5. In Theorem 1 the functions $g_1, g_2, \ldots$ may just as well be taken to be almost continuous functions.

6. Can an indecomposable continuum be an ACR of I^2?

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Discrete and continuous flows of characteristic $O^{\pm}$

by

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Abstract. The objective of this paper is to show some application of strictly almost equicontinuous transformation groups (s.a.e.) of [6] on discrete and continuous flows, making no assumption of the presence of equicontinuous points. We assume that the phase spaces X to be locally compact and connected, the set G of points of characteristic $O^{\pm}$ to be open and dispersive. Then we show that under general conditions there exist components K and L of $X-G$ which acts as positive and negative centers of attraction and sufficient conditions for G to be dispersive are also obtained.

Let (X, T) be a discrete or continuous flow on a locally compact metric space. Hence T denotes the group of integers for the case of discrete flow, the group of real numbers for the case of continuous flow. Let T^+ and T^- be respectively the set of non-negative and the set of non-positive elements of T. Let $x \in X$ and $\mathcal{U}_x$ be the neighborhood filter of x. The *positive* (*negative*) *prolongation* of $x \in X$ is defined to be the set

$$D^+(x) = \bigcap_{U \in \mathcal{U}_x} \overline{UT^+} \quad \Big(D^-(x) = \bigcap_{U \in \mathcal{U}_x} \overline{UT^-}\Big).$$

A point $x \in X$ is said to be of characteristic O^+, O^-, $O^{\pm}$, if respectively we have $D^+(x) = \overline{xT^+}$, $D^-(x) = \overline{xT^-}$; $D^+(x) = \overline{xT^+}$ and $D^-(x) = \overline{xT^-}$. The set of points of characteristic $O^{\pm}$ is invariant and is denoted by G; its complement is denoted by F. A point $y \in D^+(x)$ is characterized by that there exist sequences $\{x_n\} \subset X$ and $\{t_n\} \subset T^+$ such that $x_n \to x$ and $x_n t_n \to y$; points in $D^-(x)$ may be similarly characterized. If the group T is equicontinuous at $x \in X$, then x is of characteristic $O^{\pm}$; converse is in general not true. A flow (discrete or continuous) is said to be of *characteristic* $O^{\pm}$ if $X = G$.

In this paper we are mainly interested in the studying of F and the corresponding discrete flow when G is sufficiently large. Results for continuous flows are also obtained (Section 3) under simpler approach. The main results are application of our works [5] and [6]. Two of the main theorems which we obtain in this paper are the following.

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THEOREM 1. *Let (X, T) be a discrete flow, where X is a locally compact connected metric space. If F is non-empty, closed, totally disconnected, consisting of recurrent points, and G is connected, then F is the union of one or two fixed points.*

THEOREM 2. *Let (X, T) be a discrete flow on a compact connected metric space. If F is non-empty and compact and G is a semi-continuum which contains a point not almost periodic, then there exist invariant components A and B of F (A may equal B) such that $\alpha(x)\subset A$ and $\omega(x)\subset B$ for all $x\in X-(A\cup B)$.*

1. Flows of characteristic $O^{\pm}$. In this section we give a proof for the theorem whose case of continuous flow is proved in [2].

THEOREM 3. *Let (X, T) be a discrete or continuous flow of characteristic $O^{\pm}$ on a locally compact metric space. Then there exists a decomposition into sets which are open and closed $X = A\cup B$ such that $A = \{x\in X|\ J^+(x) = \emptyset$ and $J^-(x) = \emptyset\}$ and $B = \{x\in X|\ x$ is almost periodic$\}$.*

The *positive* (*negative*) *prolongation limit set of* $x\in X$ is defined to be

$$J^+(x) = \bigcap_{t\in T} D^+(xt) \qquad \Big(J^-(x) = \bigcap_{t\in T} D^-(xt)\Big).$$

It can be easily shown that $D^+(x) = J^+(x)\cup xT^+$ and that $y\in J^+(x)$ if and only if there exist sequences $\{x_n\}\subset X$, $\{t_n\}\subset T^+$, $t_n\to\infty$ such that $x_n\to x$ and $x_nt_n\to y$.

The proof of Theorem 3 follows from a sequence of simple lemmas.

LEMMA 1. *Let (X, T) be a discrete or continuous flow. If $x\in X$ has a compact neighborhood U and $U\subset G$, then whenever x is positively recurrent, x is positively almost periodic.*

LEMMA 2. *If (X, T) is a flow of characteristic O^+, then every orbit closure is minimal.*

Proof. Suppose there exists $\overline{yT}\subsetneq\overline{xT}$. Then $x\in D^+(y)\cup D^-(y)-\overline{yT}$, contradicting that $y\in G$.

LEMMA 3. *Let (X, T) be a flow of characteristic $O^{\pm}$. Then $x\in X$ is positively (negatively) recurrent if and only if $J^+(x)\neq\emptyset$ ($J^-(x)\neq\emptyset$).*

Proof. Obvious (cf. [1, Lemma 2]).

LEMMA 4. *Let (X, T) be a flow of characteristic $O^{\pm}$ on a locally compact metric space. Then $x\in X$ is almost periodic if and only if either $J^+(x)\neq\emptyset$ or $J^-(x)\neq\emptyset$.*

Proof. The necessity is clear. For the sufficiency we may assume $J^+(x)\neq\emptyset$. Then x is positively recurrent by Lemma 3 and x is positively almost periodic by Lemma 1. Since X is locally compact this implies that $\overline{xT^+}$ is compact, whence $\omega(x)\neq\emptyset$. By Lemma 2 we have $\overline{xT} = \omega(x)$ and $\overline{xT}$ is a compact minimal set. Hence x is almost periodic.

Proof of Theorem 3. From Lemma 4, $X = A\cup B$. That B is a closed set is clear. Now $\overline{xT}$ is a compact minimal set for $x\in B$. Take a compact neighborhood U of $\overline{xT}$. Since $D^+(x) = \overline{xT}\subset U$, there exists a sufficiently small neighborhood V of x such that $\overline{VT}\subset U$. If $y\in V$ then $y\in B$. Hence B is also open.

The proof of Theorem 3 is completed.

2. Discrete flows of characteristic $O^{\pm}$ except for a zero-dimensional set. In studying discrete flows of nearly characteristic zero we find that it is sufficient for G to have a point which is not almost periodic, with (X, T) a discrete flow on a locally compact metric space, F non-empty closed and totally disconnected and G connected. There is in general no such a point a in G, however it is desirable to seek conditions as weak as possible guaranteeing the existence of a.

LEMMA 5. *Let (X, T) be a discrete flow on a locally compact connected metric space. Let K be a closed and totally disconnected set such that $F\subset K$. Let $x\in F$. In order that there exist $a\in X-K$ and a sequence $\{t_n\}\subset T$ such that $at_n\to x$ it is sufficient that any one of the following conditions is satisfied:*

(a) *X is locally connected im kleinen at x* [4];

(b) *$X-K$ is locally connected;*

(c) *there exists an open neighborhood W of x such that $W\cap(X-K)\subset W\cap E$, where E is the set of points at which T is equicontinuous;*

(d) *K consists of recurrent points (in this case the connectedness of X is not needed).*

Proof. The proof for conditions (a), (b) and (c) are similar, cf. proof of [6, Lemma 1.11].

We prove that condition (b) is a sufficient condition. We prove by contradiction by assumming that such an a does not exist. Since $x\in F$ there exist $y\in X^*$, $z\in X$, $z\neq y$, $\{t_n\}\subset T$, $\{x_n\}\subset X$ such that

$$x_n\to x, \quad xt_n\to y, \quad x_nt_n\to z,$$

where X^* is the one-point compactification of X. Now that a locally compact connected metric space is separable, hence X^* is a compact metric space; then $\{\infty\}\cup F$ is a set of dimension zero. Let U be an open neighborhood of y such that $\overline{U}$ is compact, $\partial U\subset X-K$ and $z\notin\overline{U}$. Let $X\neq V_1\supset V_2\supset V_3\supset\ldots$ be a decreasing sequence of neighborhoods of x such that their diameters go to zero. Let A_n be the component of V_n containing x. Then there exists N_0 such that for $n>N_0$, $A_n\subset U$. For otherwise A_nt_n would intersect ∂U infinitely often and so there exists $b\in\partial U\cap(D^+(x)\cup D^-(x))$. Hence $x\in D^-(x)\cup D^+(x)\subset\overline{bT}$, which contradicts the hypothesis of the proof. Let $K_{m,n}$ be the component of x_n in $\overline{V}_m$, $n\geqslant m$. For a similar reason we can conclude that there exists M_0 such that if $m>M_0$ then $K_{m,n}t_n\cap\overline{U} = \emptyset$. For a prescribed $\varepsilon>0$ let s be chosen such that $s>M_0$, $s>N_0$, diameter $(V_s)<\frac{1}{2}\varepsilon$ and $\overline{V}_s$ is compact. Let $p_n\in K_{s,n}\cap\partial V_s$. By choosing subsequences we may assume that $p_n\to p$. Since $\underline{\lim}\{K_{s,n}|\ n = s, s+1, \ldots\}\neq\emptyset$, we have $\overline{\lim}\{K_{s,n}|\ n = s, s+1, \ldots\}$ is connected. Hence $p\in A_s$. We may assume $pt_n\to q\in\overline{U}$. Let W be a connected neighborhood of p of diameter $<\frac{1}{2}\varepsilon$. We may assume that $\{p_n\}\subset W$. Since Wt_n is connected, $pt_n\in\overline{U}$ and $p_nt_n\notin\overline{U}$, we have $r\in\partial U$ such that $r\in D^+(p)\cup D^-(p)$. Hence $p\in D^+(r)\cup D^-(r)$.

Denote r by r_ε and let $\varepsilon = 1, \frac{1}{2}, \ldots$ and assume that $c = \lim r_{1/n} \in \partial U$ exists. Then $x \in D^+(c) \cup D^-(c) \subset \overline{cT}$, which contradicts the hypothesis of the proof. Hence (b) is a sufficient condition.

For proving that (a) is a sufficient condition we use the proof of (b). The connected neighborhood W of p should be replaced by a sequence of connected sets in the $\frac{1}{2}\varepsilon$ neighborhood of x. The connected sets join p to p_n.

For proving that (c) is a sufficient condition we modify the proof of (b) by introducing a compact neighborhood U_1 of y, $U_1 \subset U$. Now the sets A_n can be shown to satisfy that $A_n \subset U_1$ for $n \geqslant s$. By the hypothesis of (c) we may assume that $p \in E$. Then $pt_n \to q \in U_1$ and $p_n t_n \notin \overline{U}$ provide a contradiction.

The proof of condition (d) is different. Since x is recurrent, $\overline{xT^+} = \overline{xT} = \overline{xT^-}$. We may assume that there exists $q \in D^+(x) - \overline{xT}$. Let A be the orbit closure of x in (X^*, T), then $q \in D^+(x) - A$. Let U be a neighborhood of A in X^* such that $q \notin \overline{U}$ and $\partial U \subset X - K$. We may assume the existence of $\{t_n\} \subset T^+$ such that $xt_n \to p \in A$ and a sequence $x_n t_n \to q$. By the fact that A is a compact invariant set and by continuity argument there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$, a sequence $\{k_i\}$, $k_i \to \infty$ such that

$$x_{n_i} s_i,\ x_{n_i}(s_i+1), \ldots, x_{n_i}(s_i+k_i-1) \in U \quad \text{and} \quad x_{n_i} k_{n_i} \notin U .$$

We may assume that $x_{n_i} k_{n_i} \to y \in X - U$. Then, $y(-1), y(-2), \ldots \in \overline{U}$. Since y is negatively recurrent we must have $y \in \partial U$. Then $y \in D^+(x) \cap (X-K)$. Hence $x \in D^-(y) \subset \overline{yT}$.

The proof of Lemma 5 is completed.

Corollary 1. *Let (X, T) be a discrete flow on a locally compact zero-dimensional metric space. If (X, T) is pointwise recurrent, then (X, T) is of characteristic $O^\pm$.*

Proof. In condition (d) put K to be X. It follows from Lemma 5 that $F = \emptyset$.

Proof of Theorem 1. We first prove for the case that X is compact. Since F consists of recurrent points, $F \neq \emptyset$, by Lemma 5 G contains a point which is not almost periodic. Applying Theorem 3 to the restricted flow (G, T) we find that $B = \emptyset$. If $x \in G$, then $\alpha(x) \cup \omega(x) \subset F$. We claim that T is equicontinuous at points in G. For suppose not there would exist sequences $\{x_n\} \subset G$, $\{t_n\} \subset T$ such that

$$x_n \to x \in G, \quad xt_n \to p \in F, \quad x_n t_n \to q \in F, \quad q \neq p .$$

We may assume $t_n \geqslant 0$ for all n. Then $p, q \in \omega(x)$. Let U be an open neighborhood of p such that $\overline{U}$ is compact, $\partial U \subset G$ and $q \notin \overline{U}$. Then there exists $\{k_n\} \subset T^+$, $k_n \to \infty$ such that

$$xt_n,\ x(t_n+1), \ldots, x(t_n+k_n-1) \in U \quad \text{and} \quad x(t_n+k_n) \notin U .$$

The proof of (d) of Lemma 5 shows that there is a contradiction to the fact that $B = \emptyset$. We have then shown that $E = G$, where E is the set of points in X where T is equicontinuous. (X, T) is then satisfied what we defined in [6[to be a *strictly almost equicontinuous transformation group* (cf. [6], p. 171 and Theorem 1.5 (5)). Since T is Abelian, by 1.17 of [6] F is the union of one or two fixed points.

It remains to consider the case that X is non-compact by considering the extended flow (X^*, T) on the one-point compactification X^*. It is easy to verify that the set of characteristic $O^\pm$ points in (X^*, T) contains G and the set of non-characteristic $O^\pm$ points $\hat{F}$ in (X^*, T) contains F. Hence (X^*, T) satisfies the hypothesis for (X, T) in the theorem. The proof for the first case then shows that $\hat{F}$ is the union of one or two fixed points. Hence F consists of one or two fixed points.

The proof of Theorem 1 is completed.

The hypothesis that F consists of recurrent points is in general necessary. However, the condition can be omitted if further connectivity is assumed on G or X, as shown in the following theorem.

Theorem 4. *Let (X, T) be a discrete flow on a locally compact connected metric space. Let F be non-empty, closed and totally disconnected and G be a semi-continuum. If G contains a point which is not almost periodic, then F is the union of one or two fixed points.*

Proof. Applying Theorem 3 to (G, T) we have $B = \emptyset$. We show that if $x \in G$ then T is equicontinuous at x. For this fact we need not assume that G is a semicontinuum. We need to assume that there exists $m \neq 0$ such that x and xm can be joined by a continuum K in G. Suppose T is not equicontinuous at x, then there exist sequences $\{x_n\} \subset G$, $\{t_n\} \subset T$ such that

$$x_n \to x \in G, \quad xt_n \to p \in F, \quad x_n t_n \to q \in F, \quad q \neq p .$$

Now let U be an open neighborhood of p such that $\overline{U}$ is compact $\partial U \subset G$ and $q \notin \overline{U}$. We may assume $t_n \geqslant 0$ for all n. Since $q \in J^+(x)$ we have $q \in \omega(x)$. Then there exists $\{k_n\} \subset T^+$, $k_n \to \infty$ such that

$$xt_n,\ x(t_n+1), \ \ldots, \ x(t_n+k_n-1) \in U \quad \text{and} \quad x(t_n+K_n) \notin U .$$

We make the assumption that $z = \lim x(t_n+k_n)$ exists, $z \notin U$. If $y \in K$, then $y(t_n+k_n) \to z$. For if not we can choose a small open neighborhood V of z such that $\overline{V}$ is compact and $\partial V \subset G$. Then $K(t_n+k_n)$ would intersect ∂V for infinitely many n. There would exist $a \in K$, $b \in \partial V$ such that $b \in J^+(a)$. This would contradict $B = \emptyset$. Hence $xm(t_n+k_n) \to z$. On the other hand $xm(t_n+k_n) = (x(t_n+k_n))m \to zm$. Then $z = zm$ is a periodic point. Now by continuity argument we have $z(-1), z(-2), \ldots \in \overline{U}$. Hence $z \in \partial U$. But then $z \in J^+(x) \cap G$ contradicts that $B = \emptyset$. Hence T is equicontinuous at x. Now set $\hat{X} = X$ if X is compact and set $\hat{X} = X^*$ if X is non-compact. Since points in G remain equicontinuous in $(\hat{X}, T)$, (X, T) is again a strictly almost equicontinuous transformation group. The theorem follows from 1.17 of [6].

Theorem 5. *Let (X, T) be a discrete flow on a locally compact connected metric space. Let F be non-empty, closed and totally disconnected and G be connected and locally connected. Then F is the union of one or two fixed points.*

Proof. Use Lemma 5 and Theorem 4.

3. Discrete flows with compact set *F*. In this section we extend results of Section 1 to the case F is compact. The techniques are those of [6, Section 2].

Proof of Theorem 2. Consider the quotient space X_* of X whose set of elements are elements of G and components of F. Since the decomposition is upper semicontinuous, the quotient topology of X_* turns X_* into a compact metric space. Let $\pi\colon X \to X_*$ be the canonical projection. Define (X_*, T) by $(x\pi)t = (xt)\pi$, $x \in X$, $t \in T$. Let G_* and F_* be the corresponding sets for G and F respectively in X_*. It is easy to verify that $G\pi \subset G_*$. The set $G\pi$ is a semi-continuum and since G is assumed to have a point not almost periodic, $G\pi$ also has a point not almost periodic. Applying Theorem 3 on $(G\pi, T)$ we have $B = \emptyset$. The proof of Theorem 4 shows that $G\pi$ consists of equicontinuous points, hence (X_*, T) is a.s.a.e. Hence the set of non-equicontinuous points of X_* consists of one or two fixed points. Let a, b denote the fixed points. Then $G_* = X - \{a, b\}$. Applying Theorem 3 on the restricted set we have $\omega(x) \cup \alpha(x) \subset \{a, b\}$ for all $x \in G_*$. Theorem 2 follows.

COROLLARY 2. *Let (X, T) be a discrete flow on a locally compact metric space X which is not compact. If F is non-empty and compact and G is a semi-continuum which contains a point not almost periodic, then F is a continuum.*

Proof. (X^*, T) satisfies Theorem 2.

THEOREM 6. *Let (X, T) be a discrete flow on a locally compact connected metric space X, F non-empty and closed. If F has two isolated fixed points and G is connected, then X is compact and F is the union of two fixed points.*

Proof. Let p, q be the isolated fixed points of F. The proof for (d) of Lemma 5 shows that there exists $a \in G$ such that $p \in \overline{aT}$. If we assume that $p \in \omega(a)$ the proof of Theorem 1 shows that $p = \omega(a)$ and T^+ is equicontinuous at a. Let $R = \{x \in G \mid \omega(x) = p\}$. In applying Theorem 3 to (G, T) we have $B = \emptyset$. It follows easily that R is both open and closed in G and we have $R = G$. Likewise there exists $b \in G$ such that $\omega(b) \cup \alpha(b) \ni q$. Then $q \in \alpha(b)$. The previous argument applied to T^- shows that $q \in \alpha(x)$ for all $x \in G$ and T^- is equicontinuous at x. It follows easily that (X_*^*, T) is a s.a.e., where X^* is again the one-point compactification of X. Let $\hat{F}$ be the set F for (X^*, T) and let $\pi\colon X^* \to X_*^*$ be the canonical projection by identifying components of $\hat{F}$ to points, as in the proof of Theorem 2. According to 1.17 and 1.22 of [6] a strictly almost equicontinuous transformation group with abelian acting group has one or two fixed points p, q and for any other point y we have $p \in \overline{yT}$ and $q \in \overline{yT}$. Since F already has two isolated fixed points a, b, it follows that X is compact and $\hat{F}$ has no other points but a, b.

4. Continuous flows. The case for continuous flows can be studied by the help of the fact that orbits are connected. Further connectivity of G is not required for this case.

THEOREM 7. *Let (X, T) be a continuous flow on a compact connected metric space, F non-empty and closed, G connected, and let G contain a point which is not almost periodic. Then F is the union of two components K and L (K may equal L) and $\omega(x) = K$, $\alpha(x) = L$ for all $x \in X - (K \cup L)$.*

Proof. We first prove for the case that F is totally disconnected. Applying Theorem 3 to (G, T) we see that $B = \emptyset$. Since T is connected, F consists of fixed points. The proof for Theorem 1 shows that G consists of points at which T is equicontinuous. Again F consists of one or two fixed points a, b. It follows easily that $\omega(x) = a$ and $\alpha(x) = b$ for all $x \in X - \{a, b\}$.

As in Section 2 we form the flow (X_*, T). The general case of Theorem 7 follows from projecting (X, T) to (X_*, T).

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