Arcwise connected and hereditarily smooth continua

by

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Abstract. We say that $X$ is smooth at the point $p \in X$ if for each convergent sequence $x_n, x_{n+1}, \ldots$ of points of $X$ and for each subcontinuum $K$ of $X$ such that $p, x \in K$, there is a sequence $K_1, K_2, \ldots$ of subcontinua of $X$ such that $p, x_n \in K_n$ for each $n = 1, 2, \ldots$ and $\lim K_n = K$.

The set of all points of a continuum $X$ at which $X$ is smooth is denoted by $I(X)$. A continuum $X$ is said to be hereditarily smooth at $p$ provided each subcontinuum of $X$ which contains $p$ is smooth at $p$. The set of all points of a continuum $X$ at which $X$ is hereditarily smooth is denoted by $HI(X)$.

It is proved that if a continuum $X$ is arcwise connected and $HI(X) \neq \emptyset$ then $X$ is hereditarily arcwise connected and $HI(X) = I(X)$; and if $C$ is the constituent of the set of all points at which $X$ is locally connected, and $C' = HI(X) \neq \emptyset$, then $C = HI(X) = I(X)$. Also other properties of an arcwise connected and hereditarily smooth continua are studied in the paper.

§ 1. Introduction. The notion of smoothness of continua in a general form has been introduced in [10]. In that paper relations are studied between this notion of smoothness and that which was introduced previously in [5] by Gerdich. In particular, it is proved that both notions coincide on metric continua which are either hereditarily unicoherent at some point or irreducible between two points, i.e., any continuum $X$ smooth in the sense of [5] is smooth in the new sense of [10]; and any continuum smooth in the new sense which is either hereditarily unicoherent at some point or irreducible is smooth in the sense of [5]. For example, smooth dendroids (see [2]) are those arcwise connected continua which are smooth in the sense of [5]. The class of arcwise connected continua which are smooth in the sense of [10] is essentially larger than the class of smooth dendroids. Any dendroid $X$ (and, more generally, any continuum $X$ hereditarily unicoherent at some point) is hereditarily smooth at $p$ (see [10], Corollary (7.1); cf. [9], Theorem (2.6)). In this paper we consider arcwise connected continua which are hereditarily smooth at some point.

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§ 2. Preliminaries. The topological spaces under consideration will be assumed to be metric and compact. If the space $X$ under consideration is established, then $d$ denotes a metric on $X$; $B(N, d)$ denotes the union of all open metric balls with the
centres in a given set \( N \) and with the radii \( r > 0 \), and \( ab \) is an arbitrary arc with endpoints \( a \) and \( b \).

The numbering of conclusions in the proofs is separate in every proof. If \( A_1, A_2, \ldots \) is a sequence of subsets of a space \( X \), then \( L(A_n) \) denotes the set of all points \( x \in X \) for which every neighbourhood intersects \( A_n \) for almost all \( n \), and \( L(A_n) \) denotes the set of all points \( x \in X \) for which every neighbourhood intersects \( A_n \) for arbitrarily large \( n \). A sequence \( A_1, A_2, \ldots \) of subsets of \( X \) is said to converge to a set \( A \) (denoted by \( \lim A_n = A \)) in the case \( \lim A_n = A = L(A_n) \).

It is known (see [8], § 47, II, Theorem 6, p. 171) that:

**Proposition 1.** If \( C_1, C_2, \ldots \) is a sequence of subcontinua of the space \( X \) such that \( L(C_n) \neq \emptyset \), then the set \( L(C_n) \) is a continuum.

It is proved (see [10], Lemma 2.2) that:

**Proposition 2.** Let \( C_1, C_2, \ldots \) be a sequence of subcontinua of the space \( X \) and \( (x, y) = L(C_n) \). If \( L(C_n) \) is irreducible between points \( x \) and \( y \), then the sequence \( C_1, C_2, \ldots \) is convergent.

We say that \( X \) is smooth at the point \( p \in X \) if for each convergent sequence \( x_1, x_2, \ldots \) of points of \( X \) and for each subcontinuum \( K \) of \( X \) such that \( p, x, x_1, \ldots \) is in \( K \) for each \( n \in N \), there exists a sequence \( x_1, x_2, \ldots \) of subcontinua of \( X \) such that \( p, x, x_1, \ldots \) is in \( K \) for each \( n \in N \) and \( \lim x_n = K \) (see [10]).

We have the following characterizations of continua which are smooth at some point (see [10], Theorems 2.4 and 3.1):

**Proposition 3.** The continuum \( X \) is smooth at the point \( p \in X \) if and only if one of the following conditions holds:

(i) for each convergent sequence \( x_1, x_2, \ldots \) of points of \( X \) and for each irreducible continuum \( I(p, x) \) between \( p \) and \( x \), there exists a sequence \( I(p, x_1), I(p, x_2), \ldots \) of irreducible continua between \( p \) and \( x_n \), respectively, such that \( \lim I(p, x_n) = I(p, x) \);

(ii) for each subcontinuum \( N \) of \( X \) and for each open set \( V \) of \( X \) there exists a continuum \( K \) such that \( p \in N \subset V \) implies \( N \subset \text{Int} K \subset K \cap V \).

We can characterize the smoothness by the notion of nonaposyndeticity of F. B. Jones (see [6], p. 104). Let \( A \in X \). Then we define \( X \setminus T(A) = \{ x \in X : \exists \text{ a subcontinuum } Q \text{ of } X \text{ such that } x \in \text{Int } Q = Q \setminus X \setminus A \} \) (see [3], p. 113), and put \( T^p(A) = T^{p^{-1}}(A) \) with \( T^p(A) = A \).

**Corollary 1.** A continuum \( X \) is smooth at \( p \in X \) if and only if for each subcontinuum \( N \) of \( X \) such that \( p \in N \), and for each closed set \( A \) in \( X \) the condition \( N \cap A = \emptyset \) implies \( N \cap T(A) = \emptyset \).

Indeed, if \( N \cap A = \emptyset \), then \( N \cap T(A) = \emptyset \). Thus, by Proposition 3(ii), there exists a subcontinuum \( Q \) of \( X \) such that \( N \cap Q = \emptyset \). Therefore \( N \cap X \setminus \text{Int } Q = \emptyset \). By the definition of \( T(A) \), i.e., \( N \cap T(A) = \emptyset \).

Conversely, if \( N \) is an arbitrary continuum in \( X \) such that \( p \in N \), and let \( T \) be an open set in \( X \) containing \( N \). Then there is a closed set \( A \) in \( X \) such that \( N \cap X \setminus A = \emptyset \). We have \( N \cap A = \emptyset \), i.e., for each \( x \in N \), there is a continuum \( Q_x \) such that \( x \in \text{Int } Q_x \subset Q_x \subset X \setminus A \). Put \( Q = \bigcup \{ Q_x : x \in N \} \).

It is easy to verify that the set \( Q \) is a continuum satisfying \( N \cap Q = \emptyset \) in \( X \). Thus condition (ii) from Proposition 3 holds. Since \( T(A) \) is closed (see [3], Lemma 1, p. 114), we have by Corollary 1.

**Corollary 1'.** Let \( X \) be an arbitrary continuum in \( X \) such that \( p \in N \). If \( N \cap T(A) = \emptyset \), then \( N \cap T(A) = \emptyset \) for each \( n = 0 \).

The set of all points of an arbitrary continuum \( X \) at which \( X \) is smooth is called the initial set of \( X \) and is denoted by \( I(X) \). If \( I(X) \neq \emptyset \), then \( X \) is said to be smooth.

The next two theorems are easy consequences of Proposition 3.

**Corollary 2.** A continuum \( X \) is locally connected at each point of \( I(X) \).

**Corollary 3.** A continuum \( X \) is locally connected if and only if \( I(X) = X \).

A continuum \( X \) is said to be hereditarily smooth at \( p \) provided each subcontinuum of \( X \) which contains \( p \) is smooth at \( p \). The set of all points of an arbitrary continuum \( X \) at which \( X \) is hereditarily smooth is called the hereditarily initial set of \( X \) and is denoted by \( H(X) \). If \( H(X) \neq \emptyset \), then \( X \) is said to be hereditarily smooth.

**Corollary 2'.** For each sub continuum \( Q \), \( X \) is locally connected at each point of \( Q \) and \( H(X) \).

**Corollary 3'.** A continuum \( X \) is hereditarily locally connected if and only if \( H(X) = X \).

§ 3. Arcwise connected continua. The main result of this section says that any arcwise connected and hereditarily smooth continuum is hereditarily arcwise connected.

**Theorem 1.** Let an arcwise connected continuum \( X \) be hereditarily smooth at a point \( p \in X \), let \( A \) be an arbitrary subcontinuum of \( X \) and let \( p_q \) be an arc in \( X \) which is irreducible between \( p \) and \( A \).

**Proof.** We have \( p_q \cap Q = (q) \). Let \( K \) be an arbitrary subcontinuum of \( Q \) such that \( q \in K \). Then \( p_q \cap K = (q) \). We will show that \( K \) is smooth at \( q \). Let \( x_1, x_2, \ldots \) be a convergent sequence of points of \( K \) and put \( x = \lim x_n \). Let \( P \) be a subcon-
tinuum of \( K \) such that \( x, q \in P \). \( X \) being hereditarily smooth at \( p \), the continuum \( pq \cup K \) is smooth at \( p \). Therefore there is a sequence \( R_1, R_2, ... \) of subcontinua of \( pq \cup K \) such that \( x_n, p \in R_n \) for each \( n = 1, 2, ... \) and \( \lim R_n = pq \cup P \) by the definition of smoothness. We define \( P_n = K \cap R_n \). Obviously \( P_n \) is a continuum for each \( n = 1, 2, ... \). Moreover \( x_n, q \in P_n \subseteq K \) for each \( n = 1, 2, ... \) and \( \lim P_n = P \). The proof of Theorem 1 is complete.

**Corollary 4.** If \( X \) is a hereditarily smooth arcwise connected continuum, then any subcontinuum of \( X \) is also hereditarily smooth.

Recall that a continuum \( X \) is said to be decomposable if there is a decomposition of \( X \) into two proper subcontinua. A continuum is said to be hereditarily decomposable if any subcontinuum of it is decomposable.

**Corollary 5.** Any hereditarily smooth arcwise connected continuum is hereditarily decomposable.

This is obvious if we observe that, by Corollaries 2 and 4, any subcontinuum of hereditarily smooth arcwise connected continuum is locally connected at some point.

It is well known that for every irreducible continuum \( X \) there exists an upper semi-continuous decomposition of \( X \) into continua (called *layers of \( X \)) (see [8], § 48, IV, p. 199) with the property that the decomposition of \( X \) into layers is the finest of all upper semi-continuous decompositions of \( X \) into continua (8), § 48, IV, Theorem 3, p. 200, [7]. Fundamental theorem, p. 259). If each layer of \( X \) has a void interior, then \( X \) is said to be of type \( \alpha \) (see [8], § 48, III, p. 197, the footnotes, and also [11], Definition 4, p. 13, where these continua are said to be of type \( \alpha \)). It is well known (see [8], § 48, VII, Theorem 3, p. 216; [11], Theorem 10, p. 15; [4], Theorem 2, p. 650) that an irreducible continuum \( X \) is of type \( \alpha \) if and only if each indecomposable subcontinuum of \( X \) has a void interior. Thus, by Corollary 1, we have

**Corollary 6.** Any irreducible subcontinuum of a hereditarily smooth arcwise connected continuum is of type \( \alpha \) (in fact, it is an arc — see Theorem 3 below).

Recall that a subcontinuum \( K \) of \( X \) is called a *continuum of convergence* (see [12], p. 127, cf. [8], § 47, VI, p. 245) provided \( X \) is a topological limit of the sequence of continua such that

\[
K = \lim_{n \to \infty} K_n \quad \text{and} \quad K \cap K_n = \emptyset \quad \text{for each } n = 1, 2, ...
\]

If \( X \) is compact, then we can assume that \( K_1, K_2, ... \) are mutually disjoint. We have

**Theorem 2.** Let \( X \) be an arcwise connected continuum which is hereditarily smooth at the point \( p \in X \). If \( K_0 \) is a continuum of convergence in \( X \) and \( pc \) is an arbitrary arc, then \( K_0 \cap pc \) is connected.

**Proof.** Suppose, on the contrary, that \( K_0 \cap pc \) is not connected. Then there is an arc \( a_0b_0 \) in \( pc \) such that

\[
a_0b_0 \cap K_0 = \{a_0, b_0\} \quad \text{and} \quad a_0 \neq b_0.
\]

Obviously we can assume \( a_0 \in pB_0 \). Since \( K_0 \) is a continuum of convergence in \( X \), \( K_0 \) is a topological limit of the sequence of continua such that

\[
K_0 = \lim_{n \to \infty} K_n \quad \text{and} \quad K_n \cap K_0 = \emptyset \quad \text{for each } m \neq n \text{ and } m, n = 0, 1, 2, ...
\]

Therefore there are sequences \( \{a_n\} \) and \( \{b_n\} \) of points of \( X \) such that

\[
\lim_{n \to \infty} a_n = a_0 \quad \text{and} \quad \lim_{n \to \infty} b_n = b_0,
\]

\[
a_n, b_n \in K_n \quad \text{for each } n = 0, 1, 2, ...
\]

Let \( p_{ab} \) be the arc in \( pc \). So we have \( p_{ab} \in K_0 \subseteq X \setminus \{b_0\} \). Let \( \varepsilon \) be a positive number such that \( \varepsilon < 1/2 \varepsilon(p_{ab}, p_{ab}, p_{ab}) \). Since \( X \) is smooth at \( p \), by Proposition 2 (i) there is a continuum \( Q \) in \( X \) such that

\[
p_{ab} \in \text{Int} Q = \{Q = B(p_{ab}, \varepsilon) \subseteq X \setminus \{b_0\}\}.
\]

By (3) and (5), and by the choice of \( \varepsilon \), we can assume that

\[
a_n \in Q \setminus \{b_n\} \quad \text{for each } n = 1, 2, ...
\]

For each \( n = 1, 2, ... \) take in \( K_n \) the continuum \( I(d_n, b_n) \) irreducible between \( Q \) and \( b_n \). Let \( e_n \) be an arc in \( a_n b_n \) such that \( e_n \cap Q = \{a_0\} \). It suffices to consider only two cases.

1. \( I(d_n, b_n) \cap e_n = \emptyset \) for each \( n = 1, 2, ... \) (or there is a subsequence \( I(d_n, b_n) \) of the sequence \( I(d_n, b_n) \) such that \( I(d_n, b_n) \cap e_n = \emptyset \) for each \( k = 1, 2, ... \), but then the proof is the same). Then we consider the following continuum

\[
R = Q \cup K_0 \cup \bigcup_{n=1}^{\infty} I(d_n, b_n).
\]

Since \( X \) is hereditarily smooth at \( p \), \( R \) is smooth at \( p \). Thus, by (3), there is a sequence of continua \( R_n \) in \( R \) such that

\[
p_n \in R_n \quad \text{for each } n = 1, 2, ...
\]

and

\[
\lim_{n \to \infty} R_n = p_{ab} \cup a_0b_0.
\]

But for each \( n = 1, 2, ... \) we have

\[
I(d_n, b_n) \subseteq R_n.
\]
Indeed, by Corollary 6 the irreducible continua $I(d_n, b_n)$ are of type $\lambda$, and by the definition of $R$ any layer of $I(d_n, b_n)$ separates $R$ between $b_n$ and $p$. Thus any layer of $I(d_n, b_n)$ is contained in $R_n$, i.e., (9) holds.

Therefore, by (7) and (9) the set $\lim R_n$ contains some irreducible continuum $z_0$ between $b_0$ and $Q$, which is contained in $K_0$, contrary to (8).

2. $I(d_n, b_n) \cap e_n \neq \emptyset$ for each $n = 1, 2, \ldots$. Then we can take, by Corollary 6, irreducible continua $I(d_n, z_n)$ in $I(d_n, b_n)$ such that

\[ z_n \in e_n \]

and

\[ \text{no proper subcontinuum of } I(d_n, z_n) \text{ containing } d_n \text{ intersects } e_n. \]

(2) and (10) imply that $\lim \, x_n = b_0$. By the standard construction we can take, by (11), irreducible continua $I(d_n, x_n)$ in $I(d_n, z_n)$ such that $\lim \, x_n = b_0$, and $I(d_n, x_n) \cap e_n = \emptyset$ for each $n = 1, 2, \ldots$. Then we obtain a contradiction as in case 1'. The proof of Theorem 2 is complete.

Let an irreducible continuum $X$ be of type $\lambda$ and let $T_n, n \in [0, 1]$, denote a layer of $X$. Thus $X = \bigcup \{T_n : 0 \leq t < 1\}$. Put

\[ L_i = \bigcup \{T_n : 0 \leq u < t\} \quad \text{and} \quad R_i = \bigcup \{T_n : t < v \leq 1\}. \]

Therefore

\[ L_i = \varphi^{-1}(0, 0) \quad \text{and} \quad R_i = \varphi^{-1}(1, 1), \]

where $\varphi$ is the canonical mapping from $X$ to the unit interval $[0, 1]$; we see that both $L_i$ and $R_i$ are connected. (Here the capital letters $L$ and $R$ stand for left and right, respectively). Adopt the following definitions (see [1], p. 46). A layer $T$ is said to be a layer of left cohesion if either $t = 0$ or $T_n = L_n \cap T_n$; and $T$ is said to be a layer of right cohesion if either $t = 1$ or $T_n = R_n \cap T_n$. One can see that $T_n$ is a layer of right cohesion ($T_n$ is a layer of left cohesion) provided the interior of $T_n$ ($T_n$) is empty. A layer $T$ is said to be a layer of cohesion if it is a layer of both left and right cohesions (see [7], p. 260; [8], § 48, IV, p. 201). We have the following (see [1], Theorem, p. 48).

**Proposition 4.** An irreducible continuum $X$ is smooth at a point $p$ if and only if all three of the following conditions are satisfied:

(i) $X$ is locally connected at $p$,

(ii) for each $t$ satisfying $0 \leq u < \varphi(p)$ the layer $T$ is of right cohesion,

(iii) for each $t$ satisfying $\varphi(p) < v \leq 1$ the layer $T$ is of left cohesion.

**Lemma 1.** For each two points $x_0$ and $y_0$ of an arbitrary layer $T_n$ of an irreducible smooth continuum $X$, there exists a continuum of convergence $K_n$ such that $\{x_0, y_0\} \subseteq K_n \subseteq T_n$.

**Proof.** By Proposition 4, the layer $T_n$ is either of right or of left cohesion. Suppose that $T_n$ is a layer of left cohesion (if $T_n$ is a layer of right cohesion the proof is the same). Then either $t_n = 0$ or $T_n = L_n \cap T_n$. If $t_n = 0$, then $T_n$ is a layer of right cohesion, and the proof is the same as for layers of right cohesion. If $t_n \neq 0$, then there are sequences $\{x_n\}$ and $\{y_n\}$ such that

1. $x_n$ and $y_n$ belongs to $L_n$ for each $n = 1, 2, \ldots$.

2. $\lim \, x_n = x_0$ and $\lim \, y_n = y_0$.

Let $\varphi$ be the canonical mapping from $X$ to the unit interval $[0, 1]$. We can assume that for each $n = 1, 2, \ldots$,

$\varphi(x_n) < \varphi(y_n) < \varphi(x_{n+1}) < t_n$.

Put

\[ K_n = \varphi^{-1}(\varphi(x_n), \varphi(y_n)) \quad \text{for each } n = 1, 2, \ldots. \]

Obviously, by (3), for each $n = 1, 2, \ldots$ the set $K_n$ is a continuum and $K_n \cap T_n = \emptyset$, and by (1), we have $\{x_n, y_n\} \subseteq K_n \subseteq T_n$. We can assume that the sequence $\{K_n\}$ is convergent. Then $K = \lim K_n$ is a continuum of convergence, and $\{x_0, y_0\} = K \subseteq T_n$. The proof of Lemma 1 is complete.

**Corollary 7.** Let a continuum $X$ be arcwise connected and hereditarily smooth at $p$. For each layer of an arbitrary irreducible submodule $A$ of $X$ and for each $x \in c$ in $X$ the set $pc \cap T$ is connected.

**Proof.** By Corollary 4, $A$ is an irreducible smooth continuum. Let $a$ be an arbitrary point of $pc \cap T$. Therefore by Lemma 1, for each $y \in T$ there is a continuum of convergence $K$ such that $\{a, y\} \subseteq K \subseteq T$. Thus $T = \bigcup \{K_y : y \in T\}$ and $pc \cap T = \bigcup \{K_y : y \in T\}$. But $K \subseteq pc$ is connected by Theorem 2, and $a \in K \subseteq pc$ for each $y \in T$. This implies that the set $pc \cap T$ is connected (see [8], § 48, II, Corollary 30), p. 132).

**Lemma 2.** Let $I(a, b)$ be an irreducible continuum between $a$ and $b$ which is smooth at the point $a$ and let $I(c, d)$ be an irreducible subcontinuum of $I(a, b)$. If $I$ is a layer of $I(c, d)$, then $T$ is a layer of $I(a, b)$.

**Proof.** Let $\varphi$ be the canonical map from $I(a, b)$ onto $I = [0, 1]$ such that $\varphi(a) = 0$. Suppose that $\varphi(c) < \varphi(d)$. (If $\varphi(c) > \varphi(d)$ the proof is the same). It follows from Theorem (5.3) in [10] that $I(a, b)$ is hereditarily unicoherent at $d$; thus $I(c, d) = \varphi^{-1}(\varphi(c), \varphi(d))$. Consider the continuum

\[ K = \varphi^{-1}(0, 0) \cup I(c, d) \cup \varphi^{-1}(0, 0). \]

Since $a, b \in K$, we have $K = I(a, b)$. Therefore $\varphi^{-1}(\varphi(c), \varphi(d)) = I(c, d)$. Thus

\[ \varphi^{-1}(\varphi(c), \varphi(d)) = \varphi^{-1}(\varphi(c), \varphi(d)) = I(c, d). \]
by Proposition 4. We infer \( I(c, d) = \psi^{-1}(\{\sigma(0), \sigma(d)\}) \). This equality implies the conclusion of the lemma.

**Lemma 3.** Let \( I(c, d) \) be an irreducible continuum between \( c \) and \( d \), which is smooth at \( d \), and let \( \psi \) be the canonical map from \( I(c, d) \) onto \([0, 1]\) such that \( \psi(c) = 0 \). If \( x_0, y_0 \) is an irreducible subcontinuum of \( I(c, d) \) such that \( \sigma(x_0) < \sigma(y_0) \), then the set \( \psi^{-1}(\{x_0, y_0\}) \) is a layer of \( I(x_0, y_0) \).

Proof. Let \( I(x_0, y_0) \) be an irreducible subcontinuum of \( I(c, d) \) such that \( \sigma(x_0) < \sigma(y_0) \). Consider the continuum \( K = \psi^{-1}(\{0, \sigma(x_0)\}) \cup I(x_0, y_0) \cup \psi^{-1}(\{\sigma(y_0), 1\}) \). Since \( c, d \in K \), we have \( K = I(c, d) \). Therefore

\[
\psi^{-1}(\{\sigma(x_0), \sigma(y_0)\}) = I(x_0, y_0) \neq \emptyset.
\]

Thus \( \psi^{-1}(\{\sigma(x_0)\}) \) and the set \( \psi^{-1}(\{\sigma(y_0)\}) \) is nowhere dense in \( I(x_0, y_0) \) by Proposition 4. This implies by Theorem 7 in [8], § 48, p. 194, that the continuum \( I(x_0, y_0) \) is irreducible between each point of the set \( \psi^{-1}(\{\sigma(x_0)\}) \) and \( y_0 \). Moreover, since for each \( \sigma(x_0) < t < \sigma(y_0) \) the set \( \psi^{-1}(\{t\}) \) separates \( I(x_0, y_0) \), we conclude that \( \psi^{-1}(\{\sigma(x_0)\}) \) is the set of all points \( a \) of \( I(x_0, y_0) \) such that \( I(x_0, y_0) \) is irreducible between \( a \) and \( y_0 \). Therefore \( \psi^{-1}(\{a\}) \) is a layer of \( I(x_0, y_0) \) (cf. [8], § 48, IV, Theorem 4, p. 202).

**Theorem 3.** If an arwise connected continuum \( X \) is hereditarily thin, then \( X \) is hereditarily arwise connected.

Proof. It suffices to prove that any irreducible continuum in \( X \) is an arc. Let \( I(a, b) \) be an arbitrary subcontinuum of \( X \) irreducible between given points \( a \) and \( b \). Then \( I(a, b) \) is of type \( I \), by Corollary 6. Therefore it suffices to show that any layer of \( I(a, b) \) degenerate. Suppose, on the other hand, that there is a nondegenerate layer of \( I(a, b) \). Since \( X \) is arwise connected, there is an arc \( p \) in \( X \) such that

\[
(1) \quad pc \cap T = \{c\}.
\]

If \( pc \cap I(a, b) = \{c\} \), then the continuum \( I(a, b) \) is smooth at \( c \), by Theorem 1. Thus \( I(a, b) \) is locally connected at \( c \). This implies that the layer \( T_c \) of the point \( c \) in \( I(a, b) \) is degenerate. But \( c \in T_c \), and hence \( T_c = T \), i.e., \( T \) is degenerate — a contradiction.

Therefore we consider the remaining case, namely that of \( pc \cap I(a, b) = \{c\} \). Take an arc \( pd \) in the arc \( pc \) such that

\[
(2) \quad pd \cap I(a, b) = \{d\}.
\]

Then \( I(a, b) \) is smooth at \( d \) by Theorem 1; and thus, by Lemma 2, if we take the continuum \( I(c, d) \) of \( I(a, b) \) irreducible between \( c \) and \( d \), then the layer of the point \( c \) in \( I(c, d) \) coincides with \( T \) by (1). Let \( \psi \) be the canonical mapping from \( I(c, d) \) to the unit interval \([0, 1]\) such that

\[
(3) \quad \psi^{-1}(0) = T \quad \text{and} \quad \psi^{-1}(1) = \{d\},
\]

and let \( cd \) mean the subarc of the arc \( pc \).

Indeed, observe firstly that

\[
\psi^{-1}(\{a, b\}) \setminus \psi^{-1}(\{c\}) = \psi^{-1}(\{c\}) \cup \psi^{-1}(\{d\}).
\]

Since \( I(c, d) \) is smooth at \( d \), the layer \( \psi^{-1}(\{c\}) \) is of right cohesion by Proposition 4. Therefore by Lemma 3 the set \( \psi^{-1}(\{a, b\}) \) is a layer of an irreducible continuum \( I(x_0, y_0) \) in \( I(c, d) \) such that \( x_0 \in \psi^{-1}(\{a\}) \) and \( y_0 \in \psi^{-1}(\{b\}) \). Then \( pc \cap I(x_0, y_0) = pc \cap \psi^{-1}(\{a, b\}) \) by the definition of \( \psi \); hence, \( x_0, y_0 \) is smooth at some point of \( \psi^{-1}(\{a, b\}) \) by Theorem 1. We infer that \( I(x_0, y_0) \) is locally connected at some point \( \psi^{-1}(\{a, b\}) \); hence

\[
(10') \quad \psi^{-1}(\{a, b\}) \text{ is a one-point set.}
\]

Suppose now that

\[
\lim_{n \to \infty} z_n = x_0 \in \psi^{-1}(\{a, b\}) \cap pc \neq \emptyset.
\]

Since \( I(c, d) \) is smooth at \( d \), the layer \( \psi^{-1}(\{b\}) \) is of right cohesion by Proposition 4. If

\[
\psi^{-1}(\{a, b\}) \setminus \psi^{-1}(\{b\}) \cap pc = \emptyset,
\]

then there is a sequence \( \{z_n\} \) of points of \( \psi^{-1}(\{a, b\}) \), i.e.,

\[
\psi(z_n) \in (a, b),
\]

such that

\[
\lim_{n \to \infty} z_n = x_0 \in \psi^{-1}(\{b\}) \cap pc.
\]
We can assume that $t < \varphi(z_n) < \beta(t)$ for each $n$. Take the arc $p z_0 < p c$, and consider the continuum $L$ of the form

$$L = p z_0 \cup \varphi^{-1}(t, \beta(0)) \cup \varphi(z_n).$$

Then, by assumption, $L$ is smooth at $p$, and thus there are, for each $n = 1, 2, \ldots$, irreducible continua $I(p, z_n) \subseteq L$ such that

$$\lim_{n \to \infty} I(p, z_n) = p z_0.$$

Since any layer $\varphi^{-1}(t')$ for $t' < \beta(t)$ separates the continuum $L$, we conclude that

$$\varphi^{-1}(\varphi(z_n), \beta(0)) = \varphi^{-1}(\varphi(z_n), \beta(0)) \cap \varphi^{-1}(\beta(0)).$$

Therefore, since

$$z_n \in \varphi^{-1}(t, \beta(0)) \cap \varphi^{-1}(\beta(0)) = \varphi^{-1}(\varphi(z_n), \beta(0)) \cap \varphi^{-1}(\beta(0)),$$

we have

$$z_n \in \lim_{n \to \infty} I(p, z_n),$$

i.e., $z_n \in p z_0 < p c$ — a contradiction.

Therefore, we can assume

$$\overline{\varphi^{-1}(\varphi(z_n), \beta(0)) \cap \varphi^{-1}(\beta(0))} \cap p c = \emptyset.$$

By (10) $\varphi^{-1}(\alpha(t))$ is a one-point set. Denote this point by $a$. Take the arc $p c$ in the arc $p c$, and take a point $e'$ in $p c$ such that $e' \cap \varphi^{-1}(\{\beta(t), 1\}) = \{e'\}$, where $e'$ is an arc in $p c$. Consider two cases.

1'. $\varphi(e') \neq \beta(t)$. Then there is a point $t_o$ such that $\beta(t) < t_o < \varphi(e')$. Consider the continuum $K$ defined as follows:

$$K = \varphi^{-1}(\varphi(e'), 1) \cup \varphi^{-1}(\varphi(z_n), t_o).$$

The continuum $K$ is irreducible between $e$ and any point of $K \cap \varphi^{-1}(t_o)$, and $K$ is smooth at $d$, because $p d \cap K = \{d\}$ (cf. Theorem 1). Therefore by Proposition 4,

$$\varphi^{-1}(\beta(t)) = \overline{\varphi^{-1}(\alpha(t), \beta(0))}.$$

By the definition of $\beta(t)$ we have $\varphi^{-1}(\beta(t)) \cap p c = \emptyset; \varphi^{-1}(\varphi(z_n), \beta(0)) \cap \varphi^{-1}(\beta(0)) \cap p c = \emptyset — a contradiction.

2'. $\varphi(e') = \beta(t)$. The layer $\varphi^{-1}(\beta(t))$ is of right cohesion of $I(c, d)$, and $\{e', z_0\} = \varphi^{-1}(\beta(t))$, we infer there are continua $K_n$ such that $K_n \cap K_n = \emptyset$ for $n \neq m$. Let $K_n = K_n \cap [e', z_0] = K_n \cap \varphi^{-1}(\beta(t))$, $\varphi(K_n) \cap [e', \beta(t)] = \emptyset$ (cf. Lemma 1). Since the continuum $p e' \cup \varphi^{-1}(\beta(t), 1)$ is smooth at $p$, there is a continuum $Q$ such that

$$p e' \subseteq \text{Int} Q = q e' \cup \varphi^{-1}(\beta(t), 1) \cup \varphi^{-1}(\varphi(z_n), \beta(0)).$$

(Cf. Proposition 3 (ii)), because $\varphi^{-1}(\varphi(z_n), \beta(0)) \cap p e' = \emptyset$ (Int $Q$ denotes the interior of $Q$ in $p e' \cup \varphi^{-1}(\beta(t), 1)$). Since $K_n \subseteq p e' \cup \varphi^{-1}(\beta(t), 1)$ and $\lim_{n \to \infty} K_n = K_0$ contains the point $e'$, we can assume that $K_0 \cap Q \neq \emptyset$ for each $n = 1, 2, \ldots$ Since $x_n \in K_0$, there is a sequence of points $\{x_n\}$ such that $\lim_{n \to \infty} x_n = x_0$ and $x_0 \in K_0$. Take the continuum $I(x_0, c_n)$ irreducible between $x_0$ and $Q$ (in $K_0$). We can assume that the sequence $\{x_n\}$ is convergent and put $c_0 = \lim x_n$. Then $c_0 \in Q$.

Consider the continuum

$$K = Q \cup e' \cup \varphi^{-1}(\alpha(t), \beta(0)) \cup \bigcup_{n \to \infty} I(x_n, c_n).$$

Since $p \in K$, $K$ is smooth at $p$. The continuum

$$I(p, z_0) = p e' \cup \varphi^{-1}(\varphi(z_n), \beta(0))$$

is irreducible between $p$ and $z_0$. Moreover, $I(p, z_0) = K_n$ and $\lim_{n \to \infty} z_n = z_0$. Then, by the smoothness of $K$ at $p$, there are irreducible continua $I(p, z_n) \subseteq K_n$ in $K$ such that

$$\lim_{n \to \infty} I(p, z_n) = I(p, z_0).$$

By the definition of $K$, we have $I(x_n, c_n) = I(p, z_n)$. Therefore

$$\lim_{n \to \infty} I(x_n, c_n) = I(p, z_0) \cup \varphi^{-1}(\beta(t)).$$

By Proposition 1 the set $M = \lim_{n \to \infty} I(x_n, c_n)$ is a continuum. Therefore, because

$$M = (p e \cap M) \cup \{\varphi^{-1}(\varphi(x_n, \beta(0)) \cap M \} \text{ and } z_0 \in \varphi^{-1}(\varphi(x_n, \beta(0)) \cap M,$$

we have either $p e \cap M = \emptyset$ or $p e \cap M \cap \varphi^{-1}(\varphi(x_n, \beta(0)) \neq \emptyset$. If $p e \cap M = \emptyset$, then

$$c_0 \in M \cap \varphi^{-1}(\varphi(x_n, \beta(0)).$$

But $c_0 \notin Q$ and, thus $c_0 \notin Q \cap \varphi^{-1}(\varphi(x_n, \beta(0)) — a contradiction, because

$$Q \cap \varphi^{-1}(\varphi(x_n, \beta(0)) \neq \emptyset,$$

and

$$\emptyset.$$
Take a sequence \( \{x_n\} \) such that \( \lim x_n = x_0 \), and \( \alpha(t_0) \in \varphi^{-1}(\varphi_x(t_0)) \). Then \( \varphi^{-1}(\varphi_x(t_0)) = T_0 \) is contained in a layer of \( \varphi^{-1}(\varphi(t_0)) \) and \( \{x_n, x_0\} \subset T_0 \). Consider the continuum \( P \) of the form

\[
P = \text{pc} \cup T_0 \cup \bigcup_{n=1}^{\infty} \varphi^{-1}(\varphi_x(t_n))
\]

Then, by assumption, \( P \) is smooth at \( p \); thus there are irreducible continua \( I(p, x_0) \subset P \) such that \( \lim I(p, x_n) = px_0 \), where \( px_0 \) is an arc in \( P \). Then any continuum \( I(p, x_n) \) must contain the point \( px_0 \), and thus \( x_0 \in px_0 \) — a contradiction by (14). The proof of Theorem 3 is complete.

§ 4. The initial set of a hereditarily smooth arcwise connected continuum. In this section we will prove that, if \( X \) is a hereditarily smooth arcwise connected continuum, then any point \( p \) of \( X \) is smooth, and there are irreducible continua \( I(p, x_0) \subset P \) such that \( \lim I(p, x_n) = px_0 \), where \( px_0 \) is an arc in \( P \). Then any continuum \( I(p, x_n) \) must contain the point \( px_0 \), and thus \( x_0 \in px_0 \) — a contradiction by (14). The proof of Theorem 3 is complete.

**Theorem 4.** Let a continuum \( X \) be arcwise connected and hereditarily smooth at the point \( p \). If \( K_0 \) is a subcontinuum of convergence of \( X \) for each point \( x_0 \) of \( X \), then they are arcs. Thus

\[
\lim_{n \to \infty} x_n = x_0 \in I(d, c) \setminus \text{pc}.
\]

Then, by assumption, \( K_0 \) is a topological limit of a sequence of disjoint continua, i.e.,

\[
K_0 = \lim_{n \to \infty} K_n \quad \text{and} \quad K_n \cap K_m = \emptyset \quad \text{for each} \quad m \neq n \quad \text{and} \quad m, n = 0, 1, 2, \ldots
\]

It follows from \( (x_n, y_n) \in K_0 \) that there are sequences \( \{x_n\} \) and \( \{y_n\} \) of points of \( X \) and a sequence \( \{x_n, y_n\} \) of arcs in \( X \) (by the hereditary arcwise connectivity of \( X \)) such that

\[
\lim_{n \to \infty} x_n = x_0 \quad \text{and} \quad \lim_{n \to \infty} y_n = y_0.
\]

We may assume (see [8], § 42, I, Theorem 1, p. 45) that the sequence \( \{x_n, y_n\} \) is convergent and

\[
\lim_{n \to \infty} x_n y_n = K_0 \subset K_0.
\]
Since $X$ is smooth at $p$, it follows by Proposition 3(ii) that there are continua $Q_a$ and $Q_b^*$ such that
\[(5) \quad p X \subseteq \text{Int } Q_a \subseteq Q_b \subseteq B(p X_0, 1/n), \]
\[(6) \quad p Y \subseteq \text{Int } Q_b^* \subseteq Q_b \subseteq B(p Y_0, 1/n). \]

We can assume by (2) that for an arbitrary but fixed $n$ and for each $i = 1, 2, \ldots$ we have $x_i Y_i \cap Q_b \neq \emptyset$ and $x_i Y_i \cap Q_b^* \neq \emptyset$. Take arcs $a_i b_i \subseteq x_i Y_i$ irreducible between $x_i Y_i \cap Q_b$ and $x_i Y_i \cap Q_b^*$ for each $i = 1, 2, \ldots$.

If the sequence $\{a_i b_i\}$ contains a subsequence $\{a_i b_i\}$ of degenerate arcs, then there is a sequence $\{x_k\}$ of points such that $x_k \in a_k b_k \cap Q_a \cap Q_b^*$. Therefore there is a point $z_k \in \lim_{k \to \infty} x_k$ such that $z_k \in Q_a \cap Q_b^* \cap K_a^*$. If each subsequence $\{a_i b_i\}$ of the sequence $\{a_i b_i\}$ is a sequence of nondegenerate arcs, then there are arcs $a_k c_k$ such that
\[(7) \quad \lim_{k \to \infty} c_k = c_0 \in Q_b^*, \]
\[(8) \quad a_k c_k \subseteq a_k b_k, \]
\[(9) \quad a_k c_k \cap Q_b^* = \emptyset \quad \text{and} \quad a_k c_k \cap Q_a = \{a_k\}. \]

Put $R = Q_a \cap Q_b^* \cap K_a^* \cup \cup_{k=1}^\infty a_k c_k$. Obviously $R$ is a continuum and $p \in R$.

Moreover, for each $k = 1, 2, \ldots$, by (4) and (9), we infer that
\[(10) \quad \text{any continuum } A \text{ in } R \text{ such that } p, c_k \in A \text{ contains } a_k. \]

Since $X$ is hereditarily smooth at $p$, the continuum $R$ is smooth at $p$. Therefore by (7) and by the definition of smoothness, there are continua $A_k$ in $R$ such that
\[(11) \quad \{p, c_k\} = \text{Lim } A_k = Q_b^* , \]
\[(12) \quad p, c_k \in A_k \quad \text{for each } k = 1, 2, \ldots \]

It follows from (10) and (12) that $a_k \in A_k$ for each $k = 1, 2, \ldots$. Let $a_k$ be a cluster point of the sequence $\{a_k\}$. We have $a_k \in K_a \cap Q_a \cap Q_b^*$ is nonempty, by (6), (9) and (11).

Thus we find that for each $n = 1, 2$, the set $K_a \cap Q_a \cap Q_b^*$ is nonempty. Therefore
\[
\lim_{k \to \infty} (K_a \cap Q_a \cap Q_b^*) = K_a \cap \text{Lim } Q_a \cap Q_b^* = K_a \cap p X_0 \cap p Y_0
\]

is nonempty. Hence $x_0 = y_0$ by (1). The proof of Theorem 4 is complete.

**Lemmata.**

**4.** Let a continuum $X$ be hereditarily arcwise connected. If the arc $A_0$ is a continuum of convergence of $X$, then any one of the arc $A_0$ is a continuum of convergence of $X$.

**Proof.** The arc $A_0$ is a continuum of convergence of $X$; thus $A_0$ is a topological limit of the sequence $A_n$ of subcontinua of $X$ such that
\[(13) \quad A_0 = \lim_{n \to \infty} A_n \quad \text{and} \quad A_0 \cap A_n = \emptyset \quad \text{for} \quad m \neq n \text{ and } m, n = 0, 1, 2, \ldots \]

Let $a_0$ and $b_0$ be endpoints of the arc $A_0$. There are sequences $\{a_n\}$ and $\{b_n\}$ of points of $X$ and a sequence $\{a_n b_n\}$ of arcs of $X$ such that
\[(14) \quad \lim_{n \to \infty} a_n = a_0 \quad \text{and} \quad \lim_{n \to \infty} b_n = b_0, \]
\[(15) \quad a_n b_n \subseteq A_n \quad \text{for each } n = 0, 1, 2, \ldots \]

by the hereditary arcwise connectedness of $X$. It follows from Proposition 2 that
\[(16) \quad \lim_{n \to \infty} a_n b_n = A_0 = a_0 b_0. \]

For each $i = 1, 2, \ldots$, there is an arc $a_i b_i$ such that
\[(17) \quad \lim_{i \to \infty} a_i b_i = A_0 \quad \text{for each } i \in a_i b_i, \]
\[(18) \quad \lim_{i \to \infty} a_i b_i = A_0 \quad \text{for each } i \in a_i b_i, \]

Let $c_i d_i$ be a subarc of the arc $A_0$ such that $a_i \leq c_i \leq d_i \leq b_i$ in the natural order of the arc $A_0$. It suffices to prove that $c_i d_i$ is a continuum of convergence of $X$.

Let $i$ be a natural number and let $a_i b_i$ be an arc determined above. Let $d_i$ be the first point in the arc $a_i b_i$, such that
\[(19) \quad \phi(d_i, d_i b_i) = 1, \]
where $d_i b_i$ is the subarc of the arc $a_i b_i$; i.e., for each $x \in a_i b_i \setminus (d_i b_i)$, we have $\phi(x, d_i b_i) > 1$. Let $c_i$ be the least point in the arc $a_i d_i$ (in $a_i b_i$) such that
\[(20) \quad \phi(c_i, a_i c_i) = 1, \]
where $a_i c_i$ is an arc in the arc $a_i b_i$. Therefore, if $c_i d_i$ is an arc in $a_i b_i$, by (4)-(7)
\[(21) \quad \phi(x, c_i d_i) > 1 \quad \text{for each } x \in c_i d_i \setminus (c_i, d_i) . \]

Put $K_n = c_n d_n$. Consider $K_0 = K_{a_n}$. By (7), any cluster point of the sequence $\{a_n\}$ is contained in $a_0 c_0$, but (4) and (8) imply that any cluster point of the sequence $\{c_n\}$ is contained in $c_0 d_0$; therefore
\[(22) \quad \lim_{i \to \infty} c_i = c_0 . \]

In a similar way we obtain
\[(23) \quad \lim_{i \to \infty} d_i = d_0. \]
Moreover (4) and (9) imply that

\[ K_0 = c_0d_0. \]

Thus, by (9), (10) and (11), we infer \( K_0 = c_0d_0 \). Therefore, by Proposition 2, we have \( \lim K_i = c_0d_0 \), i.e., the arc \( c_0d_0 \) is a continuum of convergence of \( X \) by the choice of \( K_0 \). The proof of Lemma 4 is complete.

**Lemma 5.** Let a continuum \( X \) be arcwise connected and hereditarily smooth at a point \( p \in X \). Let \( \{ p_{x_n} \} \) be a sequence of arcs of \( X \) such that \( \lim x_n = x_0 \) and \( \lim p_{x_n} \) is an arc \( p_{x_0} \). If \( \{ z_n \} \) is a sequence of points of \( p_{x_0} \) such that

(i) \( \lim_{n \to \infty} z_n = x_0 \).

(ii) If \( z_nx_n \) is an arc in \( p_{x_n} \) and \( p_{x_n} \) is an arc in \( p_{x_0} \), then \( p_{x_n} \cap z_nx_n = \{ z_n \} \),

then \( {L}_{s_0}z_nx_n \) is the arc \( z_nx_n \) in \( p_{x_0} \).

Proof. By Theorem 3 the continuum \( X \) is hereditarily arcwise connected. By assumptions, \( {L}_{s_0}z_nx_n \) is a subarc of the arc \( p_{x_0} \) and \( x_0 \in {L}_{s_0}z_nx_n \). Suppose, on the contrary, that \( z_nx_n \), \( z_0 \in p_{x_0} \) \( \\backslash \{ z_0 \} \) \( \subset p_{x_0} \). There is a sequence \( z_n' \in z_nx_n \) such that

\[ \lim_{n \to \infty} z_n' = z_0. \]

We can assume that for each \( i = 1, 2, ... \), the arc \( z_i'z_i \), contained in \( z_nx_n \), are such that \( z_nx_n \cap p_{x_0} = \emptyset \). Since \( z_nx_n \subseteq z_0x_0 \), we have \( \lim z_nx_n = z_0x_0 \) by Proposition 2. Therefore

(2) the arc \( z_0x_0 \) is a continuum of convergence of \( X \).

Take the arc \( z_i'z_i \), contained in \( z_nx_n \), for each \( i = 1, 2, ... \) Consider

\[ Q = p_{x_0} \cup \bigcup_{i=1}^{\infty} z_i'z_i. \]

Obviously \( Q \) is a continuum. Since \( X \) is hereditarily smooth at \( p \) and \( p \in Q \), \( Q \) is smooth at \( p \). Let \( e = \varepsilon(q_0, p_{x_0}) \), where \( p_{x_0} \) is an arc in \( p_{x_0} \). It follows from Proposition 3(ii) that there is a continuum \( K \) in \( Q \) such that

\[ p_{x_0} \cap \text{Int} K = K \subset B(p_{x_0}, 0). \]

By (i) of the assumptions and by (1) we can take a natural number \( n_1 \) such that \( z_i' \in K \) and \( g(z_i, z_0) \subset e \). Let \( ab \) be an arbitrary arc in \( p_{x_0} \), such that \( ab \cap K = \emptyset \) and \( p < a < b \) is the natural order of the arc \( p_{x_0} \). The continuum \( K \cup z_i'z_i' \cup b_{i_0} \), where \( b_{i_0} \) is an arc in \( p_{x_0} \), contains an arc \( pb \) by the hereditary arcwise connectedness of \( X \). Then \( pb \cap ab = \{ b \} \). By Lemma 4 and (2) \( ab \) is a continuum of convergence; thus by Theorem 4 if we take the arc \( pb \) and the arc \( pa \), which is contained in \( p_{x_0} \), then we obtain \( a = b = c \) — a contradiction.

**Theorem 5.** Let a continuum \( X \) be arcwise connected. If \( H(X) \neq \emptyset \), then \( I(X) = HI(X) \).

Proof. Obviously \( H(X) \subset I(X) \) by definition. It follows from Theorem 3 that \( X \) is hereditarily arcwise connected. Let \( X \) be smooth at \( p \) and let \( Q \) be an arbitrary subcontinuum of \( X \) such that \( p \neq Q \). By the definition of hereditary smoothness it suffices to prove that \( Q \) is smooth at \( p \). By Theorem 1, if \( r \in H(X) \) and \( rq \) is an arbitrary arc irreducible between \( r \) and \( p \), then \( qp \) is hereditarily smooth at \( q \). Let \( \{ x_n \} \) be an arbitrary sequence of points of \( Q \) such that

\[ \lim_{n \to \infty} x_n = x_0 \in p_{x_0}, \]

and let \( p_{x_0} \) be an arbitrary arc contained in \( Q \). We will prove that there is a sequence \( \{ x_n \} \) of subcontinua of \( Q \) such that \( \lim_{n \to \infty} x_n = p_{x_0} \).

Let \( g_0 \) be an arbitrary arc in \( Q \) which is irreducible between \( q \) and \( p_{x_0} \). Denote by \( y_0 \) and \( y_0x_0 \) arcs contained in \( p_{x_0} \). Put \( g_0 \in qg_0 \cup y_0x_0 \). Since \( g \) is an initial point of \( Q \), there is a sequence \( \{ g_0 \} \) of arcs in \( Q \) such that

\[ \lim_{n \to \infty} g_0 = g_0x_0. \]

For each \( n = 1, 2, ... \), let the point \( z_n \) of \( qg_0 \subset qx_0 \) be such that, if \( z_nx_n \subset x_0 \), then \( z_nx_n \) is an arc in \( g_0 \), and \( g_0x_0 \) is an arc in \( g_0x_0 \). Then \( z_nx_n \) is an arbitrary cluster point of the sequence \( \{ z_n \} \), i.e., for some subsequence \( \{ z_n \} \) of the sequence \( \{ z_n \} \) we have

\[ z_n = \lim_{n \to \infty} z_n. \]

Suppose that \( z_n \notin y_0x_0 \). Then (3) and Lemma 4 imply that

(4) any proper subarc of the arc \( x_0z_n \) is a continuum of convergence of \( X \), where \( x_0z_n \) is an arc in \( g_0z_0 \).

Let \( e = \varepsilon(q_0, p_{x_0}) \). Since \( X \) is smooth at \( p \), by Proposition 3(ii) there is a continuum \( K \) such that

\[ p_{x_0} \cap \text{Int} K = K \subset B(p_{x_0}, 0). \]

By (1) and (3) there is a natural number \( n_1 \) such that \( x_n \in K \) and \( x_n \notin K \). Let \( A \) be a non-degenerate subarc of the arc \( x_0z_n \subset g_0z_0 \) such that \( A \subset x_0z_n \backslash (K \cup \{ z_n \}) \) and denote by \( a \) and \( b \) the endpoints of \( A \) (where \( q < a < b \) in the natural ordering of the arc \( g_0 \) from \( q \) to \( x_n \)). We have two arcs \( qa \) and \( qb \) such that

\[ qa \in qg_0, \]

\[ qb \in qg_0 \cup x_0z_n \cup K \cup y_0x_0. \]
where $bY_0$ is an arc in $qX_0$. We define $ra = rq \cup qa$ and $rb = rq \cup qb$. Since $ra \cap ab = \{a\}$ and $rb \cap ab = \{b\}$, we have $a = b$ by Theorem 4 and (4). This contradicts the choice of $a$ and $b$.

Therefore any cluster point of the sequence $(z_n)$ belongs to the arc $y_0X_0$. We define $R_n = y_0a \cup y_0z_n \cup z_nx_n$, where $y_0z_n$ is an arc in $qX_0$. Then $\lim_{n \to \infty} R_n = y_0p \cup y_0x_n \cup y_0z_n = y_0z_n$ by Lemma 5. Therefore $\lim_{n \to \infty} R_n = y_0x_n$ by Proposition 2. Since the continuum $R_n$ is contained in $Q$ by the construction for each $n = 1, 2, ...$, the required condition is satisfied. The proof of Theorem 5 is complete.

**Corollary 8.** A continuum $X$ is hereditarily locally connected if and only if $HI(X) \neq \emptyset$ and $X$ is locally connected.

Indeed, if $X$ is hereditarily locally connected, then $HI(X) = X$ by Corollary 3'. In particular $HI(X) \neq \emptyset$ and $X$ is locally connected. Conversely, if $X$ is locally connected and $HI(X) = \emptyset$, then $I(X) = X$ by Corollary 3 and $HI(X) = X$ by Theorem 5, because local connectedness implies arcwise connectedness. Therefore $X$ is hereditarily locally connected by Corollary 3'.

**Corollary 9.** For every continuum $X$ the equality $HI(X) = X$ holds if and only if $I(X) = X$ and $HI(X) \neq \emptyset$.

**Theorem 6.** Let a continuum $X$ be arcwise connected. If $p, q \in HI(X)$ and if $pq$ is an arbitrary arc in $X$ with endpoints $p$ and $q$, then $pq \in HI(X)$.

**Proof.** By Theorem 3, $X$ is hereditarily arcwise connected. Take an arbitrary point $r$ of $pq$ and a convergent sequence $(z_n)$ of points of $X$. Put

$$\lim_{n \to \infty} z_n = x_0,$$

and let $R_n$ be an arbitrary arc with endpoints $r$ and $x_0$. Denote by $y_0$, such a point a point of $rX_0$ that if $y_0x_0$ is an arc in $rX_0$ then $pq \cap y_0x_0 = \{y_0\}$. Let $pr$ and $rq$ denote the arcs in $pq$. Assume $y_0 \in pr$ (if $y_0 \in rq$ the proof is the same). Since $X$ is smooth at $p$, there exists a sequence $(px_n)$ of arcs such that

$$\lim_{n \to \infty} px_n = y_0p \cup y_0x_0,$$

where $y_0p$ is an arc in $pq$. Take a sequence $(z_n)$ of points of arcs $px_n$ such that if $z_nx_0$ is an arc in $px_n$ then $(y_0p \cup y_0x_0) \cap z_nx_0 = \{z_n\}$. Let $z_0$ be an arbitrary cluster point of the sequence $(z_n)$. Suppose that $z_0 \in y_0p \setminus \{y_0\}$. Since $z_0$ is a cluster point of $(z_n)$, there is a subsequence $(z_{n_k})$ of the sequence $(z_n)$ such that

$$\lim_{k \to \infty} z_{n_k} = z_0.$$

Then, by Lemma 4, we infer that

any proper subarc of the arc $z_0x_0$ in $y_0p \cup y_0x_0$ is a continuum of convergence of $X$.

Let $e = \frac{1}{2}(z_0, y_0q \cup y_0x_0)$, where $y_0q$ is the arc in $pq$. Since $X$ is smooth at $q$, by Proposition 3(i) there is a continuum $K$ such that

$$y_0q \cup y_0x_0 \subseteq IntK = K_0 = \{y_0q \cup y_0x_0, e\}.$$

By (1) and (3) we conclude that there is a natural number $n_1$ such that $x_n \in K$ and $z_0 \notin K$. Let $A$ be an arbitrary arc in $y_0p \cup y_0x_0$, such that $A = x_n \setminus (K \cup \{z_0\})$ and denote by $a$ and $b$ the endpoints of $A$ (if $a < c < b$ in the natural ordering of the arc $y_0p \cup y_0x_0$ from $p$ to $x_0$). We have two arcs $qa$ and $qb$ such that

$$\lim_{n \to \infty} q \in qa,$$

$$\lim_{n \to \infty} q \in qb,$$

$$qa \cup K \subseteq qa \cup x_nz_n \cup z_na,$$

where $z_na$ is an arc in $pq$.

Since $qa \cap A = \{a\}$ and $qb \cap A = \{b\}$, by Theorem 4 and (4) $a = b$, which contradicts the choice of $a$ and $b$.

Therefore, any cluster point of the sequence $(z_n)$ belongs to the arc $y_0x_0$. By Lemma 5, we have

$$\lim_{n \to \infty} z_nx_0 \subseteq y_0x_0.$$

Let $rX_0$ be the arc in $rX_0$. We define $R_n = rX_0 \cup y_0z_n \cup z_nx_0$, where $y_0z_n$ is an arc in $y_0p \cup y_0x_0$. Then

$$\lim_{n \to \infty} R_n = rX_0 \cup y_0z_n \cup z_nx_0 \subseteq y_0x_0 \cup z_nx_0,$$

Therefore, by Proposition 2, we have $\lim_{n \to \infty} R_n = rX_0$.

Using the hereditary arcwise connectedness of $X$, we infer by Proposition 3(i) that $X$ is smooth at $r$. Therefore $X$ is hereditarily smooth at $r$ by Theorem 5. Thus $pq \in HI(X)$. The proof of Theorem 6 is complete.

**Theorem 7.** Let a continuum $X$ be arcwise connected and let $p, q \in X$. If $X$ is hereditarily smooth at the point $p$ and if $X$ is locally connected at each point of an arc $pq$, then $X$ is smooth at $q$.

**Proof.** By Theorem 3 we conclude that the continuum $X$ is hereditarily arcwise connected, i.e., all irreducible continua in $X$ are arcs. Let $(x_n)$ be an arbitrary sequence of points of $X$ such that

$$\lim_{n \to \infty} x_n = x_0,$$

and let $qX_0$ be an arbitrary arc joining $q$ and $x_0$. Denote by $y_0$ the point of the arc $qX_0$ such that $y_0x_0$ is an arc in $qX_0$ then $y_0x_0 \cap pq = \{y_0\}$. Since $X$ is smooth at $p$ and (1) holds, there is a sequence $(px_n)$ of arcs of $X$ such that

$$\lim_{n \to \infty} px_n = y_0p \cup y_0x_0.$$
where $p_{y_{0}}$ is an arc in $pq$. Since $X$ is locally connected in $y_{0}$, we have, for each natural number $j$, a continuum $K_{j}$ such that

$$y_{0} \in \operatorname{Int} K_{j} \subseteq K_{j} \subseteq B(y_{0}, 1/j).$$

Take for each $n = 1, 2, \ldots$, the point $x_{n}$ of $p_{x_{n}}$ such that if $x_{n} \neq x_{0}$ is an arc in $p_{x_{n}}$ then $x_{n} \cap (p_{x_{0}} \cup y_{0} x_{0}) = \{x_{n}\}$. If $x_{n} \notin y_{0} x_{0}$, then we define

$$q_{y_{0}} = y_{0} x_{0} \cup y_{0} x_{n} \cup x_{n},$$

where $q_{y_{0}}$ is an arc in $q_{y_{0}}$, and $y_{0} x_{0}$ is an arc in $y_{0} x_{0}$. If $x_{n} \notin y_{0} x_{0}$, then the arc $x_{n} y_{0}$ intersects the continuum $K_{i}$ for some $i$. Therefore we have, for a subsequence $\{x_{n}\}$ of the sequence $\{x_{n}\}$ such that $x_{n} \notin y_{0} x_{0}$, a sequence of indexes $\{i_{n}\}$ such that

$$\lim_{n \to \infty} i_{n} = 0,$$

and

$$x_{n} y_{0} \cap K_{i_{n}} \neq \emptyset.$$

Let $x_{n}$ be a point in the arc $x_{n} y_{0}$ such that if $a_{n} x_{n}$ is an arc in the arc $x_{n} y_{0}$, then $x_{n} y_{0} \cap K_{i_{n}} \neq \emptyset$. Since $X$ is hereditarily arcwise connected, there are arcs $y_{0} a_{n}$ contained in $K_{i_{n}}$. Obviously, by (3) we have

$$\lim_{n \to \infty} y_{0} a_{n} = y_{0}.$$

Consider the continua $A_{n} = (p_{y_{0}} \cup y_{0} x_{0}) \cup a_{n} x_{n}$. Obviously $\lim A_{n} = p_{y_{0}} \cup y_{0} x_{0}$. A continuum $A_{n}$ contains an arc $B_{n}$ irreducible between $x_{n}$ and $y_{0}$, and the first point in the arc $B_{n}$ which belongs to $p_{y_{0}} \cup y_{0} x_{0}$ (in the natural order in $p_{y_{0}} \cup y_{0} x_{0}$ from $x_{n}$ to $y_{0}$) is contained in $K_{i_{n}}$. Therefore we can assume that any cluster point of the sequence $\{x_{n}\}$ is contained in the arc $y_{0} x_{0}$, because we can consider arcs $B_{n}$, instead of arcs $p_{x_{n}}$. Thus, by Lemma 5, we have

$$\lim_{n \to \infty} x_{n} y_{0} = y_{0} x_{0},$$

and

$$\lim_{n \to \infty} x_{n} y_{0} x_{0} = y_{0} x_{0}.$$
It follows from (1) and (5) that \( x_n \in K \) for \( n > n_0 \), and \( z_n \notin \text{cl} \) for \( n > n_1 \). Let \( m > n_0 \) and \( m > n_1 \).

Take the arcs \( gc \) contained in \( gx_0 \) and \( qd \) contained in \( px_a \cup z_0 x_0 \cup K \cup yd \), where \( px_a \) and \( yd \) are arcs in \( px_0 \). Then \( pg \cup gc \) and \( pq \cup qd \) are both irreducible between \( p \) and the continuum of convergence \( \text{cl} D(\epsilon, f(4)) \); thus, by Theorem 4 we have \( c = d \) — a contradiction. The proof of Theorem 8 is complete.

**Theorem 9.** Let a continuum \( X \) be arcwise connected and \( HH(X) \neq \emptyset \). If \( S \subset X \) is a simple closed curve and \( p \) is an arbitrary arc which is irreducible between \( p \) and \( S \), where \( p \in HH(X) \), then \( N(X) \cap S = \{q\} \).

Proof. By Theorem 3 the continuum \( X \) is hereditarily arcwise connected. Suppose, on the contrary, that \( x_0 \in N(X) \cap S \) and \( x_0 \neq q \). Let \( x_0 \) be one of two arcs in \( S \) irreducible between \( x_0 \) and \( q \). The continuum \( X \) is not locally connected at \( x_0 \); therefore there is a closed neighborhood \( E \) of the point \( x_0 \) such that if \( C \) is a component of \( E \) which contains \( x_0 \), then \( x_0 \in E \cap C \). We infer that there is a sequence \( \{x_n\} \) of points of \( X \) such that

\[
\lim_{n \to \infty} x_n = x_0,
\]

\[
x_n \in E \cap C.
\]

Let \( p \in HH(X) \). Since \( X \) is smooth at the point \( p \) and (1) holds, there is a sequence \( \{\alpha_{x_n}\} \) of arcs of \( X \) such that

\[
\lim_{n \to \infty} \alpha_{x_n} = \alpha = \alpha_{x_0} \cup \text{cl} E.
\]

Take, for each \( n = 1, 2, \ldots \), a point \( z_n \) of the arc \( \alpha_{x_n} \) such that if \( z_n x_n \) is an arc in \( \alpha_{x_n} \), then \( z_n x_n \cap (\alpha_{x_0} \cup \alpha_{x_0}) = \{z_n\} \). Let \( z_0 \) be a cluster point of the sequence \( \{z_n\} \). Then there is a subsequence \( \{z_{n_i}\} \) of the sequence \( \{z_n\} \) such that

\[
\lim_{i \to \infty} z_{n_i} = z_0.
\]

There is an arc \( \alpha_{z_0} \) in the arc \( \alpha_{x_0} \) such that \( x_0 \notin \alpha_{z_0} \) and \( q \notin \alpha_{x_0} \cap \text{Int} E \). If \( z_n \in \alpha_{x_0} \setminus \{z\} \), then, by Lemma 5, \( x_n \in \text{Int} E \) for indexes \( i \) larger than some \( n_0 \), because \( \lim_{i \to \infty} x_{n_i} = z_0 x_0 \), where \( z_0 x_0 \) is an arc in \( \alpha_{x_0} \). Thus \( x_n \in C \) for \( i > n_0 \) — a contradiction. Therefore \( z_0 \notin \alpha_{x_0} \setminus \{a\} \). This implies that

(5) the arc \( \alpha_{x_0} \) is a continuum of convergence of \( X \).

Take the arc \( qg \) in \( x_0 q (q \neq g) \), and the arc \( I(x_0, q) \), irreducible between \( x_0 \) and \( q \), which is contained in \( S \cap x_0 q \). Then \( \alpha_{x_0} \cup \alpha_{x_0} \cup \alpha_{x_0} \cup \alpha_{x_0} \cup \alpha_{x_0} \cup \alpha_{x_0} \) arc both irreducible between \( p \) and the continuum \( \alpha_{x_0} \). It follows from (5) and Theorem 4 that \( a = q \) — a contradiction. The proof of Theorem 9 is complete.

**Corollary 11.** Let a continuum \( X \) be arcwise connected and \( HH(X) \neq \emptyset \). The continuum \( X \) is a smooth dendroid if and only if for each continuum \( C \) of the set \( X \cap N(X) \) the closure \( C \) is a dendroid.

Indeed, by Theorem 3, \( X \) is hereditarily arcwise connected. If \( X \) is a dendroid, then any subcontinuum of \( X \) is a dendroid. In particular, the closure of any continuum \( C \) of the set \( X \cap N(X) \) is a dendroid.

Conversely, if for each continuum \( C \) of the set \( X \cap N(X) \) the closure \( C \) is a dendroid, then, by Theorem 9, \( X \) fails to contain a simple closed curve. Therefore, by the hereditary arcwise connectedness of \( X \) (cf. Theorem 3), \( X \) is a dendroid.

**References**


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