Arcwise connected and hereditarily smooth continua

by

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Abstract. We say that X is *smooth* at the point $p \in X$ if for each convergent sequence $x_1, x_2, \ldots$ of points of X and for each subcontinuum K of X such that $p, x \in K$, where $x = \lim_{n\to\infty} x_n$, there is a sequence $K_1, K_2, \ldots$ of subcontinua of X such that $p, x_n \in K_n$ for each $n = 1, 2, \ldots$ and $\operatorname{Lim}_{n\to\infty} K_n = K$. The set of all points of a continuum X at which X is smooth is denoted by $I(X)$. A continuum X is said to be *hereditarily smooth at p* provided each subcontinuum of X which contains p is smooth at p. The set of all points of a continuum X at which X is hereditarily smooth is denoted by $HI(X)$. It is proved that if a continuum X is arcwise connected and $HI(X) \neq \emptyset$ then X is hereditarily arcwise connected and $HI(X) = I(X)$; and if C is the constituent of the set of all points at which X is locally connected, and $C \cap HI(X) \neq \emptyset$, then $C = HI(X) = I(X)$. Also other properties of an arcwise connected and hereditarily smooth continua are studied in the paper.

§ 1. **Introduction.** The notion of smoothness of continua in a general form has been introduced in [10]. In that paper relations are studied between this notion of smoothness and that which was introduced previously in [5] by Gordh. In particular, it is proved that both notions coincide on metric continua which are either hereditarily unicoherent at some point or irreducible between two points, i.e., any continuum X smooth in the sense of [5] is smooth in the new sense of [10]; and any continuum smooth in the new sense which is either hereditarily unicoherent at some point or irreducible is smooth in the sense of [5]. For example, smooth dendroids (see [2]) are those arcwise connected continua which are smooth in the sense of [5]. The class of arcwise connected continua which are smooth in the sense of [10] is essentially larger than the class of smooth dendroids. Any dendroid X (and, more generally, any continuum X hereditarily unicoherent at some point) is hereditarily smooth at p (see [10], Corollary (7.1); cf. [9], Theorem (2.6)). In this paper we consider arcwise connected continua which are hereditarily smooth at some point.

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§ 2. **Preliminaries.** The topological spaces under consideration will be assumed to be metric and compact. If the space X under consideration is established, then ϱ denotes a metric on X; $B(N, \varepsilon)$ denotes the union of all open metric balls with the centres in a given set N and with the radii $\varepsilon>0$, and ab is an arbitrary arc with end-points a and b.

The numbering of conclusions in the proofs is separate in every proof. If $A_1, A_2, \ldots$ is a sequence of subsets of a space X, then $\operatorname{Li}_{n\to\infty} A_n$ denotes the set of all points $x\in X$ for which every neighbourhood intersects A_n for almost all n, and $\operatorname{Ls}_{n\to\infty} A_n$ denotes the set of all points $x\in X$ for which every neighbourhood intersects A_n for arbitrarily large n. A sequence $A_1, A_2, \ldots$ of subsets of X is said to *converge* to a set A (denoted by $\operatorname{Lim}_{n\to\infty} A_n = A$) in case $\operatorname{Li}_{n\to\infty} A_n = A = \operatorname{Ls}_{n\to\infty} A_n$.

It is known (see [8], § 47, II, Theorem 6, p. 171) that:

PROPOSITION 1. *If $C_1, C_2, \ldots$ is a sequence of subcontinua of the space X such that $\operatorname{Li}_{n\to\infty} C_n \neq \emptyset$, then the set $\operatorname{Ls}_{n\to\infty} C_n$ is a continuum.*

It is proved (see [10], Lemma (2.2)) that:

PROPOSITION 2. *Let $C_1, C_2, \ldots$ be a sequence of subcontinua of the space X and $\{x, y\}\subset \operatorname{Li}_{n\to\infty} C_n$. If $\operatorname{Ls}_{n\to\infty} C_n$ is irreducible between points x and y, then the sequence $C_1, C_2, \ldots$ is convergent.*

We say that X is *smooth* at the point $p\in X$ if for each convergent sequence $x_1, x_2, \ldots$ of points of X and for each subcontinuum K of X such that $p,\ x\in K$, where $x = \lim_{n\to\infty} x_n$, there exists a sequence $K_1, K_2, \ldots$ of subcontinua of X, such that $p,\ x_n\in K_n$ for each $n = 1, 2, \ldots$ and $\operatorname{Lim}_{n\to\infty} K_n = K$ (see [10]).

We have the following characterizations of continua which are smooth at some point (see [10], Theorems (2.4) and (3.1)):

PROPOSITION 3. *The continuum X is smooth at the point $p\in X$ if and only if one of the following conditions holds:*

(i) *for each convergent sequence $x_1, x_2, \ldots$ of points of X and for each irreducible continuum $I(p, x)$ between p and x, where $x = \lim_{n\to\infty} x_n$, there exists a sequence $I(p, x_1),\ I(p, x_2), \ldots$ of irreducible continua between p and x_n, respectively, such that $\operatorname{Lim}_{n\to\infty} I(p, x_n) = I(p, x)$;*

(ii) *for each subcontinuum N of X and for each open set V of X there exists a continuum K such that $p\in N\subset V$ implies $N\subset \operatorname{Int} K\subset K\subset V$.*

We can characterize the smoothness by the notion of nonaposyndeticity of F. B. Jones (see [6], p. 104). Let $A\subset X$. Then we define $X\setminus T(A) = \{x\in X\colon$ there exists a subcontinuum Q of X such that $x\in \operatorname{Int} Q\subset Q\subset X\setminus A\}$ (see [3], p. 113), and put $T^n(A) = T(T^{n-1}(A))$ with $T^0(A) = A$.

COROLLARY 1. *A continuum X is smooth at $p\in X$ if and only if for each continuum N in X such that $p\in N$ and for each closed set A in X the condition $N\cap A = \emptyset$ implies $N\cap T(A) = \emptyset$.*

Indeed, if $N\cap A = \emptyset$, then $N\subset X\setminus A$. Thus, by Proposition 3(ii), there exists a subcontinuum Q of X such that $N\subset \operatorname{Int} Q\subset Q\subset X\setminus A$. Therefore $N\subset X\setminus T(A)$ by the definition of $T(A)$, i.e., $N\cap T(A) = \emptyset$.

Conversely, let N be an arbitrary continuum in X such that $p\in N$ and let V be an open set in X containing N. Then there is a closed set A in X such that $N\subset X\setminus A\subset \overline{X\setminus A}\subset V$. We have $N\cap T(A) = \emptyset$, i.e., for each $x\in N$, there is a continuum Q_x such that $x\in \operatorname{Int} Q_x\subset Q_x\subset X\setminus A$. Put

$$Q = \overline{\bigcup\{Q_x\colon x\in N\}}\,.$$

It is easy to verify that the set Q is a continuum satisfying $N\subset \operatorname{Int} Q\subset Q\subset \overline{X\setminus A}\subset V$. Thus condition (ii) from Proposition 3 holds, i.e., X is smooth at p.

Since $T(A)$ is closed (see [3], Lemma 1, p. 114) we have by Corollary 1.

COROLLARY 1′. *Let a continuum X be smooth at $p\in X$ and let N be a subcontinuum of X such that $p\in N$. If the set A is closed and $N\cap A = \emptyset$, then $N\cap T^n(A) = \emptyset$ for each $n = 0, 1, 2, \ldots$*

The set of all points of an arbitrary continuum X at which X is smooth is called the *initial set* of X and is denoted by $I(X)$. If $I(X) \neq \emptyset$, then X is said to be *smooth*.

The next two corollaries are easy consequences of Proposition 3.

COROLLARY 2. *A continuum X is locally connected at each point of $I(X)$.*

COROLLARY 3. *A continuum X is locally connected if and only if $I(X) = X$.*

A continuum X is said to be *hereditarily smooth at p* provided each subcontinuum of X which contains p is smooth at p. The set of all points of an arbitrary continuum X at which X is hereditarily smooth is called the *hereditarily initial set* of X and is denoted by $HI(X)$. If $HI(X) \neq \emptyset$, then X is said to be *hereditarily smooth*. We have

COROLLARY 2′. *For each subcontinuum Q of X, Q is locally connected at each point of the set $Q\cap HI(X)$.*

COROLLARY 3′. *A continuum X is hereditarily locally connected if and only if $HI(X) = X$.*

§ 3. Arcwise connected continua. The main result of this section says that any arcwise connected and hereditarily smooth continuum is hereditarily arcwise connected. Firstly we prove

THEOREM 1. *Let an arcwise connected continuum X be hereditarily smooth at a point $p\in X$, let Q be an arbitrary subcontinuum of X and let pq be an arc in X which is irreducible between p and Q. Then the continuum Q is hereditarily smooth at the point q.*

Proof. We have $pq\cap Q = \{q\}$. Let K be an arbitrary subcontinuum of Q such that $q\in K$. Then $pq\cap K = \{q\}$. We will show that K is smooth at q. Let $x_1, x_2, \ldots$ be a convergent sequence of points of K and put $x = \lim_{n\to\infty} x_n$. Let P be a subcontinuum of K such that $x, q \in P$. X being hereditarily smooth at p, the continuum $pq \cup K$ is smooth at p. Therefore there is a sequence $R_1, R_2, \ldots$ of subcontinua of $pq \cup K$ such that $x_n, p \in R_n$ for each $n = 1, 2, \ldots$ and $\operatorname{Lim}_{n\to\infty} R_n = pq \cup P$ by the definition of smoothness. We define $P_n = K \cap R_n$. Obviously P_n is a continuum for each $n = 1, 2, \ldots$ Moreover $x_n, q \in P_n \subset K$ for each $n = 1, 2, \ldots$ and $\operatorname{Lim}_{n\to\infty} P_n = P$. The proof of Theorem 1 is complete.

COROLLARY 4. *If X is a hereditarily smooth arcwise connected continuum, then any subcontinuum of X is also hereditarily smooth.*

Recall that a continuum X is said to be *decomposable* if there is a decomposition of X into two proper subcontinua. A continuum is said to be *hereditarily decomposable* if any subcontinuum of it is decomposable.

COROLLARY 5. *Any hereditarily smooth arcwise connected continuum is hereditarily decomposable.*

This is obvious if we observe that, by Corollaries 2 and 4, any subcontinuum of hereditarily smooth arcwise connected continuum is locally connected at some point.

It is well known that for every irreducible continuum X there exists an upper semi-continuous decomposition of X into continua (called *layers* of X) (see [8], § 48, IV, p. 199) with the property that the decomposition of X into layers is the finest of all linear upper semi-continuous decompositions of X into continua ([8], § 48, IV, Theorem 3, p. 200, [7], Fundamental theorem, p. 259). If each layer of X has a void interior, then X is said to be of *type* λ (see [8], § 48, III, p. 197, the footnote, and also [11], Definition 4, p. 13, where these continua are said to be of type A'). It is well known (see [8], § 48, VII, Theorem 3, p. 216; [11], Theorem 10, p. 15; [4], Theorem 2.7, p. 650) that an irreducible continuum X is of type λ if and only if each indecomposable subcontinuum of X has a void interior. Thus, by Corollary 5, we have

COROLLARY 6. *Any irreducible subcontinuum of a hereditarily smooth arcwise connected continuum is of type* λ (in fact, it is an arc — see Theorem 3 below).

Recall that a subcontinuum K of X is called a *continuum of convergence* (see [12], p. 127, cf. [8], § 47, VI, p. 245) provided K is a topological limit of the sequence of continua such that

$$K = \operatorname{Lim}_{n\to\infty} K_n \quad \text{and} \quad K \cap K_n = \emptyset \quad \text{for each } n = 1, 2, \ldots$$

If X is compact, then we can assume that $K_1, K_2, \ldots$ are mutually disjoint. We have

THEOREM 2. *Let X be an arcwise connected continuum which is hereditarily smooth at the point $p \in X$. If K_0 is a continuum of convergence in X and pc is an arbitrary arc, then $K_0 \cap pc$ is connected.*

Proof. Suppose, on the contrary, that $K_0 \cap pc$ is not connected. Then there is an arc a_0b_0 in pc such that

(1) $$a_0b_0 \cap K_0 = \{a_0, b_0\} \quad \text{and} \quad a_0 \neq b_0 .$$

Obviously we can assume $a_0 \in pb_0$. Since K_0 is a continuum of convergence in X, K_0 is a topological limit of the sequence of continua such that

(2) $$K_0 = \operatorname{Lim}_{n\to\infty} K_n \quad \text{and} \quad K_m \cap K_n = \emptyset \quad \text{for each } m \neq n \text{ and } m, n = 0, 1, 2, \ldots$$

Therefore there are sequences $\{a_n\}$ and $\{b_n\}$ of points of X such that

(3) $$\lim_{n\to\infty} a_n = a_0 \quad \text{and} \quad \lim_{n\to\infty} b_n = b_0,$$

(4) $$a_n, b_n \in K_n \quad \text{for each } n = 0, 1, 2, \ldots$$

Let pa_0 be the arc in pc. So we have $pa_0 \cap K_0 \subset X\setminus\{b_0\}$. Let ε be a positive number such that $\varepsilon < 1/2\varrho(b_0, pa_0)$. Since X is smooth at p, by Proposition 3 (ii) there is a continuum Q in X such that

(5) $$pa_0 \subset \operatorname{Int} Q \subset Q \subset B(pa_0, \varepsilon) \subset X\setminus\{b_0\} .$$

By (3) and (5), and by the choice of ε, we can assume that

(6) $$a_n \in Q \subset X\setminus\{b_n\} \quad \text{for each } n = 1, 2, \ldots$$

For each $n = 1, 2, \ldots$ take in K_n the continuum $I(d_n, b_n)$ irreducible between Q and b_n. Let eb_0 be an arc in a_0b_0 such that $eb_0 \cap Q = \{e\}$. It suffices to consider only two cases.

1′. $I(d_n, b_n) \cap eb_0 = \emptyset$ for each $n = 1, 2, \ldots$ (or there is a subsequence $I(d_{n_k}, b_{n_k})$ of the sequence $I(d_n, b_n)$ such that $I(d_{n_k}, b_{n_k}) \cap cb_0 = \emptyset$ for each $k = 1, 2, \ldots$, but then the proof is the same). Then we consider the following continuum

$$R = Q \cup K_0 \cup \bigcup_{n=1}^{\infty} I(d_n, b_n) .$$

Since X is hereditarily smooth at p, R is smooth at p. Thus, by (3), there is a sequence of continua R_n in R such that

(7) $$p, b_n \in R_n \quad \text{for each } n = 1, 2, \ldots$$

and

(8) $$\operatorname{Lim}_{n\to\infty} R_n = pa_0 \cup a_0b_0 .$$

But for each $n = 1, 2, \ldots$ we have

(9) $$I(d_n, b_n) \subset R_n .$$

Indeed, by Corollary 6 the irreducible continua $I(d_n, b_n)$ are of type λ, and by the definition of R any layer of $I(d_n, b_n)$ separates R between b_n and p. Thus any layer of $I(d_n, b_n)$ is contained in R_n, i.e., (9) holds.

Therefore, by (7) and (9) the set $\operatorname{Lim}_{n\to\infty} R_n$ contains some irreducible continuum between b_0 and Q, which is contained in K_0, contrary to (8).

2′. $I(d_n, b_n) \cap eb_0 \neq \emptyset$ for each $n = 1, 2, \ldots$ Then we can take, by Corollary 6, irreducible continua $I(d_n, z_n)$ in $I(d_n, b_n)$ such that

(10) $$z_n \in eb_0 ,$$

and

(11) no proper subcontinuum of $I(d_n, z_n)$ containing d_n intersects eb_0.

(2) and (10) imply that $\lim_{n\to\infty} z_n = b_0$. By the standard construction we can take, by (11), irreducible continua $I(d_n, x_n)$ in $I(d_n, z_n)$ such that $\lim_{n\to\infty} x_n = b_0$, and $I(d_n, x_n) \cap eb_0 = \emptyset$ for each $n = 1, 2, \ldots$ Then we obtain a contradiction as in case 1′. The proof of Theorem 2 is complete.

Let an irreducible continuum X be of type λ and let T_t, $t \in [0, 1]$, denote a layer of X. Thus $X = \bigcup \{T_t\colon 0 \leqslant t \leqslant 1\}$. Put

$$L_t = \bigcup \{T_u\colon 0 \leqslant u < t\} \quad \text{and} \quad R_t = \bigcup \{T_v\colon t < v \leqslant 1\} .$$

Therefore

$$L_t = \varphi^{-1}([0, t)) \quad \text{and} \quad R_t = \varphi^{-1}((t, 1]) ,$$

where φ is the canonical mapping from X to the unit interval $[0, 1]$; we see that both L_t and R_t are connected. (Here the capital letters L and R stand for left and right, respectively).

Adopt the following definitions (see [1], p. 46). A layer T_t is said to be a layer of *left cohesion* if either $t = 0$ or $T_t = \overline{L}_t \setminus L_t$; and T_t is said to be a layer of *right cohesion* if either $t = 1$ or $T_t = \overline{R}_t \setminus R_t$. One can see that T_0 is a layer of right cohesion (T_1 is a layer of left cohesion) provided the interior of T_0 (T_1) is empty. A lyaer T_t is said to be a layer of *cohesion* if it is a layer of both left and right cohesions (see [7], p. 260; [8], § 48, IV, p. 201). We have the following (see [1], Theorem, p. 48)

PROPOSITION 4. *An irreducible continuum X is smooth at a point p if and only if all three of the following conditions are satisfied*:

(i) *X is locally connected at p,*

(ii) *for each t satisfying $0 \leqslant t < \varphi(p)$ the layer T_t is of right cohesion,*

(iii) *for each t satisfying $\varphi(p) < t \leqslant 1$ the layer T_t is of left cohesion.*

LEMMA 1. *For each two points x_0 and y_0 of an arbitrary layer T_{t_0} of an irreducible smooth continuum X, there exists a continuum of convergence K_0 such that $\{x_0, y_0\} \subset K_0 \subset T_{t_0}$.*

Proof. By Proposition 4, the layer T_{t_0} is either of right or of left cohesion. Suppose that T_{t_0} is a layer of left cohesion (if T_{t_0} is a layer of right cohesion the proof is the same). Then either $t_0 = 0$ or $T_{t_0} = \overline{L}_{t_0} \setminus L_{t_0}$. If $t_0 = 0$, then T_0 is a layer of right cohesion, and the proof is the same as for layers of right cohesion. If $t_0 \neq 0$, then there are sequences $\{x_n\}$ and $\{y_n\}$ such that

(1) x_n and y_n belongs to L_{t_0} for each $n = 1, 2, \ldots$,

(2) $$\lim_{n\to\infty} x_n = x_0 \quad \text{and} \quad \lim_{n\to\infty} y_n = y_0 .$$

Let φ be the canonical mapping from X to the unit interval $[0, 1]$. We can assume that for each $n = 1, 2, \ldots$

(3) $$\varphi(x_n) \leqslant \varphi(y_n) < \varphi(x_{n+1}) < t_0 .$$

Put

$$K_n = \varphi^{-1}([\varphi(x_n), \varphi(y_n)]) \quad \text{for each } n = 1, 2, \ldots$$

Obviously, by (3), for each $n = 1, 2, \ldots$ the set K_n is a continuum and $K_n \cap T_{t_0} = \emptyset$, and by (1), we have $\{x_0, y_0\} \subset \operatorname{Ls}_{n\to\infty} K_n \subset T_{t_0}$. We can assume that the sequence $\{K_n\}$ is convergent. Then $K_0 = \operatorname{Lim}_{n\to\infty} K_n$ is a continuum of convergence, and $\{x_0, y_0\} \subset K_0 \subset T_{t_0}$. The proof of Lemma 1 is complete.

COROLLARY 7. *Let a continuum X be arcwise connected and hereditarily smooth at p. For each layer T of an arbitrary irreducible subcontinuum A of X and for each arc pc in X the set $pc \cap T$ is connected.*

Proof. By Corollary 4, A is an irreducible smooth continuum. Let a be an arbitrary point of $pc \cap T$. Therefore by Lemma 1, for each $y \in T$ there is a continuum of convergence K_y such that $\{a, y\} \subset K_y \subset T$. Thus $T = \bigcup \{K_y\colon y \in T\}$ and $pc \cap T = \bigcup \{K_y \cap pc\colon y \in T\}$. But $K_y \cap pc$ is connected by Theorem 2, and $a \in K_y \cap pc$ for each $y \in T$. This implies that the set $pc \cap T$ is connected (see [8], § 46, II, Corollary 3(i), p. 132).

LEMMA 2. *Let $I(a, b)$ be an irreducible continuum between a and b which is smooth at the point d, and let $I(c, d)$ be an irreducible subcontinuum of $I(a, b)$. If T is a layer of $I(c, d)$, then T is a layer of $I(a, b)$.*

Proof. Let φ be the canonical map from $I(a, b)$ onto $I = [0, 1]$ such that $\varphi(a) = 0$. Suppose that $\varphi(c) \leqslant \varphi(d)$ (if $\varphi(c) \geqslant \varphi(d)$ the proof is the same). It follows from Theorem (5.3) in [10] that $I(a, b)$ is hereditarily unicoherent at d; thus $I(c, d) \subset \varphi^{-1}([\varphi(c), \varphi(d)])$. Consider the continuum

$$K = \varphi^{-1}([0, \varphi(c)]) \cup I(c, d) \cup \varphi^{-1}([\varphi(d), 1]) .$$

Since $a, b \in K$, we have $K = I(a, b)$. Therefore $\varphi^{-1}((\varphi(c), \varphi(d))) \subset I(c, d)$. Thus

$$\varphi^{-1}([\varphi(c), \varphi(d)]) = \overline{\varphi^{-1}((\varphi(c), \varphi(d)))} \subset I(c, d)$$

by Proposition 4. We infer $I(c, d) = \varphi^{-1}([\varphi(c), \varphi(d)])$. This equality implies the conclusion of the lemma.

LEMMA 3. *Let $I(c, d)$ be an irreducible continuum between c and d, which is smooth at d, and let φ be the canonical map from $I(c, d)$ onto $I = [0, 1]$ such that $\varphi(c) = 0$. If $I(x_0, y_0)$ is an irreducible subcontinuum of $I(c, d)$ such that $\varphi(x_0) < \varphi(y_0)$, then the set $\varphi^{-1}(\varphi(x_0))$ is a layer of $I(x_0, y_0)$.*

Proof. Let $I(x_0, y_0)$ be an irreducible subcontinuum of $I(c, d)$ such that $\varphi(x_0) < \varphi(y_0)$. Consider the continuum $K = \varphi^{-1}([0, \varphi(x_0)]) \cup I(x_0, y_0) \cup \varphi^{-1}([\varphi(y_0), 1])$. Since $c, d \in K$, we have $K = I(c, d)$. Therefore

$$\varphi^{-1}\big((\varphi(x_0), \varphi(y_0))\big) \subset I(x_0, y_0).$$

Thus $\varphi^{-1}(\varphi(x_0)) \subset I(x_0, y_0)$, and the set $\varphi^{-1}(\varphi(x_0))$ is nowhere dense in $I(x_0, y_0)$ by Proposition 4. This implies by Theorem 7 in [8], § 48, II, p. 194, that the continuum $I(x_0, y_0)$ is irreducible between each point of the set $\varphi^{-1}(\varphi(x_0))$ and y_0. Moreover, since for each $\varphi(x_0) < t < \varphi(y_0)$ the set $\varphi^{-1}(t)$ separates $I(x_0, y_0)$, we conclude that $\varphi^{-1}(\varphi(x_0))$ is the set of all points a of $I(x_0, y_0)$ such that $I(x_0, y_0)$ is irreducible between a and y_0. Therefore $\varphi^{-1}(x_0)$ is a layer of $I(x_0, y_0)$ (cf. [8], § 48, IV, Theorem 4, p. 202).

THEOREM 3. *If an arcwise connected continuum X is hereditarily smooth, then X is hereditarily arcwise connected.*

Proof. It suffices to prove that any irreducible continuum in X is an arc. Let $I(a, b)$ be an arbitrary subcontinuum of X irreducible between given points a and b. Then $I(a, b)$ is of type λ, by Corollary 6. Therefore it suffices to show that any layer of $I(a, b)$ is degenerate. Suppose, on the contrary, that there is a nondegenerate layer T of $I(a, b)$. Let $p \in HI(X)$. Since X is arcwise connected, there is an arc pc in X such that

(1) $$pc \cap T = \{c\}.$$

If $pc \cap I(a, b) = \{c\}$, then the continuum $I(a, b)$ is smooth at c, by Theorem 1. Thus $I(a, b)$ is locally connected at c. This implies that the layer T_c of the point c in $I(a, b)$ is degenerate. But $c \in T$, and hence $T_c = T$, i.e., T is degenerate — a contradiction.

Therefore we consider the remaining case, namely that of $pc \cap I(a, b) \neq \{c\}$. Take an arc pd in the arc pc such that

(2) $$pd \cap I(a, b) = \{d\}.$$

Then $I(a, b)$ is smooth at d by Theorem 1; and thus, by Lemma 2, if we take the continuum $I(c, d)$ in $I(a, b)$ irreducible between c and d, then the layer of the point c in $I(c, d)$ coincides with T by (1). Let φ be the canonical mapping from $I(c, d)$ to the unit interval $[0, 1]$ such that

(3) $$\varphi^{-1}(0) = T \quad \text{and} \quad \varphi^{-1}(1) = \{d\},$$

and let cd mean the subarc of the arc pc.

We define for each $t \in [0, 1]$ the numbers $\alpha(t)$ and $\beta(t)$ by conditions:

(4) $$0 \leqslant \alpha(t) \leqslant t, \quad \varphi^{-1}(\alpha(t)) \cap cd \neq \emptyset \quad \text{and} \quad \varphi^{-1}\big((\alpha(t), t)\big) \cap cd = \emptyset,$$

(5) $$t \leqslant \beta(t) \leqslant 1, \quad \varphi^{-1}(\beta(t)) \cap cd \neq \emptyset \quad \text{and} \quad \varphi^{-1}\big((t, \beta(t))\big) \cap cd = \emptyset,$$

if $\varphi^{-1}(t) \cap cd = \emptyset$; $\alpha(t) = \beta(t) = t$ if $\varphi^{-1}(t) \cap cd \neq \emptyset$.

It is easy to check that the following conditions are satisfied.

(6) $$\alpha(t) \neq t \quad \text{if and only if} \quad \beta(t) \neq t,$$

(7) $$\alpha(\alpha(t)) = \alpha(t) = \beta(\alpha(t)) \quad \text{and} \quad \alpha(\beta(t)) = \beta(t) = \beta(\beta(t)),$$

(8) if $t' \in (\alpha(t), \beta(t))$, the $\alpha(t') = \alpha(t)$ and $\beta(t') = \beta(t)$,

(9) if $t' \notin [\alpha(t), \beta(t)]$ and $\alpha(t) \neq \beta(t)$, then $(\alpha(t'), \beta(t')) \cap [\alpha(t), \beta(t)] = \emptyset$.

Moreover,

(10) if $\alpha(t) \neq t$, then $\overline{\varphi^{-1}\big((\alpha(t), \beta(t))\big)} \setminus \varphi^{-1}\big((\alpha(t), \beta(t))\big) \subset pc$.

Indeed, observe firstly that

$$\overline{\varphi^{-1}\big((\alpha(t), \beta(t))\big)} \setminus \varphi^{-1}\big((\alpha(t), \beta(t))\big) \subset \varphi^{-1}(\alpha(t)) \cup \varphi^{-1}(\beta(t)).$$

Since $I(c, d)$ is smooth at d, the layer $\varphi^{-1}(\alpha(t))$ is of right cohesion by Proposition 4. Therefore by Lemma 3 the set $\varphi^{-1}(\alpha(t))$ is a layer of an irreducible continuum $I(x_0, y_0)$ in $I(c, d)$ such that $x_0 \in \varphi^{-1}(\alpha(t))$ and $y_0 \in \varphi^{-1}(t)$. Then $pc \cap I(x_0, y_0) = pc \cap \varphi^{-1}(\alpha(t))$ by the definition of $\alpha(t)$ (see (4)), thus $I(x_0, y_0)$ is smooth at some point of $\varphi^{-1}(\alpha(t))$ by Theorem 1. We infer that $I(x_0, y_0)$ is locally connected at some point of $\varphi^{-1}(\alpha(t))$; hence

(10′) $\varphi^{-1}(\alpha(t))$ is a one-point set.

Suppose now that

$$z_0 \in \Big\{\Big[\overline{\varphi^{-1}\big((\alpha(t), \beta(t))\big)} \setminus \varphi^{-1}\big((\alpha(t), \beta(t))\big)\Big] \cap \varphi^{-1}(\beta(t))\Big\} \setminus pc.$$

Since $I(c, d)$ is smooth at d, the layer $\varphi^{-1}(\beta(t))$ is of right cohesion by Proposition 4. If

$$\overline{\varphi^{-1}\big((\alpha(t), \beta(t))\big)} \cap \varphi^{-1}(\beta(t)) \cap pc \neq \emptyset,$$

then there is a sequence $\{z_n'\}$ of points of $\varphi^{-1}\big((\alpha(t), \beta(t))\big)$, i.e.,

$$\varphi(z_n') \in (\alpha(t), \beta(t)),$$

such that

$$\lim_{n \to \infty} z_n' = z_0' \in \varphi^{-1}(\beta(t)) \cap pc.$$

We can assume that $t<\varphi(z_n')<\beta(t)$ for each n. Take the arc $pz_0'\subset pc$, and consider the continuum L of the form

$$L = pz_0' \cup \overline{\varphi^{-1}(t,\beta(t))}.$$

Then, by assumption, L is smooth at p, and thus there are, for each $n = 1, 2, \ldots$, irreducible continua $I(p, z_n')\subset L$ such that

$$\operatorname*{Lim}_{n\to\infty} I(p, z_n') = pz_0'.$$

Since any layer $\varphi^{-1}(t')$ for $t<t'<\beta(t)$ separates the continuum L, we conclude that

$$\overline{\varphi^{-1}\big((\varphi(z_n'),\beta(t))\big)}\subset I(p, z_n').$$

Therefore, since

$$z_0\in\overline{\varphi^{-1}(t,\beta(t))}\cap\varphi^{-1}(\beta(t)) = \overline{\varphi^{-1}(\varphi(z_n'),\beta(t))}\cap\varphi^{-1}(\beta(t)),$$

we have

$$z_0\in\operatorname*{Lim}_{n\to\infty} I(p, z_n'),$$

i.e., $z_0\in pz_0'\subset pc$ — a contradiction.

Therefore, we can assume

$$\overline{\varphi^{-1}\big((\alpha(t),\beta(t))\big)}\cap\varphi^{-1}(\beta(t))\cap pc = \emptyset.$$

By (10′) $\varphi^{-1}(\alpha(t))$ is a one-point set. Denote this point by e. Take the arc pe in the arc pc, and take a point e' in pe such that $ee'\cap\varphi^{-1}([\beta(t),1]) = \{e'\}$, where ee' is an arc in pe. Consider two cases.

1′. $\varphi(e')\neq\beta(t)$. Then there is a point t_0 such that $\beta(t)<t_0<\varphi(e')$. Consider the continuum K defined as follows:

$$K = \varphi^{-1}([\varphi(e'),1])\cup ee'\cup\overline{\varphi^{-1}([\alpha(t),t_0))}.$$

The continuum K is irreducible between d and any point of $K\cap\varphi^{-1}(t_0)$, and K is smooth at d, because $pd\cap K = \{d\}$ (cf. Theorem 1). Therefore by Proposition 4,

$$\varphi^{-1}(\beta(t))\subset\overline{\varphi^{-1}\big([\alpha(t),\beta(t))\big)}.$$

By the definition of $\beta(t)$ we have $\varphi^{-1}(\beta(t))\cap pc\neq\emptyset$; thus

$$\overline{\varphi^{-1}\big((\alpha(t),\beta(t))\big)}\cap\varphi^{-1}(\beta(t))\cap pc\neq\emptyset$$

— a contradiction.

2′. $\varphi(e') = \beta(t)$. The layer $\varphi^{-1}(\beta(t))$ is of right cohesion of $I(c, d)$, and $\{e', z_0\}\subset\varphi^{-1}(\beta(t))$, we infer there are continua K_n such that $K_n\cap K_m = \emptyset$ for $n\neq m$, $\operatorname*{Lim}_{n\to\infty} K_n = K_0$, $\{e', z_0\}\subset K_0\subset\varphi^{-1}(\beta(t))$, $\varphi(K_n)\cap[\alpha(t),\beta(t)] = \emptyset$ (cf. Lemma 1). Since the continuum $pe'\cup\varphi^{-1}([\beta(t),1))$ is smooth at p, there is a continuum Q such that

$$pe'\subset\operatorname{Int}Q\subset Q\subset pe'\cup\varphi^{-1}([\beta(t),1))\setminus\overline{\varphi^{-1}\big([\alpha(t),\beta(t))\big)}$$

(cf. Proposition 3 (ii)), because $\overline{\varphi^{-1}\big((\alpha(t),\beta(t))\big)}\cap pe' = \emptyset$ (Int Q denotes the interior of Q in $pe'\cup\varphi^{-1}([\beta(t),1])$. Since $K_n\subset pe'\cup\varphi^{-1}([\beta(t),1])$ and $\operatorname*{Lim}_{n\to\infty} K_n = K_0$ contains the point e', we can assume that $K_n\cap Q\neq\emptyset$ for each $n = 1, 2, \ldots$ Since $z_0\in K_0$, there is a sequence of points $\{z_n\}$ such that $\lim_{n\to\infty} z_n = z_0$, and $z_n\in K_n$. Take the continua $I(z_n, c_n)$ irreducible between z_n and Q (in K_n). We can assume that the sequence $\{c_n\}$ is convergent and put $c_0 = \lim_{n\to\infty} c_n$. Then $c_0\in Q$.

Consider the continuum

$$K = Q\cup ee'\cup\varphi^{-1}([\alpha(t),\beta(t)])\cup\bigcup_{n=1}^{\infty} I(z_n, c_n).$$

Since $p\in K$, K is smooth at p. The continuum

$$I(p, z_0) = pe\cup\overline{\varphi^{-1}\big([\alpha(t),\beta(t))\big)}$$

is irreducible between p and z_0. Moreover, $I(p, z_0)\subset K$, and $\lim_{n\to\infty} z_n = z_0$. Then, by the smoothness of K at p, there are irreducible continua $I(p, z_n)$ in K such that $\operatorname*{Lim}_{n\to\infty} I(p, z_n) = I(p, z_0)$.

By the definition of K, we have $I(z_n, c_n)\subset I(p, z_n)$. Therefore

$$\operatorname*{Ls}_{n\to\infty} I(z_n, c_n)\subset I(p, z_0)\cap\varphi^{-1}(\beta(t)).$$

By Proposition 1 the set $M = \operatorname*{Ls}_{n\to\infty} I(z_n, c_n)$ is a continuum. Therefore, because

$$M = (pe\cap M)\cup\Big(\overline{\varphi^{-1}\big([\alpha(t),\beta(t))\big)}\cap M\Big)\quad\text{and}\quad z_0\in\overline{\varphi^{-1}\big([\alpha(t),\beta(t))\big)}\cap M,$$

we have either $pe\cap M = \emptyset$ or $pe\cap M\cap\overline{\varphi^{-1}\big((\alpha(t),\beta(t))\big)}\neq\emptyset$. If $pe\cap M = \emptyset$, then

$$c_0\in M\subset\overline{\varphi^{-1}\big([\alpha(t),\beta(t))\big)}.$$

But $c_0\in Q$, and thus $c_0\in Q\cap\overline{\varphi^{-1}\big((\alpha(t),\beta(t))\big)}$ — a contradiction, because

$$Q\cap\overline{\varphi^{-1}\big([\alpha(t),\beta(t))\big)} = \emptyset,$$

If

$$pe\cap M\cap\overline{\varphi^{-1}\big([\alpha(t),\beta(t))\big)}\neq\emptyset,$$

then
$$pc \cap \varphi^{-1}(\beta(t)) \cap \overline{\varphi^{-1}\big((\alpha(t), \beta(t))\big)} \neq \emptyset$$
— a contradiction.

Therefore the case
$$\overline{\varphi^{-1}\big((\alpha(t), \beta(t))\big)} \cap \varphi^{-1}(\beta(t)) \cap pc = \emptyset$$
is also impossible. Thus (10) holds.

Obviously by (6)-(9)

(11) the number of nondegenerate intervals of the form $[\alpha(t), \beta(t)]$ in $[0, 1]$ is countable.

Let $t_1, t_2, \ldots$ be points of $[0, 1]$ which determine all the intervals mentioned in (11). Put
$$R = (I(d, c) \cap pc) \cup \bigcup_{i=1}^{\infty} \varphi^{-1}\big((\alpha(t_i), \beta(t_i))\big).$$

Observe that if $t \in [0, 1]$ and $\varphi^{-1}(t) \cap pc = \emptyset$, then $t \in (\alpha(t_i), \beta(t_i))$ for some t_i, and thus $\varphi^{-1}(t) \subset R$; if $t \in [0, 1]$ and $\varphi^{-1}(t) \cap pc \neq \emptyset$, then $\varphi^{-1}(t) \cap R = \varphi^{-1}(t) \cap pc$, and therefore, by Corollary 7, $\varphi^{-1}(t) \cap R$ is connected. If R is closed, then $\varphi|R$ is monotone; thus R is a continuum, and since $d, c \in R \subset I(d, c) \setminus (T \setminus \{c\})$, we have a contradiction by the irreducibility of $I(d, c)$. We conclude that

(12) R is not closed.

Therefore, there is a sequence $\{q_n\}$ of points of R such that

(13) $$\lim_{n\to\infty} q_n = q_0 \in I(d, c) \setminus R.$$

Thus

(14) $$\varphi^{-1}(\varphi(q_0)) \cap pc \neq \emptyset \quad \text{and} \quad q_0 \notin pc.$$

Since $I(d, c) \cap pc$ is closed, we can assume that all points q_n are contained in $\bigcup_{n=1}^{\infty} \varphi^{-1}\big((\alpha(t_i), \beta(t_i))\big)$.

Suppose that there is a subsequence $\{q_{n_k}\}$ of the sequence $\{q_n\}$ such that $q_{n_k} \in \varphi^{-1}\big((\alpha(t_{i_0}), \beta(t_{i_0}))\big)$ for some i_0. Then $\lim_{k\to\infty} \varphi(q_{n_k}) = \varphi(q_0)$ and, by (14), either $\varphi(q_0) = \alpha(t_{i_0})$ or $\varphi(q_0) = \beta(t_{i_0})$. But then
$$q_0 \in \overline{\varphi^{-1}\big((\alpha(t_{i_0}), \beta(t_{i_0}))\big)} \setminus \varphi^{-1}\big((\alpha(t_{i_0}), \beta(t_{i_0}))\big) \subset pc$$
by (10), contrary to (14).

Therefore there is no such subsequence. Let $q_n \in \varphi^{-1}\big((\alpha(t_{i_n}), \beta(t_{i_n}))\big)$. Since $\varphi^{-1}(\alpha(t_{i_n}))$ is a one-point set for each $n = 1, 2, \ldots$ (by (10′)), we can assume that the sequence $\{\varphi^{-1}(\alpha(t_{i_n}))\}$ is convergent and $\lim_{n\to\infty} \varphi^{-1}(\alpha(t_{i_n})) = x_0 \in pc$ by (10).

Take a sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x_0$, and $\alpha(t_{i_n}) < \varphi(x_n) < \varphi(q_n) < \beta(t_{i_n})$. Then $\operatorname{Ls}_{n\to\infty} \varphi^{-1}([\varphi(x_n), \beta(t_{i_n})]) = T_0$ is contained in a layer of $I(c, d)$, and $\{x_0, q_0\} \subset T_0$. Consider the continuum P of the form
$$P = pc \cup T_0 \cup \bigcup_{n=1}^{\infty} \varphi^{-1}([\varphi(x_n), \beta(t_{i_n})]).$$

Then, by assumption, P is smooth at p; thus there are irreducible continua $I(p, x_n) \subset P$ such that $\operatorname{Lim}_{n\to\infty} I(p, x_n) = px_0$, where px_0 is an arc in pc. Then any continuum $I(p, x_n)$ must contain the point q_n, and thus $q_0 \in px_0$ — a contradiction by (14). The proof of Theorem 3 is complete.

§ 4. The initial set of a hereditarily smooth arcwise connected continuum. In this section we will prove that, if X is a hereditarily smooth arcwise connected continuum, then any point p of X at which X is smooth is such that X is hereditarily smooth at p; and that the intial set of X is equal to the constituent C of the set of all points of X at which X is locally connected provided C contains some point of the initial set of X. Firstly we will prove some theorems which are needed in the proofs of the theorems mentioned above, and which also show the structure of hereditarily smooth arcwise connected continua. We have

THEOREM 4. *Let a continuum X be arcwise connected and hereditarily smooth at the point p. If K_0 is a subcontinuum of convergence of X, then for each two continua px_0 and py_0 which are irreducible between p and K_0 we have $x_0 = y_0$.*

Proof. It follows from Theorem 3 that X is hereditarily arcwise connected. Therefore, if px_0 and py_0 are arbitrary irreducible continua between p and K_0, then they are arcs. Thus

(1) $$K_0 \cap px_0 = \{x_0\} \quad \text{and} \quad K_0 \cap py_0 = \{y_0\}.$$

By assumption K_0 is a topological limit of a sequence of disjoint continua, i.e.,
$$K_0 = \operatorname{Lim}_{n\to\infty} K_n \quad \text{and} \quad K_m \cap K_n = \emptyset \quad \text{for each } m \neq n \text{ and } m, n = 0, 1, 2, \ldots$$

It follows from $\{x_0, y_0\} \subset K_0$ that there are sequences $\{x_n\}$ and $\{y_n\}$ of points of X and a sequence $\{x_n y_n\}$ of arcs in X (by the hereditary arcwise connectivity of X) such that

(2) $$\lim_{n\to\infty} x_n = x_0 \quad \text{and} \quad \lim_{n\to\infty} y_n = y_0,$$

(3) $$x_n y_n \subset K_n \quad \text{for each } n = 1, 2, \ldots$$

We may assume (see [8], § 42, I, Theorem 1, p. 45) that the sequence $\{x_n y_n\}$ is convergent and

(4) $$\operatorname{Lim}_{n\to\infty} x_n y_n = K_0' \subset K_0.$$

Since X is smooth at p, it follows by Proposidition 3(ii) that there are continua Q_n and Q'_n such that

(5) $$px_0 \subset \operatorname{Int} Q_n \subset Q_n \subset B(px_0, 1/n),$$

(6) $$py_0 \subset \operatorname{Int} Q'_n \subset Q'_n \subset B(py_0, 1/n).$$

We can assume by (2) that for an arbitrary but fixed n and for each $i = 1, 2, \ldots$ we have $x_i y_i \cap Q_n \neq \emptyset$ and $x_i y_i \cap Q'_n \neq \emptyset$. Take arcs $a_i b_i \subset x_i y_i$ irreducible between $x_i y_i \cap Q_n$ and $x_i y_i \cap Q'_n$ for each $i = 1, 2, \ldots$

If the sequence $\{a_i b_i\}$ contains a subsequence $\{a_{i_k} b_{i_k}\}$ of degenerate arcs, then there is a sequence $\{z_{i_k}\}$ of points such that $z_{i_k} \in a_{i_k} b_{i_k} \cap Q_n \cap Q'_n$. Therefore there is a point $z_0 \in \operatorname{Lim}_{k\to\infty} \{z_{i_k}\}$ such that $z_0 \in Q_n \cap Q'_n \cap K_0$.

If each subsequence $\{a_{i_k} b_{i_k}\}$ of the sequence $\{a_i b_i\}$ is a sequence of nondegenerate arcs, then there are arcs $a_{i_k} c_{i_k}$ such that

(7) $$\lim_{k\to\infty} c_{i_k} = c_0 \in Q'_n,$$

(8) $$a_{i_k} c_{i_k} \subset a_{i_k} b_{i_k},$$

(9) $$a_{i_k} c_{i_k} \cap Q'_n = \emptyset \quad \text{and} \quad a_{i_k} c_{i_k} \cap Q_n = \{a_{i_k}\}.$$

Put $R = Q_n \cup Q'_n \cup K'_0 \cup \bigcup_{k=1}^{\infty} a_{i_k} c_{i_k}$. Obviously R is a continuum and $p \in R$. Moreover, for each $k = 1, 2, \ldots$, by (4) and (9), we infer that

(10) any continuum A in R such that $p, c_{i_k} \in A$ contains a_{i_k}.

Since X is hereditarily smooth at p, the continuum R is smooth at p. Therefore by (7) and by the definition of smoothness, there are continua A_{i_k} in R such that

(11) $$\{p, c_0\} \subset \operatorname{Lim}_{k\to\infty} A_{i_k} = Q'_n,$$

(12) $$p, c_{i_k} \in A_{i_k} \quad \text{for each } k = 1, 2, \ldots$$

It follows from (10) and (12) that $a_{i_k} \in A_{i_k}$ for each $k = 1, 2, \ldots$ Let a_0 be a cluster point of the sequence $\{a_{i_k}\}$. We have $a_0 \in K'_0 \cap Q_n \cap Q'_n$ is nonempty, by (4), (9) and (11).

Thus we find that for each $n = 1, 2$, the set $K'_0 \cap Q_n \cap Q'_n$ is nonempty. Therefore

$$\operatorname{Lim}_{n\to\infty} (K'_0 \cap Q_n \cap Q'_n) = K'_0 \cap \operatorname{Lim}_{n\to\infty} Q_n \cap \operatorname{Lim}_{n\to\infty} Q'_n = K'_n \cap px_0 \cap py_0$$

is nonempty. Hence $x_0 = y_0$ by (1). The proof of Theorem 4 is complete.

Lemma 4. *Let a continuum X be hereditarily arcwise connected. If the arc A_0 is a continuum of convergence of X, then any subarc of the arc A_0 is a continuum of convergence of X.*

Proof. The arc A_0 is a continuum of convergence of X; thus A_0 is a topological limit of the sequence A_n of subcontinua of X such that

$$A_0 = \operatorname{Lim}_{n\to\infty} A_n \quad \text{and} \quad A_m \cap A_n = \emptyset \quad \text{for} \quad m \neq n \text{ and } m, n = 0, 1, 2, \ldots$$

Let a_0 and b_0 be endpoints of the arc A_0. There are sequences $\{a_n\}$ and $\{b_n\}$ of points of X and a sequence $\{a_n b_n\}$ of arcs of X such that

(1) $$\lim_{n\to\infty} a_n = a_0 \quad \text{and} \quad \lim_{n\to\infty} b_n = b_0,$$

(2) $$a_n b_n \subset A_n \quad \text{for each } n = 0, 1, 2, \ldots$$

by the hereditary arcwise connectedness of X. It follows from Proposition 2 that

(3) $$\operatorname{Lim}_{n\to\infty} a_n b_n = A_0 = a_0 b_0.$$

For each $i = 1, 2, \ldots$ there is an arc $a_{n_i} b_{n_i}$ such that

(4) $$\varrho(x, A_0) \leqslant 1/i \quad \text{for each } x \in a_{n_i} b_{n_i},$$

(5) $$\varrho(a_{n_i}, a_0) \leqslant 1/i \quad \text{and} \quad \varrho(b_{n_i}, b_0) \leqslant 1/i.$$

Let $c_0 d_0$ be an subarc of the arc A_0 such that $a_0 \leqslant c_0 \leqslant d_0 \leqslant b_0$ in the natural order of the arc A_0. It suffices to prove that $c_0 d_0$ is a continuum of convergence of X.

Let i be a natural number and let $a_{n_i} b_{n_i}$ be an arc determined above. Let d_{n_i} be the first point in the arc $a_{n_i} b_{n_i}$ such that

(6) $$\varrho(d_{n_i}, d_0 b_0) \leqslant 1/i,$$

where $d_0 b_0$ is the subarc of the arc $a_0 b_0$; i.e., for each $x \in a_{n_i} d_{n_i} \setminus \{d_{n_i}\} \subset a_{n_i} b_{n_i}$ we have $\varrho(x, d_0 b_0) > 1/i$. Let c_{n_i} be the least point in the arc $a_{n_i} d_{n_i}$ (in $a_{n_i} b_{n_i}$) such that

(7) $$\varrho(c_{n_i}, a_0 c_0) \leqslant 1/i,$$

where $a_0 c_0$ is an arc in the arc $a_0 b_0$. Therefore, if $c_{n_i} d_{n_i}$ is an arc in $a_{n_i} b_{n_i}$, by (4)-(7)

(8) $$\varrho(x, a_0 c_0 \cup d_0 b_0) > 1/i \quad \text{for each } x \in c_{n_i} d_{n_i} \setminus \{c_{n_i}, d_{n_i}\}.$$

Put $K_{n_i} = c_{n_i} d_{n_i}$. Consider $K_0 = \operatorname{Ls}_{i\to\infty} K_{n_i}$. By (7), any cluster point of the sequence $\{c_{n_i}\}$ is contained in $a_0 c_0$, but (4) and (8) imply that any cluster point of the sequence $\{c_{n_i}\}$ is contained in $c_0 d_0$; therefore

(9) $$\lim_{i\to\infty} c_{n_i} = c_0.$$

In a similar way we obtain

(10) $$\lim_{i\to\infty} d_{n_i} = d_0.$$

Moreover (4) and (9) imply that

(11) $$K_0 \subset c_0 d_0 .$$

Thus, by (9), (10) and (11), we infer $K_0 = c_0 d_0$. Therefore, by Proposition 2, we have $\operatorname{Lim}_{i\to\infty} K_{n_i} = c_0 d_0$, i.e., the arc $c_0 d_0$ is a continuum of convergence of X by the choice of K_{n_i}. The proof of Lemma 4 is complete.

LEMMA 5. *Let a continuum X be arcwise connected and hereditarily smooth at a point $p \in X$. Let $\{px_n\}$ be a sequence of arcs of X such that $\lim_{n\to\infty} x_n = x_0$ and $\operatorname{Lim}_{n\to\infty} px_n$ is an arc px_0. If $\{z_n\}$ is a sequence of points of px_0 such that*

(i) $\lim_{n\to\infty} z_n = z_0$,

(ii) *if $z_n x_n$ is an arc in px_n and pz_n is an arc in px_0, then $pz_n \cap z_n x_n = \{z_n\}$, then $\operatorname{Ls}_{n\to\infty} z_n x_n$ is the arc $z_0 x_0$ in px_0.*

Proof. By Theorem 3 the continuum X is hereditarily arcwise connected. By assumptions, $\operatorname{Ls}_{n\to\infty} z_n x_n$ is a subarc of the arc px_0 and $x_0 \in \operatorname{Ls}_{n\to\infty} z_n x_n$. Suppose, on the contrary, that $\operatorname{Ls}_{n\to\infty} z_n x_n = z_0' x_0$ and $z_0' \in pz_0 \setminus \{z_0\} \subset px_0$. Then there is a sequence $z_{n_i}' \in z_{n_i} x_{n_i}$ such that

(1) $$\lim_{n\to\infty} z_{n_i}' = z_0' .$$

We can assume that for each $i = 1, 2, \ldots$ the arcs $z_{n_i}' x_{n_i}$ contained in $z_{n_i} x_{n_i}$ are such that $z_{n_i} x_{n_i} \cap px_0 = \emptyset$. Since $\operatorname{Ls}_{i\to\infty} z_{n_i} x_{n_i} = z_0' x_0$, we have $\operatorname{Lim}_{i\to\infty} z_{n_i} x_{n_i} = z_0' x_0$ by Proposition 2. Therefore

(2) the arc $z_0' x_0$ is a continuum of convergence of X.

Take the arc $z_{n_i} z_{n_i}'$ contained in $z_{n_i} x_{n_i}$ for each $i = 1, 2, \ldots$ Consider

$$Q = px_0 \cup \bigcup_{i=1}^{\infty} z_{n_i} z_{n_i}' .$$

Obviously Q is a continuum. Since X is hereditarily smooth at p and $p \in Q$, Q is smooth at p. Let $\varepsilon = \frac{1}{3}\varrho(z_0, pz_0')$, where pz_0' is an arc in px_0. It follows from Proposition 3(ii) that there is a continuum K in Q such that

(3) $$pz_0' \subset \operatorname{Int} K \subset K \subset B(pz_0', \varepsilon) .$$

By (i) of the assumptions and by (1) we can take a natural number n_i such that $z_{n_i}' \in K$ and $\varrho(z_{n_i}, z_0) < \varepsilon$. Let ab be an arbitrary arc in pz_{n_i} such that $ab \cap K = \emptyset$ and $p < a < b$ in the natural order of the arc px_0. The continuum $K \cup z_{n_i} z_{n_i}' \cup bz_{n_i}$, where bz_{n_i} is an arc in pz_{n_i}, contains an arc pb by the hereditary arcwise connectedness of X. Then $pb \cap ab = \{b\}$. By Lemma 4 and (2) ab is a continuum of convergence; thus by Theorem 4 if we take the arc pb and the arc pa, which is contained in px_0, then we obtain $a = b$ — a contradiction.

THEOREM 5. *Let a continuum X be arcwise connected. If $HI(X) \neq \emptyset$, then $I(X) = HI(X)$.*

Proof. Obviously $HI(X) \subset I(X)$ by definition. It follows from Theorem 3 that X is hereditarily arcwise connected. Let X be smooth at p and let Q be an arbitrary subcontinuum of X such that $p \in Q$. By the definition of hereditary smoothness it suffices to prove that Q is smooth at p. By Theorem 1, if $r \in HI(X)$ and rq is an arbitrary arc irreducible between r and Q, then Q is hereditarily smooth at q. Let $\{x_n\}$ be an arbitrary sequence of points of Q such that

(1) $$\lim_{n\to\infty} x_n = x_0 \in Q ,$$

and let px_0 be an arbitrary arc contained in Q. We will prove that there is a sequence $\{R_n\}$ of subcontinua of Q such that $\operatorname{Lim}_{n\to\infty} R_n = px_0$.

Let qy_0 be an arbitrary arc in Q which is irreducible between q and px_0. Denote by py_0 and $y_0 x_0$ arcs contained in px_0. Put $qx_0 = qy_0 \cup y_0 x_0$. Since q is an initial point of Q, there is a sequence $\{qx_n\}$ of arcs in Q such that

(2) $$\operatorname{Lim}_{n\to\infty} qx_n = qx_0 .$$

For each $n = 1, 2, \ldots$, let the point z_n of $qx_0 \cap qx_n$ be such that, if $z_n x_n$ is an arc in qx_n and qz_n is an arc in qx_0, then $qz_n \cap z_n x_n = \{z_n\}$. Let z_0 be an arbitrary cluster point of the sequence $\{z_n\}$, i.e., for some subsequence $\{z_{n_i}\}$ of the sequence $\{z_n\}$ we have

(3) $$z_0 = \lim_{i\to\infty} z_{n_i} .$$

Suppose that $z_0 \notin y_0 x_0$. Then (3) and Lemma 4 imply that

(4) any proper subarc of the arc $z_0 x_0$ is a continuum of convergence of X, where $z_0 x_0$ is an arc in qx_0.

Let $\varepsilon = \frac{1}{3}\varrho(z_0, px_0)$. Since X is smooth at p, by Proposition 3(ii) there is a continuum K such that

(5) $$px_0 \subset \operatorname{Int} K \subset K \subset B(px_0, \varepsilon) .$$

By (1) and (3) there is a natural number n_i such that $x_{n_i} \in K$ and $z_{n_i} \notin K$.

Let A be a non-degenerate subarc of the arc $z_{n_i} x_0 \subset qx_0$ such that $A \subset z_{n_i} x_0 \setminus (K \cup \{z_{n_i}\})$ and denote by a and b the endpoints of A (where $q < a < b$ in the natural ordering of the arc qx_0 from q to x_0). We have two arcs qa and qb such that

(6) $$qa \subset qx_0 ,$$

(7) $$qb \subset qz_{n_i} \cup z_{n_i} x_{n_i} \cup K \cup by_0 ,$$

where by_0 is an arc in qx_0. We define $ra = rq \cup qa$ and $rb = rq \cup qb$. Since $ra \cap ab = \{a\}$ and $rb \cap ab = \{b\}$, we have $a = b$ by Theorem 4 and (4). This contradicts the choice of a and b.

Therefore any cluster point of the sequence $\{z_n\}$ belongs to the arc y_0x_0. We define $R_n = py_0 \cup y_0z_n \cup z_nx_n$, where y_0z_n is an arc in qx_0. Then $\operatorname{Ls}_{n\to\infty} R_n = py_0 \cup \operatorname{Ls}_{n\to\infty} y_0z_n \cup \operatorname{Ls}_{n\to\infty} z_nx_n \subset py_0 \cup y_0x_0 = px_0$ by Lemma 5. Therefore $\operatorname{Lim}_{n\to\infty} R_n = px_0$ by Proposition 2. Since the continuum R_n is contained in Q by the construction for each $n = 1, 2, \ldots$, the required condition is satisfied. The proof of Theorem 5 is complete.

COROLLARY 8. *A continuum X is hereditarily locally connected if and only if $HI(X) \neq \emptyset$ and X is locally connected.*

Indeed, if X is hereditarily locally connected, then $HI(X) = X$ by Corollary 3′. In particular $HI(X) \neq \emptyset$ and X is locally connected. Conversely, if X is locally connected and $HI(X) \neq \emptyset$, then $I(X) = X$ by Corollary 3 and $HI(X) = X$ by Theorem 5, because local connectedness implies arcwise connectedness. Therefore X is hereditarily locally connected by Corollary 3′.

COROLLARY 9. *For every continuum X the equality $HI(X) = X$ holds if and only if $I(X) = X$ and $HI(X) \neq \emptyset$.*

THEOREM 6. *Let a continuum X be arcwise connected. If $p, q \in HI(X)$ and if pq is an arbitrary arc in X with endpoints p and q, then $pq \subset HI(X)$.*

Proof. By Theorem 3, X is hereditarily arcwise connected. Take an arbitrary point r of pq and a convergent sequence $\{x_n\}$ of points of X. Put

(1) $$\lim_{n\to\infty} x_n = x_0 ,$$

and let rx_0 be an arbitrary arc with endpoints r and x_0. Denote by y_0 such a point of rx_0 that if y_0x_0 is an arc in rx_0 then $pq \cap y_0x_0 = \{y_0\}$. Let pr and rq denote the arcs in pq. Assume $y_0 \in pr$ (if $y_0 \in rq$ the proof is the same). Since X is smooth at p, there exists a sequence $\{px_n\}$ of arcs such that

(2) $$\operatorname{Lim}_{n\to\infty} px_n = py_0 \cup y_0x_0 ,$$

where py_0 is an arc in pq. Take a sequence $\{z_n\}$ of points of arcs px_n such that if z_nx_n is an arc in px_n then $(py_0 \cup y_0x_0) \cap z_nx_n = \{z_n\}$. Let z_0 be an arbitrary cluster point of the sequence $\{z_n\}$. Suppose that $z_0 \in py_0 \setminus \{y_0\}$. Since z_0 is a cluster point of $\{z_n\}$, there is a subsequence $\{z_{n_i}\}$ of the sequence $\{z_n\}$ such that

(3) $$\lim_{i\to\infty} z_{n_i} = z_0 .$$

Then, by Lemma 4, we infer that

(4) any proper subarc of the arc z_0x_0 in $py_0 \cup y_0x_0$ is a continuum of convergence of X.

Let $\varepsilon = \frac{1}{3}\varrho(z_0, y_0q \cup y_0x_0)$, where y_0q is the arc in pq. Since X is smooth at q, by Proposition 3(ii) there is a continuum K such that

(5) $$y_0q \cup y_0x_0 \subset \operatorname{Int} K \subset K \subset B(y_0q \cup y_0x_0, \varepsilon) .$$

By (1) and (3) we conclude that there is a natural number n_i such that $x_{n_i} \in K$ and $z_{n_i} \notin K$. Let A be an subarc of the arc $z_{n_i}x_0$ in $py_0 \cup y_0x_0$ such that $A \subset z_{n_i}x_0 \setminus (K \cup \{z_{n_i}\})$ and denote by a and b the endpoints of A ($p<a<b$ in the natural ordering of the arc $py_0 \cup y_0x_0$ from p to x_0). We have two arcs qa and qb such that

(6) $$qb \subset pq ,$$

(7) $$qa \subset K \cup x_{n_i}z_{n_i} \cup z_{n_i}a ,$$

where $z_{n_i}a$ is an arc in pq.

Since $qa \cap A = \{a\}$ and $qb \cap A = \{b\}$, by Theorem 4 and (4) $a = b$, which contradicts the choice of a and b.

Therefore, any cluster point of the sequence $\{z_n\}$ belongs to the arc y_0x_0. By Lemma 5, we have

(8) $$\operatorname{Ls}_{n\to\infty} z_nx_n \subset y_0x_0 .$$

Let ry_0 be the arc in rx_0. We define $R_n = ry_0 \cup y_0z_n \cup z_nx_n$, where y_0z_n is an arc in $py_0 \cup y_0x_0$. Then

$$\operatorname{Ls}_{n\to\infty} R_n = ry_0 \cup \operatorname{Ls}_{n\to\infty} y_0z_n \cup \operatorname{Ls}_{n\to\infty} z_nx_n \subset ry_0 \cup y_0x_0 = rx_0 .$$

Therefore, by Proposition 2, we have $\operatorname{Lim}_{n\to\infty} R_n = rx_0$.

Using the hereditary arcwise connectedness of X, we infer by Proposition 3(i) that X is smooth at r. Therefore X is hereditarily smooth at r by Theorem 5. Thus $pq \subset HI(X)$. The proof of Theorem 6 is complete.

THEOREM 7. *Let a continuum X be arcwise connected and let $p, q \in X$. If X is hereditarily smooth at the point p and if X is locally connected at each point of an arc pq, then X is smooth at q.*

Proof. By Theorem 3 we conclude that the continuum X is hereditarily arcwise connected, i.e., all irreducible continua in X are arcs. Let $\{x_n\}$ be an arbitrary sequence of points of X such that

(1) $$\lim_{n\to\infty} x_n = x_0$$

and let qx_0 be an arbitrary arc joining q and x_0. Denote by y_0 the point of the arc qx_0 such that if y_0x_0 is an arc in qx_0 then $y_0x_0 \cap pq = \{y_0\}$. Since X is smooth at p and (1) holds, there is a sequence $\{px_n\}$ of arcs of X such that

(2) $$\operatorname{Lim}_{n\to\infty} px_n = py_0 \cup y_0x_0 ,$$

where py_0 is an arc in pq. Since X is locally connected in y_0, we have, for each natural number j, a continuum K_j such that

(3) $$y_0 \in \operatorname{Int} K_j \subset K_j \subset B(y_0, 1/j).$$

Take for each $n = 1, 2, \ldots$ the point z_n of px_n such that if $z_n x_n$ is an arc in px_n then $z_n x_n \cap (py_0 \cup y_0 x_0) = \{z_n\}$. If $z_n \in y_0 x_0$, then we define

$$qx_n = qy_0 \cup y_0 z_n \cup z_n x_n,$$

where qy_0 is an arc in qx_0, and $y_0 z_n$ is an arc in $y_0 x_0 \subset qx_0$. If $z_n \notin y_0 x_0$, then the arc $z_n x_n$ intersects the continuum K_i for some i. Therefore we have, for a subsequence $\{z_{n_i}\}$ of the sequence $\{z_n\}$ such that $z_{n_i} \notin y_0 x_0$, a sequence of indexes $\{j_{n_i}\}$ such that $\lim_{i\to\infty} 1/j_{n_i} = 0$, and

(4) $$z_{n_i} x_{n_i} \cap K_{j_{n_i}} \neq \emptyset.$$

Let a_{n_i} be a point in the arc $z_{n_i} x_{n_i}$ such that if $a_{n_i} x_{n_i}$ is an arc in the arc $z_{n_i} x_{n_i}$ then $K_{j_{n_i}} \cap a_{n_i} x_{n_i} = \{a_{n_i}\}$. Since X is hereditarily arcwise connected, there are arcs $y_0 a_{n_i}$ contained in $K_{j_{n_i}}$. Obviously, by (3) we have

(5) $$\operatorname{Lim}_{i\to\infty} y_0 a_{n_i} = y_0.$$

Consider the continua $A_{n_i} = py_0 \cup y_0 a_{n_i} \cup a_{n_i} x_{n_i}$. Obviously $\operatorname{Lim}_{i\to\infty} A_{n_i} = py_0 \cup y_0 x_0$. A continuum A_{n_i} contains an arc B_{n_i} irreducible between x_{n_i} and p, and the first point in the arc B_{n_i} which belongs to $py_0 \cup y_0 x_0$ (in the natural order in $py_0 \cup y_0 x_0$ from x_0 to p) is contained in $K_{j_{n_i}}$. Therefore we can assume that any cluster point of the sequence $\{z_n\}$ is contained in the arc $y_0 x_0$, because we can consider arcs B_{n_i} instead of arcs px_{n_i}. Thus, by Lemma 5, we have $\operatorname{Ls}_{n\to\infty} z_n x_n \subset y_0 x_0$ and $\operatorname{Ls}_{n\to\infty} y_0 z_n \subset y_0 x_0$, where $y_0 z_n$ is an arc in $py_0 \cup y_0 x_0$.

We define, as before, $R_n = qy_0 \cup y_0 z_n \cup z_n x_n$. Then

$$\operatorname{Ls}_{n\to\infty} R_n = qy_0 \cup \operatorname{Ls}_{n\to\infty} y_0 z_n \cup \operatorname{Ls}_{n\to\infty} z_n x_n \subset qx_0 \cup y_0 x_0 = qx_0.$$

Therefore, by Proposition 2, we have $\operatorname{Lim}_{n\to\infty} R_n = qx_0$, i.e., X is smooth at q. The proof of Theorem 7 is complete.

Let $N(X)$ be the set of all points of X at which X is not locally connected. We have the following (cf. [2], Theorem 2, p. 299).

COROLLARY 10. *If a continuum X is arcwise connected and hereditarily smooth at p, then the constituent of the set $X \setminus N(X)$ containing p is the initial set of X.*

Proof. Obviously $I(X) \subset X \setminus N(X)$ by Corollary 2. Denote by C the constituent of the set $X \setminus N(X)$ which contains p. To prove $I(X) \subset C$ take a point $q \in I(X)$ and an arbitrary arc pq. By Theorem 5 we have $q \in HI(X)$, and thus, by Theorem 6, $pq \subset HI(X) \subset I(X)$. Therefore X is locally connected at each point of the arc pq by Corollary 2; thus $q \in C$, i.e., $I(X) \subset C$.

To prove $C \subset I(X)$ take a point $q \in C$ and an arbitrary arc $pq \subset C$ (such an arc does exist by the definition of the constituent since X is hereditarily arcwise connected by Theorem 3). Thus X is locally connected at each point of the arc pq, and therefore X is smooth at q by Theorem 7. Thus $q \in I(X)$, i.e., $C \subset I(X)$. The proof of Corollary 10 is complete.

§ 5. **The set** $N(X)$. In this section we study properties of the set $N(X)$. We have

THEOREM 8. *If a continuum X is arcwise connected and $HI(X) \neq \emptyset$, then for any continuum Y*

$$Y \subset X \quad \textit{implies} \quad N(Y) \subset N(X).$$

Proof. By Theorem 3 the continuum X is hereditarily arcwise connected. Let X be hereditarily smooth at a point p, let Y be an arbitrary subcontinuum of X and let pq be an arbitrary arc in X such that $pq \cap Y = \{q\}$. Then, by Theorem 1, Y is hereditarily smooth at q. Suppose, on the contrary, that Y is not locally connected at the point x_0 and X is locally connected at this point. Therefore there is a closed neighbourhood E of the point x_0 in Y and a component C of E containing x_0 such that $x_0 \in \overline{E \setminus C}$. We infer that there exists a sequence $\{x_n\}$ of points of X such that

(1) $$x_0 = \lim_{n\to\infty} x_n,$$

(2) $$x_n \in E \setminus C.$$

Let qx_0 be an arbitrary arc in Y. Since Y is smooth at q, by (1) there is a sequence $\{qx_n\}$ of arcs of Y such that

(3) $$\operatorname{Lim}_{n\to\infty} qx_n = qx_0.$$

Take, for each $n = 1, 2, \ldots$, a point z_n of qx_n such that if $z_n x_n$ is an arc in qx_n then $z_n x_n \cap qx_0 = \{z_n\}$. Let z_0 be a cluster point of the sequence $\{z_n\}$. Then there is a subsequence $\{z_{n_i}\}$ of the sequence $\{z_n\}$ such that $\lim_{i\to\infty} z_{n_i} = z_0$. There is an arc ax_0 in the arc qx_0 such that $a \neq x_0$ and $ax_0 \subset \operatorname{Int} E$. If $z_0 \in ax_0 \setminus \{a\}$, then, by Lemma 5, $z_{n_i} x_{n_i}$ is contained in E for indexes i larger than some i_0, because $\lim_{i\to\infty} z_{n_i} x_{n_i} = z_0 x_0$, where $z_0 x_0$ is the subarc of qx_0; thus $x_{n_i} \in C$ for $i > i_0$ — a contradiction. Therefore $z_0 \notin ax_0 \setminus \{a\}$. This implies by Lemma 4 that

(4) any subarc of the arc ax_0 is a continuum of convergence of X.

Let cd be an subarc of the arc ax_0 such that $cd \subset ax_0 \setminus \{a, x_0\}$ and $c \in ad \setminus \{a, d\}$. Let $\varepsilon = \frac{1}{2}\varrho(x_0, cd \cup pq)$. Since X is locally connected at x_0, there is a continuum K such that

(5) $$x_0 \in \operatorname{Int} K \subset K \subset B(x_0, \varepsilon).$$

It follows from (1) and (5) that $x_n \in K$ for $n > n_0$, and $z_n \notin cd$ for $n > n_1$. Let $m > n_0$ and $m > n_1$.

Take the arcs qc contained in qx_0 and qd contained in $pz_m \cup z_m x_m \cup K \cup x_0 d$, where pz_m and $x_0 d$ are arcs in px_0. Then $pq \cup qc$ and $pq \cup qd$ are both irreducible between p and the continuum of convergence cd of X (cf. (4)); thus, by Theorem 4 we have $c = d$ — a contradiction. The proof of Theorem 8 is complete.

THEOREM 9. *Let a continuum X be arcwise connected and $HI(X) \neq \emptyset$. If $S \subset X$ is a simple closed curve and pq is an arbitrary arc which is irreducible between p and S, where $p \in HI(X)$, then $N(X) \cap S \subset \{q\}$.*

Proof. By Theorem 3 the continuum X is hereditarily arcwise connected. Suppose, on the contrary, that $x_0 \in N(X) \cap S$ and $x_0 \neq q$. Let $x_0 q$ be one of two arcs in S irreducible between x_0 and q. The continuum X is not locally connected at x_0; therefore there is a closed neighbourhood E of the point x_0 such that if C is a component of E which contains x_0 then $x_0 \in \overline{E \setminus C}$. We infer that there is a sequence $\{x_n\}$ of points of X such that

(1) $$\lim_{n\to\infty} x_n = x_0\,,$$

(2) $$x_n \in E \setminus C\,.$$

Let $p \in HI(X)$. Since X is smooth at the point p and (1) holds, there is a sequence $\{px_n\}$ of arcs of X such that

(3) $$\operatorname*{Lim}_{n\to\infty} px_n = pq \cup qx_0\,.$$

Take, for each $n = 1, 2, \ldots$, a point z_n of the arc px_n such that if $z_n x_n$ is an arc in px_n then $z_n x_n \cap (pq \cup px_0) = \{z_n\}$. Let z_0 be a cluster point of the sequence $\{z_n\}$. Then there is a subsequence $\{z_{n_i}\}$ of the sequence $\{z_n\}$ such that

(4) $$\lim_{i\to\infty} z_{n_i} = z_0\,.$$

There is an arc ax_0 in the arc qx_0 such that $a \neq x_0$ and $q \notin ax_0 \subset \operatorname{Int} E$. If $z_0 \in ax_0 \setminus \{a\}$, then, by Lemma 5, $z_{n_i} x_{n_i}$ is contained in E for indexes i larger than some i_0, because $\operatorname*{Lim}_{i\to\infty} z_{n_i} x_{n_i} = z_0 x_0$, where $z_0 x_0$ is an arc in qx_0. Thus $x_{n_i} \in C$ for $i > i_0$ — a contradiction. Therefore $z_0 \notin ax_0 \setminus \{a\}$. This implies that

(5) the arc ax_0 is a continuum of convergence of X.

Take the arc aq in $x_0 q\,(a \neq q)$, and the arc $I(x_0, q)$, irreducible between x_0 and q, which is contained in $\overline{S \setminus x_0 q}$. Then $pq \cup aq$ and $pq \cup I(x_0, q)$ are both irreducible between p and the continuum ax_0. It follows from (5) and Theorem 4 that $a = q$ — a contradiction. The proof of Theorem 9 is complete.

COROLLARY 11. *Let a continuum X be arcwise connected and $HI(X) \neq \emptyset$. The continuum X is a smooth dendroid if and only if for each constituent C of the set $X \setminus N(X)$ the closure $\overline{C}$ is a dendroid.*

Indeed, by Theorem 3, X is hereditarily arcwise connected. If X is a dendroid, then any subcontinuum of X is a dendroid. In particular, the closure of any constituent C of the set $X \setminus N(X)$ is a dendroid.

Conversely, if for each constituent C of the set $X \setminus N(X)$ the closure $\overline{C}$ is a dendroid, then, by Theorem 9, X fails to contain a simple closed curve. Therefore, by the hereditary arcwise connectedness of X (cf. Theorem 3), X is a dendroid.

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