which is Δ_2 in A. Furthermore, the prime model M_T of T has $\mathrm{SSy}(M_T) = S$ by construction. ∎

Theorem 12. (i) *K_1 and K_2 are rich, strongly complete reducts, K_1 uniformly rich, and Γ_1 a countable K_1-isomorphism type. The unique K_2-isomorphism type compatible with Γ_1 is arithmetical in Γ_1.* (ii) *Neither $\{+\}$ nor $\{\cdot, |\}$ can have recursive isomorphism types in non-standard models.*

Proof. (i) The standard system S associated with Γ_1 can be universally coded in a set A which is Δ_m in Γ_1 where m is as given in Definition 4A of uniformly rich. So by Lemma 4.8 there is a model M whose full diagram is Δ_{m+1} in Γ_1, and such that M has K_1-isomorphism type Γ_1 by Lemma 4.5. The K_2-isomorphism type of M is recursive in the full diagram of M. For (ii) first notice that the standard system of a non-standard model always includes a non-recursive set A (e.g. a separation of certain recursively inseparable sets). For a countable non-standard $\{+\}$ isomorphism type Γ, if we are given the element 1, then $\overline{n}$ is recursive in Γ, as is the nth prime, p_n, while $\overline{p_n}|a$ is r.e. in Γ. Let a and b be elements which realize Φ_A and $\Phi_{\neg A}$ respectively (the φ_n's and Φ_A as in the proof of Lemma 4.1). Then A is recursive in Γ, by checking for either $\overline{p_n}|a$ or $\overline{p_n}|b$. For a $\{\cdot, |\}$ isomorphism type Γ, let a and b be elements which realize Φ_A and $\Phi_{\neg A}$ as given in the proof of Lemma 4.1. Let c be an element such that a prime $p|c$ if $p|a$ or $p|b$, but $p^2|c$ for any prime p. Then $n \in A$ iff

$$\exists x(x|c \wedge x^{\overline{n+1}}|a \wedge x^{\overline{n+2}}|a) \quad \text{iff} \quad \forall x(x|c \wedge x^{\overline{n+1}}|b \rightarrow x^{\overline{n+2}}|b).$$

So A is recursive in Γ. (Using c avoids having to know the primes). ∎

The results of Theorem 12 are only illustrative of the problems concerning connection between isomorphism types of different reducts. Perhaps a general method can be devised to comprehend (ii) for a certain class of reducts. We note that the method of (i) also implies that any finite set of symbols define in P has an isomorphism type compatible with Γ_1 and Δ_{m+1} in Γ_1. Lemma 4.9 illustrates one way to amalgamate compatible reducts, but other sorts of amalgamations might be informative for other purposes. Of course, there are too many distinct amalgamations of a given reduct pair for them all to be arithmetical, analytic, etc.

§ 5. **Problems.** In this section we pose a number of problems, and in some cases give our related conjectures or hypotheses. A few of the questions might be answered by purely technical results. But more often we tried to phrase a general problem which yields many particular instances. We hope that investigations arising from some of these questions may be useful in furthering our understanding of the relation between model theory and practical proof theory for arithmetic. We have divided the problems into four groups, but the first two problems are intended to provide a theme underlying the rest.

Problem 1. How much of the structure of a model M do we need to know in order to determine: (a) the isomorphism type of M? (b) the elementary type of M? or, (c) whether M satisfies a given sentence σ?

As a simple example, we can omit finite (and certain infinite) sets of elements and still retrieve the isomorphism type of M. However, we have intended that by "part" of a structure, several different measures might be used. Examples where we have answered this question (negatively) are where we have taken "part" to be the set of initial segments (Theorem 2), and also the isomorphism types under each of + and · (Theorem 10). What about other measures? (cf. Problems 3 and 8).

PROBLEM 2. (a) Is there any structural property of a model M which will determine the elementary type of M, without determining its isomorphism type? (b) Replace "elementary type of M" in (a) by "whether M satisfies a given sentence σ."

We conjecture that for "useful" structural properties, the answer to (a) is no (evidence provided in the above theorems). The situation for (b) is more hopeful. For example, a large enough initial substructure will suffice for an $\exists_1$-sentence. However, we hope that even more useful properties based on various combinations of reducts might be given for certain σ (cf. Problem 12).

The next group of six problems deals with structures under the full language L of arithmetic.

PROBLEM 3. If M_1 and M_2 have final segments which are isomorphic, what properties of M_1 are preserved in M_2?

Ask the same question with "final segments" replaced by "closed segments in the non-standard part", or replaced by other structural properties.

PROBLEM 4. What is the theory of the non-standard parts of all models of P? (Is it decidable? arithmetical? analytic? note that it does not contain P).

Such a theory might be considered as the "eventual arithmetic" which is uneffected by the irregularities caused by small numbers. Also, what is the set C of sentences σ of L which satisfy: for every M, there is a cut in the $<$-order of M such that σ holds in the structure above the cut, and also in the structure below the cut? Is C consistent? closed under $\wedge$?

PROBLEM 5. Determine the properties of the fine structure of the set of cuts in the $<$-order of M which yield submodels.

Many such properties may be determined by refining the techniques of Propositions 1.4 and 2.2, Theorems 1,4, and 8, etc. But are any significant properties preserved in certain limits of cuts? Note that the intersection of some downward limits are not models of P (Theorem 4), while on-the-other-hand, every $M \neq N$ has a sequence of non-standard initial segment submodels whose intersection is N (one case follows from Theorem 8 and the other case by a refinement of Proposition 1.4).

PROBLEM 6. (Intuitively, are any "interesting" problems which may be given in number theory neither $\forall_1$ nor $\exists_1$?) Specifically, given a particular sentence σ, which is not apparently $\forall_1$ or $\exists_1$, can we prove: if σ is not decided by P, then σ is not implied by any consistent $\forall_1$ or $\exists_1$ sentence?

The hypothesis avoids a full independence result for σ. This question is intended to determine the potential seriousness of Theorem 2 for a particular problem such as the twin-prime problem. Because notice that many famous problems are $\forall_1$ and thus not subject to Theorem 2 (e.g. Fermat's Last Theorem, the four color problem, the Goldbach conjecture). Undoubtedly $\forall_1$ and $\exists_1$ questions are common because they are concrete, and correspondingly they could be subject to special techniques.

PROBLEM 7. Determine the structure of the embedding tree, E, of Theorem 6: (a) What are the isomorphism types of the branches? (b) What are the splitting points? (c) Does the following hold for any or all $\exists_1$-sentences φ which are independent in P: there is an $\exists_1$-sentence ψ such that whenever α is an $\forall_1$-sentence and $P \vdash (\alpha \wedge \psi) \to \varphi$, then $P \vdash \varphi \to \neg\alpha$?

It would be most interesting, and possibly of proof theoretic significance, if E were *not* homogeneous (homogeneous: if there is an automorphism of E exchanging any two elements which are neither maximal nor minimal). However, there seem to be many possibilities for showing E to be homogeneous. For example, if the syntactic property (c) held for all φ, then no branch would be well-ordered (compare with Lemma 3.6 but note that ψ is chosen "uniformly"). Similar uniform syntactic properties correspond to other features of the structure of E. Even if these uniform properties fail, techniques analogous to those of Lemma 3.6 might apply. The points of E beyond which there are no splitting points correspond to "amalgamation bases" for models of P, while maximal elements of E correspond to "existentially closed" structures for P.

PROBLEM 8. What possibilities are there for the number of C^1EP's satisfied by: (a) submodels; (b) end-extensions, of a model $M \neq N$?

In either case it can be 1 (Theorem 6) and for submodels it can be $\aleph_0$ (Theorem 8). If each branch of the embedding tree of Theorem 6 is isomorphic to a closed segment of the real line, then for countable models, the only possibilities for (a) and (b) are 1 and $\aleph_0$ (proof using Lemma 2.4 in a fashion similar to Theorem 8).

The second group of four problems concern reducts of P.

PROBLEM 9. Characterize those reducts K of P for which the K-isomorphism type of every non-standard model is not recursive. Does any rich reduct have a recursive non-standard isomorphism type?

PROBLEM 10. Assume K is a reduct with $\leqslant \aleph_0$ isomorphism types amongst countable models of P. Can K have more than 2 isomorphism types amongst countable models? Ask the same question, but with the weaker assumption that there are $\leqslant \aleph_0$ n-types under K.

We suggest that in each case the answer might be no. One might attempt to prove such a result by arguing that any non-standard model must be saturated under K (the Vaught theorem concerning theories with countably many n-types might be useful here). If such a result were obtained, it would suggest why number theory has not led to any nice algebraic theories. And alternatively, any K which gave a positive answer to Problem 10 could be quite interesting.

PROBLEM 11. Exhibit a reduct of P which is complete in P, but not strongly complete.

We think that such reducts should exist, but they might not be easy to produce. For one thing, completeness could not be shown by an effective elimination of quantifiers. Secondly, completeness in P gives a formula γ of L such that

$$P \vdash \gamma(\bar{\varphi}(\dot{\bar{n}})) \leftrightarrow \varphi(n)$$

for each formula φ of L_K. So if K is not strongly complete some

$$\forall x\big(\gamma(\bar{\varphi}(\dot{\bar{x}})) \leftrightarrow \varphi(x)\big)$$

will be independent in P.

PROBLEM 12. (a) Characterize those reducts whose countable isomorphism types cannot all be embedded in a single countable isomorphism type. (b) When does the image of an embedding of a reduct K-isomorphism type in a model M_1 yield a submodel M_2?

The isomorphism types under $\{+\}$ could not be so embedded by Lemma 4.6. Although it is of separate interest, (b) is also related to (a). This is because under such embeddings a standard set associated with the embedded isomorphism type might also need to be realized in M_1.

The third group of problems deal with amalgamations of pairs of reducts. We believe that this topic could be quite promising as it would permit us to "switch relations" (i.e. select particular primitive symbols in our language) depending on the particular number theoretic question at hand.

DEFINITION 5A. We call a set of reducts $K_1, \ldots, K_n$, *full* if the full language L of arithmetic can be defined from the K_i.

PROBLEM 13. Give useful criteria for a reduct pair to be full.

Before we expect any criteria, more examples need to be given. Most of the currently available examples of interesting reduct pairs are due to Julia Robinson. These include $\{+1, |\}$ and $\{+, x^y\}$. Are $+$ and hyperexponentiation full? or are these two concepts "too far apart"?

PROBLEM 14. (a) Is there a pair (or set) of reducts which is full, but where each reduct is not rich? (b) The same question, but instead of "not rich", ask for a compatible set containing a recursive non-standard isomorphism type for each reduct.

Even more important than knowing what can be taken for full reduct pairs, would be to know how isomorphism types can be amalgamated. Ideally, for given isomorphism types for a full pair of reducts, we would hope for an interesting class of amalgamations which have useful descriptions. Different pairs of reducts might behave differently in this regard.

PROBLEM 15. Let K_1 and K_2 be a full pair of reducts. Are there models M_1 and M_2 with $M_1 \neq M_2$ such that M_2 is interpretable in M_1 where the interpretation is the identity for K_1?

There are non-trivial interpretations of models in other models based on formalizations of the completeness theorem. But here we are interested in useful interpretations where the symbols of interpreted model are kept relatively close to those of the interpreting model. For example, in Problem 15, think of K_1 and K_2 both rich and strongly complete. Then the problem is whether there are f_1 and f_2 which are K_1- and K_2-isomorphisms, respectively, f_i: $M_1 \to M_2$, such that $f_2^{-1} \circ f_1$ is definable in M_1 (so we can use this composition M_1 check what K_2 properties elements have in M_2).

DEFINITION 5B. Let K_1 and K_2 be a full pair of reducts and let $J = \{J_1, J_2\}$ give a class of formulas J_i of L_{K_i} (e.g. open formulas all formulas, etc.). An n, J-amalgamation of compatible isomorphism types Γ_1 and Γ_2 (under K_1 and K_2, respectively) is a *pair of n-types* $\{t_1, t_2\}$ under K_1 and K_2.

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