On a question of Borsuk concerning non-continuous retracts II

by

Kenneth R. Kellum * (Birmingham, Ala.)

Abstract. We prove a rather complicated theorem giving sufficient conditions for a set to be an almost continuous retract of an n-cube. This theorem is then used to construct examples of pathological almost continuous retracts of the unit square. It is shown that there exist both closed and non-closed almost continuous retracts of the unit square which are not arc wise connected and that the closure of an almost continuous retract of the unit square need not be such a retract. On the other hand, we show that if M is an almost continuous retract of I^n, where M is interior to I^n and $n>1$, then M does not separate I^n. Also it is proved that an almost continuous retract of a Peano continuum is almost arc wise connected. This last result affords a simple example of an acylic plane continuum which is not an almost continuous retract of a 2-cell. Lastly, we answer negatively a question of Naimpally and Pareek by giving an example of a Γ_1-almost continuous function on the unit internal which lacks a fixed point.

1. Preliminaries. This paper is a continuation of work begun in [3]. We have attempted to make the present paper as self-contained as possible. However, the reader may wish to refer to [3] for further background.

No distinction is made between a function and its graph. The letter I denotes the interval $[-1, 1]$.

DEFINITION 1. The statement that the function $f\colon X \to Y$ is *almost continuous* means that if $D \subset (X \times Y)$ is an open set containing f, then D contains a continuous function with domain X.

DEFINITION 2. Suppose N is a subset of M. We say that N is an *almost continuous retract* (ACR) *of* M if there exists an almost continuous function from M onto N which leaves each point of N fixed.

Our primary result is Theorem 1, which will be proved in Section 2.

THEOREM 1. *Suppose M is contained in I^n. The following conditions are together sufficient for M to be an* ACR *of I^n*:

(i) *M has at most finitely many arc components, $A_0, A_1, \ldots, A_k$, such that A_0 is dense in M and A_i is no-where dense if $1 \leqslant i \leqslant k$,*

(ii) *M contains no simple closed curve, and*

---

* Author supported in part by NSF Grant GY-10729.

(iii) *There exists a function* f_0: $\mathrm{Cl}(M)\to M$ *which leaves each point of* M *fixed and a sequence of continuous functions* $g_1, g_2, \ldots$ *such that* g_i: $I^n\to A_0$ *and such that if* $P_1, P_2, \ldots$ *is a sequence of points of* $\mathrm{Cl}(M)$ *converging to* P, *then* $g_1(P_1), g_2(P_2), \ldots$ *converges to* $f_0(P)$.

Theorem 2 follows immediately from Theorem 6.1 of [2].

THEOREM 2. *An* ACR *of* I^n *has the fixed point property.*

COROLLARY 1. *Under the hypotheses of Theorem* 1, M *has the fixed point property.*

DEFINITION 3. The statement that the subset K of $X\times Y$ is a *blocking set* of f: $X\to Y$ means that K contains no point of f, K is closed and K intersects g whenever g: $X\to Y$ is continuous. If no proper subset of K is a blocking set of f, we say that K is a *minimal blocking set* (MBS) *of* f.

Various versions of Theorem 3 have been used by the author in earlier papers. We give a statement and proof here for the sake of completeness.

THEOREM 3. *Suppose* f: $X\to Y$ *is not almost continuous and* X *is compact. Then there exists a minimal blocking set* K *of* f.

Proof. Since f is not almost continuous, there exists an open set D containing f such that D contains no continuous function with domain X. Then $(X\times Y)-D$ is a blocking set of f. Let θ be a collection of blocking sets of f, linearly ordered by inclusion. Suppose g: $X\to Y$ is continuous. The *set* g is a homeomorph of X and is compact. For each C in θ, $g\cap C\neq\emptyset$, so $\{g\cap C\colon C \text{ is in } \theta\}$ is a nested collection of closed non-empty subsets of g must have a non-empty intersection. It follows that $\bigcap\{C\colon C \text{ is in } \theta\}$ is a blocking set of f. The result now follows from Zorn's lemma.

2. Proof of Theorem 1. We break the proof into a series of lemmas. We will need some notation. Suppose K is a subset of $I^n\times M$. The projection of K into I^n is denoted by $p_x(K)$ and the projection of K into M is denoted by $p_y(K)$. If A is a subset of $p_x(K)$, then $K|A=\{(x,y) \text{ in } K\colon x \text{ is in } A\}$, and if A consists of a single point, z, we write $K|z$.

LEMMA 1. *Suppose* K *is a* MBS *of a function* f: $I^n\to M$, *where* M *satisfies conditions* (i) *and* (ii) *of Theorem* 1. *if* z *is an isolated point of* $p_x(K)$, *then* $p_y(K|z)$ *contains some* A_i.

Proof. Let U be a neighborhood of z such that $p_x(K)\cap U=\{z\}$. By the minimality of K there exists a continuous function g: $I^n\to M$ such that $g\cap K$ consists of the point $(z, g(z))$. Let A_i be the arc component of M containing $g(I^n)$. Assume that $p_y(K|z)$ does not contain A_i. Let (z, y) be in the complement of $K|z$ where y is in A_i. Let C be an arc in A_i joining y and $g(I^n)$. Since M contains no simple closed curve, $g(I^n)\cup C$ is a dendrite ([6], p. 300) and hence an AR ([6], p. 335). So, the function h: $(I^n-U)\cup\{z\}\to g(I^n)\cup C$ defined by $h|I^n-U=g|I^n-U$ and $h(z)=y$ has a continuous extension h': $I^n\to g(I^n)\cup C$. But then $h'\cap K=\emptyset$, a contradiction. Thus $A_i\subset p_y(K|z)$.

LEMMA 2. *Under the hypotheses of Lemma* 1, $p_y(K|z)\cap A_0=\emptyset$.

Proof. First note that since A_0 is dense in M, K is closed and $K\cap f=\emptyset$, A_0 cannot be contained in $K|z$. Thus $A_i\neq A_0$. Assume that (z, x) is in K where x is in A_0. Let V be a neighborhood of x which intersects none of $A_1, A_2, \ldots, A_k$. Now, $K-(K\cap(U\times V))$ is a closed proper subset of K, so there exists a continuous function g: $I^n\to M$ such that $g\cap K$ consists of a point $(z, g(z))$ where $g(z)$ is in V. But then $g(I^n)$ is contained in A_0. Let y be in $A_0-(A_0\cap p_y(K|z))$. A contradiction can now be reached in the same way as in the proof of Lemma 1.

LEMMA 3. *Under the hypotheses of Lemma* 1: *if* z_1 *and* z_2 *are two different isolated points of* $p_x(K)$, *then* $p_y(K|z_1)$ *and* $p_y(K|z_2)$ *do not both contain the same arc component of* M.

Proof. Note that each of $K-K|z_1$ and $K-K|z_2$ intersects each continuous function from I^n into A_0. Assume that $p_y(K|z_1)$ and $p_y(K|z_2)$ contain the same arc component, say A_1. Let V be a neighborhood of y which meets none of $A_2, A_3, \ldots, A_k$. There exists a continuous function g: $I^n\to M$ such that $(g\cap K)\subset(\{z_1\}\times V)$. But, since $g(I^n)\cap A_0=\emptyset$, we have $g(I^n)\subset A_1$ and $g\cap K|z_1\neq\emptyset$, a contradiction.

Completion of the proof. Denote by θ the set of all closed subsets of $I^n\times M$ such that $p_x(L)$ has c-many points not in $\mathrm{Cl}(M)$. Using transfinite induction, we may define a function f: $I^n\to M$ such that $f|\mathrm{Cl}(M)=f_0$ and f intersects each member of θ. We complete the proof by showing that f is almost continuous. Assume that it is not. By Theorem 2, there exists a MBS K of f. Then K intersects each g_i in a point $(P_i, g_i(P_i))$. Let T be the set of all isolated points of $p_x(K)$. By Lemmas 1 and 3, T is finite, so $p_x(K)-T$ is perfect. By the construction of f, $p_x(K)-T\subset\mathrm{Cl}(M)$. For each i, $g_i(I^n)\subset A_0$, so, by Lemma 2, P_i is not in T, so P_i is in $\mathrm{Cl}(M)$. We may assume that $P_1, P_2, \ldots$ converges to some point P. But then, by hypothesis, $g_1(P_1), g_2(P_2), \ldots$ converges to $f_0(P)=f(P)$. Since K is closed $(P, f(P))$ is in K, a contradiction. This completes the proof.

3. Examples. In this section, Theorem 1 is used to show that the following sets are ACR's of I^2.

M_1: Let $M_1=\{(x, (\sin 1/x))\colon 0<x\leqslant\frac{1}{2}\}\cup\{(0,0)\}$.

M_2: Let $M_2=\mathrm{Cl}(M_1)$.

M_3: Let $S=\{(r\cos\alpha, r\sin\alpha)\colon r=\alpha/(1+2\alpha);\ 0\leqslant\alpha\}$, so that S is a spiral beginning at $(0,0)$ closing outward on the circle $C=\{(x,y)\colon x^2+y^2=\frac{1}{2}\}$. Let M_3 be the set S together with the single point $P=(\frac{1}{2},0)$ of C.

Clearly, each of M_1, M_2, and M_3 satisfy conditions (i) and (ii) of Theorem 1. All that is required to prove that these sets are ACR's of I^2 is to construct f_0 and $g_1, g_2, \ldots$ We indicate the construction of these functions but omit the details of showing that condition (iii) is satisfied.

M_1 *is an* ACR *of* I^2.

Proof. Let f_0 leave points of M_0 fixed and for each P in $\mathrm{Cl}(M_1)-M_1$, let $f_0(P)=(0,0)$. Let g_i be a retraction of I^2 onto the arc $\{(x, \frac{1}{2}(\sin 1/x))\colon 1/2i\pi\leqslant x\leqslant\frac{1}{2}\}$ such that if $-1\leqslant x\leqslant 1/2i\pi$, then $g_i(x,y)=(1/2i\pi, 0)$.

M_2 *is an* ACR *of* I^2.

Proof. Let f_0 be the identity of M_2. Let g_i be a retraction of I^2 onto $B_i = \{(x, \frac{1}{2}(\sin 1/x))\colon 1/\pi(2i+\frac{3}{2}) \leqslant x < \frac{1}{2}\}$, such that if $-\frac{1}{2} \leqslant y \leqslant \frac{1}{2}$ and (x, y) lies to the left of B_i, then $g_i(x, y)$ is the left-most point of $(I \times \{y\}) \cap B_i$.

M_3 *is an* ACR *of* I^2.

Proof. Let $f_0(Q) = P$ if Q is in $\mathrm{Cl}(M_3) - M_3$. Let $C_1, C_2, \ldots$ be a sequence of circles each with center $(0, 0)$ such that $C_1 \cap M_3, C_2 \cap M_3, \ldots$ is a sequence of points $P_1, P_2, \ldots$ converging to P. Denote by C_i^* the circle C_i together with its interior. Let g_i be a retraction of I^2 onto $M_3 \cap C_i^*$ such that $g_i(T) = P_i$ if T is in $I^2 - \mathrm{int}(C_i^*)$.

Note that $\mathrm{Cl}(M_3)$ does not have the fixed point property, so $\mathrm{Cl}(M_3)$ is not an ACR of I^2. In summary, we have that an ACR of I^2 need not be arc wise connected even if it is closed, and the closure of an ACR of I^2 may or may not be an ACR of I^2.

4. ACR's of Peano continua. In this section we obtain some properties of ACR's of Peano continua.

The proof of Theorem 4 is a simple modification of the proof of Theorem 3.16 of [1], and is omitted.

THEOREM 4. *Suppose M is an* ACR *of I^n, where M is closed and in the interior of I^n and $n > 1$. Then M does not separate I^n.*

DEFINITION 4. The space M is said to be *almost arc wise connected* (AAWC) if for each pair of disjoint open sets U and V in M there exists an arc in M which meets both U and V.

THEOREM 5. *An* ACR *of a Peano continuum is* AAWC.

Proof. Suppose M is an ACR of the Peano continuum N, and let $f\colon N \to M$ be an almost continuous function which leaves points of M fixed. Let x be in U and let y be in V where U and V are disjoint open sets in M. Then

$$D = (N \times M) - \big((\{x\} \times (M - U)) \cup (\{y\} \times (M - V))\big)$$

is an open set containing f. So D contains a continuous function g with domain N. Then $g(N)$ is a Peano continuum, $g(x)$ is in U, $g(y)$ is in V, $g(N)$ is arc wise connected, and the result follows.

Note. The set $M_2 \cup ([-\frac{1}{2}, 0] \times \{0\})$ is not AAWC and cannot be an ACR of I^2, even though it is a continuum with the fixed point property.

THEOREM 6. *The continuous image of an* AAWC *space is* AAWC.

Proof. Suppose f is continuous with domain the AAWC space N. Suppose U and V are disjoint open sets in $f(N)$. Let A be an arc in N which meets both $f^{-1}(U)$ and $f^{-1}(V)$. Then $f(A)$ contains the desired arc.

Using Theorem 6 and a modification of the proof of Theorem 5 we can now prove:

THEOREM 7. *An* ACR *of an* AAWC *space is* AAWC.

We can now shed some light on the problem suggested in Remark 1 of [3].

COROLLARY 2. *An* ACR *of an* ACR *of a Peano continuum is* AAWC.

5. Γ_1-almost continuity. We now digress a bit. It would be of interest to study types of functions and retracts, more general than almost continuous, which have the fixed point property when defined on an n-cube. The notion of Γ_1-almost continuity was introduced by Naimpally and Pareek [7]. They leave open the question as to whether a Hausdorff space with the fixed point property for continuous functions also has the fixed point property for Γ_1-almost continuous functions. We will give a counter-example.

DEFINITION 5. The function $f\colon X \to Y$ is *Γ_1-almost continuous* if each open set of the form $\bigcap_{i=1}^{n} ((X \times Y) - (A_i \times B_i))$, where A_i and B_i are closed, contains a continuous function with domain X.

EXAMPLE. Let $f\colon I \to I$ be a function which has no fixed point such that if U is an open subset of I, then $f(U) = I$. Then f is Γ_1-almost continuous. To see that this is true, assume not. Then there exists an open set $\bigcap_{i=1}^{n} I^2 - (A_i \times B_i)$ which contains no continuous function. Then $\bigcup_{i=1}^{n} (A_i \times B_i)$ contains a MBS, K. By Theorem 1 of [5], the X-projection of K is connected and non-degenerate, so $\bigcup_{i=1}^{n} A_i$ contains an open set. But then some A_i contains an open set and $f \cap (A_i \times B_i) \neq \emptyset$, a contradiction.

6. Questions and remarks.

1. It would be of interest to know something about AAWC spaces.

2. Note that Theorems 5 and 7 and Corollary 2 all hold for Γ_1-almost continuous functions.

3. Under what conditions is a subset K of the plane a MBS of some real function? It is known that K must be a Cantor set with connected X-projection [4].

4. Call M a *weak almost continuous retract* (WACR) of I^n if there exists an almost continuous function $f\colon I^n \to I^n$ such that $f(I^n) = M$ and f leaves each point of M fixed. In Question 10 of [8] it is asserted, in effect, that if the continuum M is a WACR of I^2, then M has the fixed point property. This author has thus far been unable to determine if this is true.

The point is, when considering the almost continuity of a function f, one must be carefull to specify the space that is to be considered the range of f. Let $f\colon X \to Y$ be any function. Let Y' be the set Y together with a point P and declair a non-empty subset U of Y' to be open if P is in U and $U - \{P\}$ is open in Y. Then f, considered as a function from X into Y', is almost continuous. This raises the following question. Suppose $f\colon X \to Y$ is almost continuous and Z is a subspace of Y such that $f(X) \subset Z$. Under what conditions is $f\colon X \to Z$ almost continuous? One obvious condition is that Z be open in Y. Another condition is that Z be a retract of Y.

5. In Theorem 1 the functions $g_1, g_2, \ldots$ may just as well be taken to be almost continuous functions.

6. Can an indecomposable continuum be an ACR of I^2?

References

[1] S. K. Hildebrand and D. E. Sanderson, *Connectivity functions and retracts*, Fund. Math. 57 (1965), pp. 237–245.

[2] H. B. Hoyle III, *Connectivity maps and almost continuous functions*, Duke Math. J. 37 (1970), pp. 671–680.

[3] K. R. Kellum, *On a question of Borsuk concerning non-continuous retracts I*, Fund. Math. 87 (1975), pp. 89–92.

[4] — *Sums and limits of almost continuous functions*, Colloq. Math. 31 (1974), pp. 125–128.

[5] — and B. D. Garrett, *Almost continuous real functions*, Proc. Amer. Math. Soc. 33 (1972), pp. 181–184.

[6] K. Kuratowski, *Topology*, vol. 2, New York–London–Warszawa 1968.

[7] S. A. Naimpally and C. M. Pareek, *Graph topologies for function spaces II*, Prace Mat. 13 (1970), pp. 221–231.

[8] J. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. 47 (1959), pp. 249–263.

MILES COLLEGE
Birmingham, Alabama

Accepté par la Rédaction le 9. 9. 1974

Discrete and continuous flows of characteristic $O^{\pm}$

by

Ping-Fun Lam * (Columbia, Miss.)

Abstract. The objective of this paper is to show some application of strictly almost equicontinuous transformation groups (s.a.e.) of [6] on discrete and continuous flows, making no assumption of the presence of equicontinuous points. We assume that the phase spaces X to be locally compact and connected, the set G of points of characteristic $O^{\pm}$ to be open and dispersive. Then we show that under general conditions there exist components K and L of $X-G$ which acts as positive and negative centers of attraction and sufficient conditions for G to be dispersive are also obtained.

Let (X, T) be a discrete or continuous flow on a locally compact metric space. Hence T denotes the group of integers for the case of discrete flow, the group of real numbers for the case of continuous flow. Let T^+ and T^- be respectively the set of non-negative and the set of non-positive elements of T. Let $x \in X$ and $\mathcal{U}_x$ be the neighborhood filter of x. The *positive* (*negative*) *prolongation* of $x \in X$ is defined to be the set

$$D^+(x) = \bigcap_{U \in \mathcal{U}_x} \overline{UT^+} \quad \Big(D^-(x) = \bigcap_{U \in \mathcal{U}_x} \overline{UT^-}\Big).$$

A point $x \in X$ is said to be of characteristic O^+, O^-, $O^{\pm}$, if respectively we have $D^+(x) = \overline{xT^+}$, $D^-(x) = \overline{xT^-}$; $D^+(x) = \overline{xT^+}$ and $D^-(x) = \overline{xT^-}$. The set of points of characteristic $O^{\pm}$ is invariant and is denoted by G; its complement is denoted by F. A point $y \in D^+(x)$ is characterized by that there exist sequences $\{x_n\} \subset X$ and $\{t_n\} \subset T^+$ such that $x_n \to x$ and $x_n t_n \to y$; points in $D^-(x)$ may be similarly characterized. If the group T is equicontinuous at $x \in X$, then x is of characteristic $O^{\pm}$; converse is in general not true. A flow (discrete or continuous) is said to be of *characteristic* $O^{\pm}$ if $X = G$.

In this paper we are mainly interested in the studying of F and the corresponding discrete flow when G is sufficiently large. Results for continuous flows are also obtained (Section 3) under simpler approach. The main results are application of our works [5] and [6]. Two of the main theorems which we obtain in this paper are the following.

---

* Partially supported by NSF Grant GP-38885.