Complete exact sequences

by

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Abstract. All groups in this paper are Abelian and have the p-adiac topology. A subgroup A of a group B is a *subspace of* B if the relative topology on A coincides with the p-adiac topology on A. For each group A, A^* denotes the completion of $A/p^{\omega}A$ as a metric space.

If $f: A \to B$, then f induces $f^*: A^* \to B^*$. Moreover, * is a function on the category of Abelian groups. However, * is neither left nor right exact.

This paper is a study of the class of short exact sequences $0 \to A \to B \to C \to 0$ with A a subspace of B and for which the induced sequences $0 \to A^* \to B^* \to C^* \to 0$ remains exact.

1. Basic concepts. All groups in this paper are Abelian and have the p-adiac topology. The basic results of such groups are found in [1] and [2].

A subgroup A of a group B is a *subspace of* B if the relative topology on A coincides with the p-adiac topology on A. If A is p-pure in B, i.e. $p^nA = A \cap p^nB$ for $n < \omega$, then A is a subspace of B. However, p^nB is a subspace of B that need not be p-pure in B. The class of short exact sequences

$$0 \to A \to B \to C \to 0$$

with A a subspace of B is a *proper class* as defined in [3].

It is well known that a group A is metrizable if and only if $p^{\omega}A = \bigcap_{n<\omega} p^nA = 0$. For each group A, the quotient group $A/p^{\omega}A$ is always metrizable. If A^* denotes the completion of $A/p^{\omega}A$ as a metric space, then A^* can be made into a group and the completion topology is the p-adiac topology. If $p^{\omega}A = 0$, then A is a dense p-pure subspace of A^*.

If $f: A \to B$, then f induces $f^*: A^* \to B^*$. Moreover, * is a function on the category of Abelian groups. However, * is neither left nor right exact.

This paper is a study of the class of short exact sequences $0 \to A \to B \to C \to 0$ with A a subspace of B and for which the induce sequences $0 \to A^* \to B^* \to C^* \to 0$ remains exact.

2. Complete exact sequences of metrizable groups. In this section we study exact sequences of groups without elements of infinite p-height.

DEFINITION 2.1. Let E: $0 \to A \to B \to C \to 0$ be exact. Then E is a *complete exact sequence* if the induced sequence E^*: $0 \to A^* \to B^* \to C^* \to 0$ is exact.

THEOREM 2.2. *Let E: $0\to A\to B\to C\to 0$ be an exact sequence of groups. Then E is a complete exact sequence of metrizable groups if and only if A is a closed subspace of B.*

Proof. Suppose A is a closed subspace of the metrizable group B. Then C is metrizable and the sequence $0\to A^*\to B^*\to B^*/A^*\to 0$ is exact. We will show that B^*/A^* is a complete group that contains B/A as a dense p-pure subgroup. The sequences $0\to A\to A^*\to A^*/A\to 0$ and $0\to B\to B^*\to B^*/B\to 0$ are exact. Since A is closed in B, then the 3×3 Lemma shows that the sequence

$$0\to B/A\to B^*/A^*\to (B^*/B)/(A^*/A)\to 0$$

is exact. Now B/A is p-pure and dense in B^*/A^* since B^*/B is p-divisible and $A = A^*\cap B$. Finally, since $p^\omega C = 0$, then $C^* \simeq (B/A)^* \simeq B^*/A^*$. Thus, E is a complete exact sequence of metrizable groups.

Suppose E is a complete exact sequence of metrizable groups. Since E^* is exact, by Theorem 13.1 in [1], A^* is a closed subspace of B^*. Thus, $p^n A^* = U\cap A^*$ with U open in B^*. Therefore, $p^n A = p^n A^*\cap A = U\cap B\cap A$ with $U\cap B$ open in B. Thus, A is a subspace of B. Since C is metrizable, then A is closed in B.

COROLLARY 2.3. *Let B and C be metrizable, with f: $B\to C$ an epimorphism. Then $\operatorname{Ker} f$ is a subspace if and only if $\operatorname{Ker} f^* = (\operatorname{Ker} f)^*$.*

Proof. Suppose $\operatorname{Ker} f$ is a subspace. Since 0 is closed in C, then $0\to(\operatorname{Ker} f)^*\to B^*\xrightarrow{f^*} C^*\to 0$ is exact. Thus $\operatorname{Ker} f^* = (\operatorname{Ker} f)^*$ if and only if $\operatorname{Ker} f$ is a subspace of B.

PROPOSITION 2.4. *Let B be a metrizable group and f: $B\to C$. If $\operatorname{Im} f$ is a subspace then $\operatorname{Im} f^* = (\operatorname{Im} f)^*$.*

Proof. Suppose $\operatorname{Im} f$ is a subspace. If $y\in(\operatorname{Im} f)^*$, then there are $y_n\in\operatorname{Im} f$ such that $y_n\to y$ in C^*, hence in $(\operatorname{Im} f)^*$. In the p-adiac topology we can choose a Couchy sequence $b_n\in B$ with $f(b_n) = y_n$. Let $b_n\to b\in B^*$. Then $f^*(b) = y$ and $y\in\operatorname{Im} f^*$. Since $\operatorname{Im} f^*\subseteq(\operatorname{Im} f)^*$, then $\operatorname{Im} f^* = (\operatorname{Im} f)^*$.

If A is p-pure in B then A^* is a summand of B^*, thus A is p-pure closed if and only if $B^* = A^*\oplus(B/A)^*$.

COROLLARY 2.5. *Let B be a metrizable group and f: $B\to C$. Then $B^* = \operatorname{Ker} f^*\oplus\operatorname{Im} f^*$ if and only if $\operatorname{Ker} f$ is a closed p-pure subgroup.*

THEOREM 2.6. *There is a one to one correspondence between p-pure closed subgroups of a metrizable group B and summands of B^*: given $A\subseteq B$, let A^* correspond to A; given X a summand of B^*, let $X\cap B$ correspond to X.*

Proof. Suppose $B^* = X\oplus Y$. Then the restriction to B of the projection onto Y has kernel $X\cap B$. Since Y is metrizable, and $p^n B\cap X = B\cap p^n X$, $X\cap B$ is p-pure and closed. Therefore, $B^* = (B\cap X)^*\oplus(B/X\cap B)^*$. Since $X\cap(B/X\cap B)^* = 0$, $X = (X\cap B)^*$ and the correspondence $X\to X$, $B\to(X\cap B)^*$ is 1-1.

If A is a p-pure subgroup of B, then A^* is a summand of B^*. If A is closed then $A = A^*\cap B$. Thus, the correspondence $A\to A^*\to A^*\cap B$ is one to one.

COROLLARY 2.6. *If B has only trivial p-pure subgroups, then B^* has only trivial p-pure closed subgroups.*

COROLLARY 2.7. *Z^* is indecomposable.*

3. Complete exact sequences. In this section we use the results in Section 2 to classify complete exact sequences for arbitrary groups.

THEOREM 3.1. *Let E: $0\to A\xrightarrow{\alpha} B\xrightarrow{\beta} C\to 0$ be exact with A a subspace of B. Then E^*: $0\to A^*\xrightarrow{\alpha^*} B^*\xrightarrow{\beta^*} C^*\to 0$ is exact if and only if $p^\omega A = p^\omega B\cap A$.*

Proof. Suppose $p^\omega A = p^\omega B\cap A$. Then $0\to A/p^\omega A\to B/p^\omega B$ is exact. Hence $0\to A^{*\alpha^*}\to B^*$ is exact. By Corollary 2.3, $\operatorname{Ker}\beta^* = (\operatorname{Ker}\beta)^* = A^*$ and by Proposition 2.4, $\operatorname{Im}\beta^* = (\operatorname{Im}\beta)^* = C^*$. Thus, E^* is exact.

If E^* is exact, then the diagram

$$\begin{array}{ccc} A^* & \xrightarrow{\alpha^*} & B^* \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ A/p^\omega A & \xrightarrow[\alpha']{} & B/p^\omega B \end{array}$$

commutes with α^*, i and j monic. Hence α' is monic and $p^\omega A = p^\omega B\cap A$.

COROLLARY 3.2 (Fuchs Theorem 39.8 [1]). *Let E: $0\to A\to B\to C\to 0$ be a p-pure exact sequence, then E^*: $0\to A^*\to B^*\to C^*\to 0$ is split.*

Proof. If A is p-pure in B, then $p^\omega A = p^\omega B\cap A$ and A^* is a summand of B^*.

COROLLARY 3.3. *Let E: $0\to A\to B\to C\to 0$ be exact. If C is complete, then $B^*/A^* \simeq C\simeq (B/A)^*$ if and only if $p^\omega A = p^\omega B$.*

COROLLARY 3.4. *For all k, $B/p^k B\simeq B^*/p^k B^*$.*

References

[1] L. Fuchs, *Infinite Abelian Groups*, New York and London 1970.
[2] I. Kaplanski, *Infinite Abelian Groups*, The University of Michigan Press 1968.
[3] S. Mac Lane, *Homology*, Berlin 1963.

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