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Equitable partitions of the continuum

by

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Abstract. The real line contains no dense homogeneous subset which is isomorphic to its complement.

A partition of the real line R into two subsets S and $R\setminus S$ is *equitable* if S is dense in R, S is homogeneous (in the sense of order-preserving permutations) and $S\approx R\setminus S$ (as ordered sets). It became of interest to find such a partition during the study of automorphisms of certain ordered permutation groups [2]. However, it seems to be of independent set-theoretic interest whether such a partition exists. It will be shown in this paper that no such partition exists.

Suppose then that S is such a subset. First note that $R\setminus S$ must also be a dense subset of R. For consider a maximal real interval J contained in S; J cannot be open, for then its end points are adjacent points of $R\setminus S$, which implies $R\setminus S$, and hence also S, is discrete, an obvious impossibility. Thus, J must contain at least one of its end points. Suppose that J is not just a single point. Then since S is homogeneous, there must exist an order-preserving permutation f of S which maps an end point of J into the interior of J. Because of the density of S in R, f has a unique extension to an order-preserving permutation of R. Then $J\cup Jf^{-1}$ is a real interval contained in S, denying the maximality of J. Hence J is just a single point and $R\setminus S$ is dense in R.

Let $\varphi\colon S\to R\setminus S$ denote the assumed order isomorphism and observe that also $\varphi\colon R\setminus S\to S$. Because of the density of S and $R\setminus S$, φ has a unique extension to an order-preserving permutation of R. The same is true of any order-preserving permutation of S or of $R\setminus S$. No notational distinction will be made between a map and its extension.

It is now useful to study the group $A(S)$ of all order-preserving permutations of S. A convex subset $B\subseteq S$ is an *o-block* if for each $g\in A(S)$, $Bg=B$ or $Bg\cap B$ is empty. $A(S)$ is *o-primitive* if there are no non-trivial o-blocks. It is known [1] that for an o-primitive $A(S)$, there are just these two possibilities: (i) $A(S)$ is *o-2-transitive*; for each $x<y$, $z<w$, $x,y,z,w\in S$, there exists $h\in A(S)$ such that $xh=z$ and

$yh = w$, or (ii) $A(S)$ is *uniquely transitive*; for each $x, z \in S$, there exists exactly one $h \in A(S)$ such that $xh = z$. In this second case, $A(S)$ is isomorphic (as a group) to a subgroup of the additive group of real numbers, which in turn is isomorphic (as an ordered set) to S (Ohkuma [3]).

It will be shown that the $A(S)$ of this study falls in case (ii). First, $A(S)$ is o-primitive. For consider any o-block $B \subseteq S$. Let $\bar{B}$ be the convexification of B in R. If B were a proper block, let $a \in R$ be an end point of B. Whether $a \in S$ or $a \in R \setminus S$, there exists $f \in A(S)$ such that af is in the interior of B, so that $Bf \neq B$, yet $Bf \cap B$ is not empty, a contradiction. Hence there is no proper o-block, and $A(S)$ is o-primitive.

Next, if $A(S)$ were o-2-transitive choose points $a<x<y<b$ with $a, b \in R \setminus S$, $x, y \in S$. Then $a\varphi, b\varphi \in S$ and $a\varphi < b\varphi$. Hence there exists $g \in A(S)$ such that $a\varphi g = x$ and $b\varphi g = y$. Then the continuous map φg must have a real fixed point in the interval $[a, b]$, which is impossible since $\varphi g\colon S \to R \setminus S$ and $\varphi g\colon R \setminus S \to S$.

Hence $A(S)$ is uniquely transitive on S, and S is isomorphic (as an ordered set) to a subgroup of the additive group of real numbers. If this isomorphism is ψ, then since $S\psi$ is a non-discrete subgroup of R, $S\psi$ is dense in R, so that ψ has an extension to an order-preserving permutation of R. Thus, $S\psi \approx R \setminus S\psi$, and it may be assumed henceforth that $S\psi = S$ is a subgroup of R such that $A(S)$ is uniquely transitive and $S \approx R \setminus S$.

For each $s \in S$, let τ_s denote the translation $x\tau_s = x+s$ of S. The only order-preserving automorphism of S is the identity map, for if σ is such, then σ and $\tau_{0\sigma}$ are members of $A(S)$ which agree at 0 and so must be equal; but the only τ_s which is a group homomorphism is the identity map.

It is clear that the map $g \mapsto \varphi^{-1} g \varphi$ is an order-preserving group automorphism of $A(S)$. Therefore, $g = \varphi^{-1} g \varphi$ for all $g \in A(S)$.

If $x\varphi < xh$ for some $x \in S$, $h \in A(S)$, then $y\varphi < yh$ for all $y \in S$; for by homogeneity, there exists $g \in A(S)$ such that $xg = y$, which gives rise to $y\varphi = xg\varphi = x\varphi g < xhg = xgh = yh$, using the fact that φ commutes with all $g \in A(S)$ from the previous paragraph, and that $A(S)$, being isomorphic to S as a group, is abelian.

The map φ can now be described arithmetically. In fact $s\varphi = s+0\varphi$ for every $s \in S$. For otherwise, say $s\varphi > s+0\varphi$ for some $s \in S$. Then $0\varphi < s\varphi - s$, and by density, there exists $z \in S$, $0\varphi < z = 0\tau_z < s\varphi - s$. By the previous paragraph, $s\varphi < s\tau_z = s+z < s\varphi$, a contradiction. Similarly, it cannot be that $s\varphi < s+0\varphi$, and so $s\varphi = s+0\varphi$.

The conclusion is that $R \setminus S = S\varphi = S+0\varphi$ is a coset of S in the group R, and in fact is the only other coset of S; that is, S has index two in R. But as R is a divisible group, it has no subgroups of finite index, and so no such S can exist.

Two further unsolved questions of this sort may be posed.

Is there an equitable partition of R into three or more subsets?

Is there an equitable partition of R^n?

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