The Lefschetz fixed point theorem for multi-valued maps of non-metrizable spaces

by

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Abstract. Let $\varphi\colon X\to X$ be a multi-valued map from a Hausdorff space X into itself; φ is called *admissible* provided the following two conditions are satisfied:

(i) φ is upper semi-continuous and compact,

(ii) there exist a Hausdorff space Y and two continuous (single-valued) maps $p, q\colon Y\to X$ such that p is a Vietoris map and $q(p^{-1}(x))\subset\varphi(x)$ for each $x\in X$.

A Hausdorff space X is called a *Lefschetz multi-space* (denoted by $X\in\mathfrak{L}_M$) if for every such map φ and for every such pair (p, q) the Lefschetz number $\Lambda(q_* p_*^{-1})$ is defined and if $\Lambda(q_* p_*^{-1})\neq 0$ for some such pair (p, q), then φ has a fixed point.

In this paper we prove the following results:

(i) a Hausdorff space X which is α-dominated, for every $\alpha\in\mathrm{Cov}(X)$, by a Lefschetz multi-space Y is also a Lefschetz multi-space,

(ii) every open subset of an admissible space (cf., [2]), and in particular every open subset a locally convex topological space, is a Lefschetz multi-space,

(iii) every NES (compact) space (cf., [2]) is a Lefschetz multi-space.

It is known [3] that the Lefschetz fixed point theorem is true for admissible multi-valued maps of arbitrary metric ANR-spaces. In this note, being concerned with the extension of the above result to the non-metrizable case, we show that for the following types of spaces the Lefschetz fixed point theorem for admissible multi-valued maps is true:

(i) open subsets in admissible linear topological spaces (in the sense of Klee [4]) or, in particular, open subsets of locally convex topological spaces,

(ii) all NES (compact) spaces.

In the single-valued case these results were given by G. Fournier and A. Granas in [2].

1. Preliminaries. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers Q from the category of Hausdorff spaces and continuous maps to the category of graded vector spaces over Q and en iar maps of degree zero. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q-dimensional Čech homology group with compact carriers of X. For a continuous map $f\colon X\to Y$, $H(f)$ is the induced linear map $f_* = \{f_{*q}\}$, where $f_{*q}\colon H_q(X)\to H_q(Y)$.

A non-empty space X is called *acyclic* provided: (i) $H_q(X) = 0$ for all $q\geqslant 1$ and (ii) $H_0(X)\approx Q$.

A continuous map $p\colon Y\to X$ is called a *Vietoris map* provided (i) p is proper (i.e., the counter image $p^{-1}(C)$ of every compact subset $C\subset X$ is also compact) and (ii) the set $p^{-1}(x)$ is acyclic for every $x\in X$.

We observe that

(1.1) *If $p\colon Y\to X$ is a Vietoris map, then for every subset A of X the contraction $\tilde{p}$ of p to the pair $(p^{-1}(A), A)$ is also a Vietoris map.*

(1.2) VIETORIS MAPPING THEOREM. *If $p\colon Y\to X$ is a Vietoris map, then the induced map $p_*\colon H(Y)\to H(X)$ is an isomorphism.*

Theorem (1.2) clearly follows from the original statement of the Vietoris Mapping Theorem for compacta (e.g., [1]).

Let $f\colon E\to E$ be an endomorphism of an arbitrary vector space E. Let us put $N(f) = \{x\in E\colon f^n(x) = 0\}$ (f^n is the nth iterate of f) and $\tilde{E} = \dfrac{E}{N(f)}$. Since $f(N(f))\subset N(f)$, we have the induced endomorphism $\tilde{f}\colon \tilde{E}\to\tilde{E}$. Call f *admissible* provided $\dim\tilde{E}<\infty$. Let $f = \{f_q\}\colon E\to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. Call f a *Leray endomorphism* if (i) all f_q are admissible and (ii) almost all $\tilde{E}_q$ are trivial. For such f we define the (generalized) *Lefschetz number* $\Lambda(f)$ by putting

$$\Lambda(f) = \sum_q (-1)^q \operatorname{tr}(\tilde{f}_q).$$

The following important property of the Leray endomorphisms is a consequence of the well-known formula $\operatorname{tr}(u\circ v) = \operatorname{tr}(v\circ u)$ for trace:

(1.3) *Assume that in the category of graded vector spaces the following diagram commutes:*

$$\begin{array}{ccc} E' & \xrightarrow{\;u\;} & E'' \\ {\scriptstyle f'}\downarrow & \nwarrow^{v} & \uparrow{\scriptstyle f''} \\ E' & \xrightarrow[\;u\;]{} & E'' \end{array}$$

Then, if f' or f'' is a Leray endomorphism, then so is the other and in that case $\Lambda(f') = \Lambda(f'')$.

Let X be a Hausdorff space. A continuous map $f\colon X\to X$ is called *a Lefschetz map* provided $f_*\colon H(X)\to H(X)$ is a Leray endomorphism. For such f we define the *Lefschetz number* $\Lambda(f)$ of f by putting $\Lambda(f) = \Lambda(f_*)$. Clearly, if f and g are homotopic maps, $f\sim g$, then $\Lambda(f) = \Lambda(g)$.

2. Multi-valued maps. Let X and Z be two Hausdorff spaces and assume that for every point $x\in X$ a non-empty subset $\varphi(x)$ of Z is given; in this case we say that φ is a *multi-valued map* from X to Z and we write $\varphi\colon X\to Z$. In what follows the symbols φ, ψ, χ will be reserved for multi-valued maps; the single-valued maps will be denoted by $f, g, h, p, q, \ldots$ Let $\varphi\colon X\to Z$ be a multi-valued map. We associate with φ the diagram of continuous maps

$$X \xleftarrow{p_\varphi} \Gamma_\varphi \xrightarrow{q_\varphi} Z$$

in which

$$\Gamma_\varphi = \{(x, z)\in X\times Z\colon z\in\varphi(x)\}$$

is the *graph* of φ and the *natural projections* p_φ and q_φ are given by $p_\varphi(x, z) = x$ and $q_\varphi(x, z) = z$.

The point-to-set map φ extends to a set-to-set map by putting $\varphi(A) = \bigcup_{a\in A}\varphi(a)\subset Z$ for $A\subset X$; $\varphi(A)$ is said to be the *image* of A under φ. If $\varphi(A)\subset B\subset Z$, then the *contraction* of φ to the pair (A, B) is the multi-valued map $\tilde{\varphi}\colon A\to B$ defined by $\tilde{\varphi}(a) = \varphi(a)$ for each $a\in A$. The *counter-image* of a subset $B\subset Z$ under φ is $\varphi^{-1}(B) = \{x\in X\colon \varphi(x)\subset B\}$.

A multi-valued map $\varphi\colon X\to Z$ is called *upper semi continuous* (u.s.c.) provided (i) $\varphi(x)$ is compact for each $x\in X$ and (ii) for each open set $V\subset Z$ the counter-image $\varphi^{-1}(V)$ is an open subset of X.

The following fact is evident:

(2.1) PROPOSITION. *If $\varphi\colon X\to Z$ is a* u.s.c. *map and A a compact subset of X, then the image $\varphi(A)$ of A under φ is compact.*

A multi-valued map $\varphi\colon X\to Z$ is called *compact* provided the image $\varphi(X)$ of X under φ is contained in a compact subset of Z.

From (2.1) we deduce

(2.2) PROPOSITION. *Let $\varphi\colon X\to X_1$ and $\psi\colon X_1\to X_2$ be two* u.s.c. *maps. If φ or ψ is compact, then the composition $\psi\circ\varphi$ of φ and ψ is compact and* u.s.c.

Let $\varphi\colon X\to X$ be a multi-valued map. A point $x\in X$ is called a *fixed point* for φ whenever $x\in\varphi(x)$. Let $\varphi, \psi\colon X\to Z$ be two multi-valued maps. If $\varphi(x)\subset\psi(x)$ for each $x\in X$, then we say that φ is a *selector* of ψ and indicate this by $\varphi\subset\psi$.

Let $p\colon Y\to X$ be a single-valued map from the space Y onto X; we associate with such p the multi-valued map $\varphi_p\colon X\to Y$ given by $\varphi_p(x) = p^{-1}(x)$ for each $x\in X$.

We have the following

(2.3) PROPOSITION. *If $p\colon Y\to X$ is a continuous closed map such that the set $p^{-1}(x)$ is a non-empty compact set for each $x\in X$, then the multi-valued map $\varphi_p\colon X\to Y$ is* u.s.c.

3. Admissible maps. A u.s.c. compact map $\varphi\colon X\to Z$ is said to be *acyclic* provided the set $\varphi(x)$ is acyclic for every point $x\in X$. We observe that if $\varphi\colon X\to Z$

is an acyclic map, then the natural projection $p_\varphi\colon \Gamma_\varphi \to X$ is a Vietoris map. Using the Vietoris Mapping Theorem, we define the linear map

$$\varphi_*\colon H(X)\to H(Z)$$

by putting

$$\varphi_* = q_{\varphi *}\circ p_{\varphi *}^{-1};$$

φ_* is said to be induced by the multi-valued (acyclic) map φ. It is easily seen that if $\varphi = f$ (i.e., φ is single-valued), then $\varphi_* = f_*$.

Let $\varphi\colon X\to Z$ be a multi-valued map. A pair (p, q) of single-valued continuous maps of the form $X \overset{p}{\leftarrow} Y \overset{q}{\rightarrow} Z$ is called a *selected pair* of φ (written $(p, q)\subset\varphi$) if the following two conditions are satisfied:

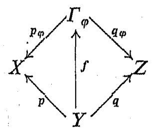
(i) p is a Vietoris map,

(ii) $q(p^{-1}(x))\subset\varphi(x)$ for each $x\in X$.

(3.1) Remark. We observe that if φ is a compact map and $(p, q)\subset\varphi$, then q is a compact map.

(3.2) Proposition. *If $\varphi\colon X\to Z$ is an acyclic map and $(p, q)\subset\varphi$ then $q_*p_*^{-1} = \varphi_*$.*

Proof. Let (p, q) be a selected pair of φ of the form $X \overset{p}{\leftarrow} Y \overset{q}{\rightarrow} Z$. Consider the commutative diagram



in which $f(y) = (p(y), q(y))$ for every $y\in Y$.

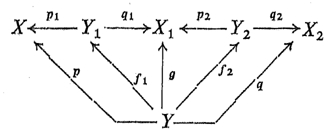
The condition $q(p^{-1}(x))\subset\varphi(x)$ implies that $(p(y), q(y))\in\Gamma_\varphi$. Applying to the above diagram the functor H, we deduce that $q_*p_*^{-1} = q_{\varphi *}p_{\varphi *}^{-1}$ and the proof is completed.

(3.3) Definition. An u.s.c. compact map $\varphi\colon X\to Z$ is called *admissible* provided there exists a selected pair (p, q) of φ.

Every acyclic map and, in particular, every continuous compact single-valued map is admissible. Then, for example, the pair (p_φ, q_φ) is a selected pair of such map φ.

(3.4) Theorem. *Let $\varphi\colon X\to X_1$ and $\psi\colon X_1\to X_2$ be two u.s.c. maps. Assume further that $(p_1, q_1)\subset\varphi$ and $(p_2, q_2)\subset\psi$. If φ or ψ is a compact map, then the composition $\psi\circ\varphi\colon X\to X_2$ of φ and ψ is an admissible map and there exists a selected pair (p, q) of $\psi\circ\varphi$ such that $q_{2*}p_{2*}^{-1}q_{1*}p_{1*}^{-1} = q_*p_*^{-1}$.*

Proof. From (2.2) we infer that $\psi\circ\varphi$ is u.s.c. and compact. Consider the diagram



in which $Y = \{(y_1, y_2)\in Y_1\times Y_2\colon q_1(y_1) = p_2(y_2)\}$; $p(y_1, y_2) = p_1(y_1)$, $q(y_1, y_2) = q_2(y_2)$, $f_1(y_1, y_2) = y_1$, $f_2(y_1, y_2) = y_2$, $g(y_1, y_2) = q_1(y_1)$ for each $(y_1, y_2)\in Y$.

Since $f_1^{-1}(y_1)$ is homeomorphic to $p_2^{-1}(q_1(y_1))$ and p_2 is a Vietoris map, we deduce that f_1 is a Vietoris map. Moreover, we have $q(p^{-1}(x))\subset(\psi\circ\varphi)(x)$ for each $x\in X$. Applying to the above diagram the functor H, we obtain $q_{2*}p_{2*}^{-1}q_{1*}p_{1*}^{-1} = q_*p_*^{-1}$. The proof of (3.4) is completed.

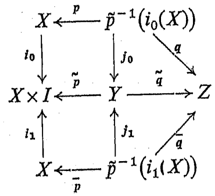
In particular, Theorem (3.4) implies that the composition of acyclic maps is admissible.

(3.5) Definition. Let $\varphi, \psi\colon X\to Z$ be two admissible maps. We say that φ and ψ are *homotopic* (written $\varphi\sim\psi$) provided there exists an admissible map $\chi\colon X\times I\to Z$, where $I = [0, 1]$, such that

$$\chi(x, 0)\subset\varphi(x) \quad\text{and}\quad \chi(x, 1)\subset\psi(x) \quad\text{for each } x\in X.$$

(3.6) Theorem. *Let $\varphi, \psi\colon X\to Z$ be two admissible maps. Then $\varphi\sim\psi$ implies that there exist selected pairs $(p, q)\subset\varphi$ and $(\bar p, \bar q)\subset\psi$ such that $q_*p_*^{-1} = \bar q_*\bar p_*^{-1}$.*

Proof. Let $(\tilde p, \tilde q)$ be a selected pair of χ. Consider the following commutative diagram:



in which $i_0(x) = (x, 0)$, $i_1(x) = (x, 1)$ for each $x\in X$, j_0, j_1 are inclusions and $p, \bar p$ are given as the first coordinate of $p(y)$ for every $y\in\tilde p^{-1}(i_0(X))$ or $y\in\tilde p^{-1}(i_1(X))$ respectively. Then $p, \bar p$ are Vietoris maps and we have $(p, q)\subset\varphi$, $(\bar p, \bar q)\subset\psi$. We observe that $i_{0*} = i_{1*}$ is a linear isomorphism. This and the commutativity of the above diagram imply $q_*p_*^{-1} = \bar q_*\bar p_*^{-1}$. This proves Theorem (3.6).

(3.7) Definition. An admissible map $\varphi\colon X\to X$ is called a *Lefschetz map* provided for each selected pair $(p, q)\subset\varphi$ the linear map $q_*p_*^{-1}\colon H(X)\to H(X)$ is a Leray endomorphism.

If $\varphi\colon X\to X$ is a Lefschetz map, then we define the *Lefschetz set* $\Lambda(\varphi)$ of φ by putting

$$\Lambda(\varphi) = \{\Lambda(q_*p_*^{-1})\colon (p, q)\subset\varphi\}.$$

Remark. If φ is an acyclic Lefschetz map or in particular $\varphi = f$ is a single-valued Lefschetz map, then from (3.2) we deduce that the set $\Lambda(\varphi)$ is a singleton and in this case we shall denote it by $\Lambda(\varphi)$.

The following two theorems are immediate consequences of (3.6) and the above remark.

(3.8) THEOREM. *Let $\varphi, \psi\colon X \to X$ be two admissible Lefschetz maps. Then* (i) *$\varphi \subset \psi$ implies $\Lambda(\varphi) \subset \Lambda(\psi)$,* (ii) *$\varphi \sim \psi$ implies $\Lambda(\varphi) \cap \Lambda(\psi) \neq \emptyset$.*

(3.9) THEOREM. *Let $\varphi, \psi\colon X \to X$ be two acyclic Lefschetz maps. If $\varphi \subset \psi$ or $\varphi \sim \psi$, then $\Lambda(\varphi) = \Lambda(\psi)$.*

4. Almost fixed points. Let X be a Hausdorff space and $Y \subset X$. By $\mathrm{Cov}_X(Y)$ we denote the directed set of all open coverings of Y in X. We let $\mathrm{Cov}_X(X) = \mathrm{Cov}(X)$. Let Z be a Hausdorff space and $\alpha \in \mathrm{Cov}(Z)$. Two multi-valued maps $\varphi, \psi\colon X \to Z$ are said to be *α-close* provided for each $x \in X$ there exists a member $U_x \in \alpha$ such that $\varphi(x) \cap U_x \neq \emptyset$ and $\varphi(x) \cap U_x \neq \emptyset$.

Let $\varphi\colon X \to X$ be a multi-valued map and $\alpha \in \mathrm{Cov}(X)$. A point $x \in X$ is said, to be an *α-fixed point* for φ provided there exists a member $U \in \alpha$ such that (i) $x \in U$ and (ii) $\varphi(x) \cap U \neq \emptyset$. Clearly, if $\alpha, \beta \in \mathrm{Cov}\, X$ and α refines β, then every α-fixed point for φ is also a β-fixed point for φ.

(4.1) LEMMA. *Let $\varphi\colon X \to X$ be a* u.s.c. *map. The statements are equivalent:*

(i) *φ has a fixed point,*

(ii) *there is a cofinal family of coverings $\mathscr{B} = \{\alpha\} \subset \mathrm{Cov}_X(Y)$ of $Y = \varphi(X)$ in X such that φ has an α-fixed point for every $\alpha \in \mathscr{B}$.*

Proof. It is evident that (i) implies (ii). Assume that φ has no fixed points. Then for each $x \in X$ there are open neighbourhoods V_x and $U_{\varphi(x)}$ of x and $\varphi(x)$ respectively such that $V_x \cap U_{\varphi(x)} = \emptyset$. From the fact that φ is u.s.c. we deduce that set $V = \varphi^{-1}(U_{\varphi(x)})$ is an open neighbourhood of x in X. Let $W_x = V_x \cap V$: then $\varphi(W_x) \subset U_{\varphi(x)}$ and $W_x \cap U_{\varphi(x)} = \emptyset$. Putting $\alpha = \{W_x\}_{x \in Y}$ we get a covering of Y in X such that φ has no α-fixed point. If β is a member of $\mathscr{B}$ that refines α, then φ has no β-fixed point and thus we obtain a contradiction of (ii). The proof of (4.1) is completed.

5. Lefschetz multi-spaces. A Hausdorff space X is called a *Lefschetz multi-space* (denoted by $X \in \mathscr{L}_M$) provided that every admissible map $\varphi\colon X \to X$ is a Lefschetz map and if $\Lambda(\varphi) \neq \{0\}$, then φ has a fixed point.

(5.1) THEOREM (cf. [3]). *If X is a metric* ANR, *then $X \in \mathscr{L}_M$.*

(5.2) THEOREM. *Let Y be a Hausdorff space and $X \in \mathscr{L}_M$. Assume that $p\colon Y \to X$ is a Vietoris, closed map and $q\colon Y \to X$ is a compact map. Then $q_* p_*^{-1}$ is a Leray endomorphism and if $\Lambda(q_* p_*^{-1}) \neq 0$ then there exists a point $y \in Y$ such that $p(y) = q(y)$.*

Theorem (5.2) clearly follows from (2.3). Moreover, from (5.2) we deduce

(5.3) COROLLARY. *Let Y, X, p, q be as in (5.2). If X is an acyclic space, then there exists a point $y \in Y$ such that $p(y) = q(y)$.*

In particular, for acyclic Lefschetz multi-spaces we have

(5.4) THEOREM. *Every acyclic Lefschetz multi-space has the fixed point property within the class of admissible maps.*

Finally, from (3.9) we deduce

(5.5) COROLLARY. *Let $X \in \mathscr{L}_M$ and assume that $\varphi, \psi\colon X \to X$ are two acyclic maps which satisfy one of the following conditions:*

(i) *φ is a selector of ψ,*

(ii) *φ and ψ are homotopic.*

Then $\Lambda(\varphi) = \Lambda(\psi)$.

Let X and Y be two Hausdorff spaces. We say that X is *r-dominated* by Y if there exist two maps, $s\colon X \to Y$ and $r\colon Y \to X$, such that $r \circ s = \mathrm{Id}_X$; let $\alpha \in \mathrm{Cov}\, X$; we say that X is *α-dominated* by Y provided there exist two maps, $s_\alpha\colon X \to Y$ and $r_\alpha\colon Y \to X$, such that $r_\alpha \circ s \underset{\alpha}{\sim} \mathrm{Id}_X$, i.e., there exists a single-valued homotopy h joining $r_\alpha \circ s_\alpha$ and Id_X such that for every $x \in X$ there is a $U_x \in \alpha$ such that $h(x, t) \in U_x$ for all $t \in I$. Clearly, if $f \underset{\alpha}{\sim} g$, then f and g are α-close.

By $\mathscr{R}(\mathscr{L}_M)$ we denote the class of all Hausdorff spaces which are r-dominated by a space in $\mathscr{L}_M$. We say that $X \in \mathscr{D}(\mathscr{L}_M)$, if for every $\alpha \in \mathrm{Cov}\, X$ there exists a space $Y_\alpha \in \mathscr{L}_M$ such that X is α-dominated by Y. Clearly, $\mathscr{L}_M \subset \mathscr{R}(\mathscr{L}_M) \subset \mathscr{D}(\mathscr{L}_M)$.

(5.6) THEOREM. $\mathscr{D}(\mathscr{L}_M) = \mathscr{R}(\mathscr{L}_M) = \mathscr{L}_M$.

Proof. It is sufficient to prove $\mathscr{D}(\mathscr{L}_M) \subset \mathscr{L}_M$. Let $X \in \mathscr{D}(\mathscr{L}_M)$ and $\varphi\colon X \to X$ be an admissible map. Consider a selected pair $(p, q) \subset \varphi$. We prove that $q_* p_*^{-1}$ is a Leray endomorphism. Let $\alpha \in \mathrm{Cov}(X)$. Then from the definition we obtain a space $Y_\alpha \in \mathscr{L}_M$ and two maps $s_\alpha\colon X \to Y_\alpha$ and $r_\alpha\colon Y_\alpha \to X$ such that $r_\alpha \circ s_\alpha \underset{\alpha}{\sim} \mathrm{Id}_X$. Define the map $\varphi_\alpha\colon Y_\alpha \to Y_\alpha$ by putting $\varphi_\alpha = s_\alpha \circ \varphi \circ r_\alpha$. Applying to φ_α Theorem (3.4), we obtain a selected pair $(p_\alpha, q_\alpha) \subset \varphi_\alpha$ such that $q_{\alpha*} p_{\alpha*}^{-1} = s_{\alpha*} q_* p_*^{-1} r_{\alpha*}$. Since $Y_\alpha \in \mathscr{L}_M$, we infer that φ_α is a Lefschetz map, and so $q_{\alpha*} p_{\alpha*}^{-1}$ is a Leray endomorphism. Since $r_\alpha \circ s_\alpha \sim \mathrm{Id}_X$, we infer that $r_{\alpha*} s_{\alpha*} = \mathrm{Id}_{H(X)}$, and so from (1.3) we conclude that $q_* p_*^{-1}$ is a Leray endomorphism and

$$\Lambda(q_* p_*^{-1}) = \Lambda(r_{\alpha*} s_{\alpha*} q_* p_*^{-1}) = \Lambda(s_{\alpha*} q_* p_*^{-1} r_{\alpha*}) = \Lambda(q_{\alpha*} p_{\alpha*}^{-1})$$

(i.e., φ is a Lefschetz map).

Assume that $\Lambda(\varphi) \neq \{0\}$; then there exists a selected pair $(p, q) \subset \varphi$ such that $\Lambda(q_* p_*^{-1}) \neq 0$. This implies ($Y_\alpha \in \mathscr{L}_M$) that φ_α has a fixed point. Since $r_\alpha \circ s_\alpha \underset{\alpha}{\sim} \mathrm{Id}_X$, we infer that φ has an α-fixed point for arbitrary $\alpha \in \mathrm{Cov}(X)$. Finally, from Lemma (4.1) we infer that φ has a fixed point and the proof is completed.

6. The Lefschetz fixed point theorem for admissible spaces. Let U be a neighbourhood of the origin in a linear topological space E. Then U is *shrinkable* provided for any $x \in \mathrm{cl}(U)$ and $0 < \lambda < 1$ the point $\lambda \cdot x$ lies in U. It is known (cf., Klee [4]) that shrinkable neighbourhoods form the base of E at 0. It follows that given an arbitrary neighbourhood W of 0 there is a shrinkable neighbourhood V of 0 such that $V+V\subset W$ and such that any interval $tx+(1-t)y$ $(0\leqslant t\leqslant 1)$ with x and y in V is entirely contained in W. From this, since the topological structure of E is determined by a base of the neighbourhoods of the origin, we deduce the following

(6.1) Lemma (cf., [2]). *Let U be an open subset of a linear topological space E. Then for each $\alpha\in \mathrm{Cov}(U)$ there exists a refinement $\beta\in\mathrm{Cov}(U)$ such that any two β-close maps of any space X into U are stationary α-homotopic, i.e., there exists an α-homotopy h such that $h_t(x)$ is constant $(0\leqslant t\leqslant 1)$ whenever $f(x)=g(x)$.*

(6.2) Definition. Let E be a linear topological space. We say (following Klee [4]) that E is *admissible* provided for any compact $K\subset E$ and any $\alpha\in\mathrm{Cov}_E(K)$ there is a map $\pi_\alpha\colon K\to E$ such that (i) $\pi_\alpha(K)$ is contained in a finite-dimensional subspace of E and (ii) the inclusion $i\colon K\to E$ and $\pi_\alpha\colon K\to E$ are α-close.

(6.3) Theorem. *Let E be an admissible linear topological space; then every open subset of E is a Lefschetz multi-space.*

Proof. Let $V\subset E$ be an open set and let $\varphi\colon V\to V$ be an admissible map. Denote by K a compact subset of V which contains $\varphi(V)$. Let $\mathscr{B}=\{\alpha\}$ be a cofinal family of coverings in $\mathrm{Cov}_V(K)$ such that each member of $\alpha\in\mathscr{B}$ is of the form $y+U$, where $y\in K$ and U is a shrinkable neighbourhood of the origin in E. Let $\alpha\in\mathscr{B}$, and take $\pi_\alpha\colon K\to E$ satisfying $\pi_\alpha(K)\subset E^n\subset E$ for some n and such that $i\colon K\to E$ and $\pi_\alpha\colon K\to E$ are α-close. Define the multi-valued map $\varphi_\alpha\colon V\cap E^n\to V\cap E^n$ by putting $\varphi_\alpha(x)=\pi_\alpha(\varphi(x))$. Let (p,q) be a selected pair of φ where $p,q\colon Y\to V$; define:

$$p'\colon p^{-1}(V\cap E^n)\to V\cap E^n,$$

$$q'\colon p^{-1}(V\cap E^n)\to K,$$

$$q''\colon Y\to K,$$

$$\pi'_\alpha\colon K\to V\cap E^n$$

as contractions of the respective maps. Since p' is a Vietoris map, $(p',\pi'_\alpha\circ q')$ is is a selected pair of φ_α, which is therefore admissible. Let $i\colon V\cap E^n\to V$, $i_K\colon K\to V$ and $j\colon p^{-1}(V\cap E^n)\to Y$ be the inclusions. In view of Lemma (6.1) we may assume without loss of generality that $i\circ\pi'_\alpha$ is homotopic to i_K, and hence that $i_{K*}=i_*\pi'_{\alpha*}$. We have $i\circ p'=p\circ j$, and so $p_*^{-1}i_*=j_*p_*'^{-1}$. Since $q=i_K\circ q''$ and $q''\circ j=q'$, we obtain

$$q_*p_*^{-1}=i_{K*}q''_*p_*^{-1}=i_*\pi'_{\alpha*}q''_*p_*^{-1}\quad\text{and}\quad \pi'_{\alpha*}q'_*p_*^{-1}i_*=\pi'_{\alpha*}q'_*j_*p_*'^{-1}=\pi'_{\alpha*}q'_*p_*'^{-1}.$$

So by (1.3) $q_*p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_*p_*^{-1})=\Lambda(q'_*p_*'^{-1})$. Hence φ is a Lefschetz map and $\Lambda(\varphi)\subset\Lambda(\varphi_\alpha)$.

Now $\Lambda(\varphi)\neq\{0\}$ implies $\Lambda(\varphi)\neq\{0\}$, and so by Theorem (5.1) φ_α has a fixed point and, since π_α and i are α-close, φ has an α-fixed point for every $\alpha\in\mathscr{B}$. Finally, from (4.1) we deduce that φ has a fixed point.

7. The Lefschetz fixed point theorem for neighbourhood extension spaces. A Hausdorff space X is a *neighbourhood extension space for compact spaces* (resp. *for compact metric spaces*) provided for any pair (Y,A), where Y is a compact Hausdorff space and A a closed subset of Y (resp. Y is a compact metric space and A a closed subset of Y), and any map $f_0\colon A\to X$ there is an extension $f\colon U\to X$ of f_0 over an open neighbourhood of A in Y. The class of the neighbourhood extension spaces for compact spaces (resp. for compact metric spaces) will be denoted by NES (compact) (resp. NES (compact metric)). Clearly, NES (compact $\subset$ NES (compact metric).

(7.1) Lemma (cf. [2]). *Every Tychonoff cube is a retract of a locally convex topological space.*

(7.2) Theorem. *Every open subset of a Tychonoff cube is a Lefschetz multi-space.*

Theorem (7.2) clearly follows from (5.6), (6.3) and (7.1).

(7.3) Theorem. NES (*compact*)$\subset\mathscr{L}_M$.

Proof. Let $X\in$ NES(compact) and let $\varphi\colon X\to X$ be an admissible map. Denote by K a compact set containing $\varphi(X)$. Embed K into a Tychonoff cube T and denote by $s\colon K\to\tilde{K}$ the homeomorphism of K onto $\tilde{K}\subset T$. Let $i\colon K\to X$ be the inclusion. Consider $i\circ s^{-1}\colon\tilde{K}\to X$; since $X\in$ NES (compact), there exists an open subset U of T containing $\tilde{K}$ and an extension $h\colon U\to X$ of $i\circ s^{-1}$ over U; thus if $i\colon\tilde{K}\to U$ is the inclusion, we have $h\circ j=i\circ s^{-1}$. Define $\psi\colon U\to U$ by putting $\psi=j\circ s\circ\tilde{\varphi}\circ h$, where $\tilde{\varphi}$ is the contraction of φ to the pair (X,K). Let (p,q) be a selected pair of φ, where $p,q\colon Y\to X$. Let q' be the contraction of q to the pair (Y,K). Then the pair (p,q') is a selected pair of admissible map $\tilde{\varphi}$. By Theorem (3.4) we infer that $\tilde{\varphi}\circ h$ is an admissible map and has a selected pair $(\bar{p},\bar{q})$ such that $\bar{q}_*\bar{p}_*^{-1}=q'_*p_*^{-1}h_*$. Since $(p,j\circ s\circ\bar{q})$ is a selected pair of ψ, we conclude that ψ is an admissible map and therefore, in view of (7.2), ψ is a Lefschetz map. We have $j_*s_*\bar{q}_*\bar{p}_*^{-1}=j_*s_*q_*p_*^{-1}h_*$ and $h_*j_*s_*q'_*p_*^{-1}=i_*s_*^{-1}s_*q'_*p_*^{-1}=i_*q'_*p_*^{-1}=q_*p_*^{-1}$, and so by (1.3) we conclude that $q_*p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_*p_*^{-1})=\Lambda(j_*s_*\bar{q}_*\bar{p}_*^{-1})$. Hence φ is a Lefschetz map and $\Lambda(\varphi)\subset\Lambda(\psi)$.

If $\Lambda(\varphi)\neq\{0\}$, then $\Lambda(\psi)\neq\{0\}$. So by (7.2) there exists a point $y\in U$ such such that $y\in\psi(y)=(j\circ s\circ\tilde{\varphi}\circ h)(y)$; hence $h(y)\in h(\psi(y))=(h\circ j\circ s\circ\tilde{\varphi}\circ h)(y)$ $=\varphi(h(y))$, i.e., $h(y)$ is a fixed point of φ.

(7.4) Theorem. *Let $X\in$ NES (compact, metric) and let $\varphi\colon X\to X$ be an admissible map such that $\varphi(X)$ is contained in a compact metrizable subset of X. Then φ is a Lefschetz map and $\Lambda(\varphi)\neq\{0\}$ implies that φ has a fixed point.*

The proof of (7.4) is strictly analogous to that of Theorem (7.3).

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Some problem in elementary arithmetics *

by

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Abstract. Three different questions concerning peano arithmetic P are considered. (1) How large can the set of theories of the submodels or end extensions of some fixed non-standard model of P be? (2) What are the properties of the partial ordering of embeddibility between complete extensions of P? (3) How is the isomorphism type of a model of P related to the isomorphism types of its reducts?

This paper is concerned with (complete) extensions of elementary arithmetic P. The bulk of the paper is contained in §§ 2, 3, 4 and each one of these sections is concerned with a separate idea.

Let M be a non-standard model of P, and let M' be a submodel or end extension of M. What can $\mathrm{Th}(M')$ be, and how well does this family of theories characterize M? These questions are considered in § 2.

In § 3 a partial ordering of complete extensions of P is introduced. ($T_1 \leqslant T_2$ if each model of T_1 is embeddable in a model of T_2.) This ordering is shown to be a tree, and several of its other properties are considered.

It is well known that for elementary arithmetic the similarity type of the language used is relatively unimportant. In § 4 a study is made of the relationship between the isomorphism type of a model of P and the isomorphism type of certain of its reducts.

The paper is completed by § 1, which contains the required preliminaries, and § 5, which contains a collection of open problems.

§ 1. Preliminaries.

1A. P denotes the theory of *elementary Peano arithmetic*. When technicalities arise we may assume that the *basic* language L for P is suitably formalized with variables, logical symbols (e.g. $\neg$, $\wedge$, $\exists$, $=$), and the traditional symbols $<$, 0, $'$,

* This paper was prepared by Don Jensen during 1973 at the University of Waterloo and the University of Aberdeen. It resulted from work he had done in collaboration with Professor Ehrenfeucht. The paper was near completion when Don Jensen was killed. §§ 1, 2, 3, 4 had been completed and § 5 was in the form of an unfinished manuscript. This last section has been left unfinished. The introduction was written by H. Simmons.

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