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INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

Accepté par la Rédaction le 8. 7. 1974

Two model theoretic ideas in independence proofs

by

David Pincus (Cambridge, Mass.)

Abstract. Some new Fraenkel-Mostowski models are built on universal homogeneous structures. Also a connection is established between indiscernability theorems and models for the compactness theorem.

I. Introduction

This paper will illustrate the model theoretic ideas underlying some set theoretical independence proofs. The results include conceptual simplifications of known independence proofs, new independence proofs, and a new theorem in model theory.

In § II we discuss Fraenkel-Mostowski models built on universal homogeneous structures. The idea dates back to Mostowski's proof, [17], of the independence of the axiom of choice, (AC), from the ordering principle. Mathias [16] reawakened interest in the idea with his proof of the independence of the order extension principle from the ordering principle. Others followed, notably Plotkin ([23] and [24]), and Felgner ([3] and [4]) as well as the author. Except for [17] the work cited above is set up in the language of forcing. Arguments here and in [13] demonstrate that only Fraenkel-Mostowski ideas are involved.

In § IIA we indicate what, besides the universality and homogeneity of the structure, is involved in proving the support intersection lemma of Mostowski [17]. These results are applied in the remainder of § II. § IIB contains a conceptual proof of the combinatorial group-theoretic lemma of Läuchli [15]. The resulting Fraenkel-Mostowski model is then used to settle a question of Halpern [9]. In § IIC we eliminate forcing from Gauntt's solution ([7]) to Mostowski's problem on the axiom of choice for finite sets. A by-product is that these results, and related ones of Truss [27], transfer automatically to ZF set theory (1). § IID is a brief mention of other applications. These are from the author's thesis and are more fully exposited by Jech in [13].

(1) Our set theories incorporate classes when desirable. ZF is the usual Zermelo Fraenkel set theory. ZFA is the usual weakening (see [17]) of ZF to permit a set of atoms. E is Godel's axiom of strong choice. ZFE is ZF+E. ZFE is a conservative extension of ZF+AC. We assume that our standard universe, Std, satisfies ZFE.

In § III we demonstrate a heuristic equivalence between indiscernability theorems in model theory and models illustrating the independence of AC from the compactness theorem for logic. First the Erenfeucht-Mostowski theorem is used to give a straightforward proof of the compactness theorem in the model of Mostowski [17]. Halpern [8], proved the Boolean prime ideal theorem (a set-theoretic equivalent to the compactness theorem) using algebraic and combinatorial methods. The "equivalence" of the two facts is next shown i.e. the Erenfeucht-Mostowski theorem is deduced from the fact that the compactness theorem holds in Mostowski's model. Finally we apply this argument to the model of Halpern and Levy [12], where the compactness theorem also holds. This gives a new indiscernability principle for model theory (²).

II. Fraenkel-Mostowski models based on universal homogeneous structures

IIA. Sufficient conditions for Mostowski's intersection lemma. The conditions given below are not appealing. They do seem to be what is involved.

IIA1. Assumptions on the category $\mathscr{C}$. We fix a category $\mathscr{C}$ of first order structures with all structure preserving homomorphisms. Notions such as embedding, isomorphism, etc., are understood to be with reference to $\mathscr{C}$. For example if $A \in \mathscr{C}$ then B is a *substructure* of A if $B \in \mathscr{C}$ and $\mathrm{Id}\, B$, the identity on B, is a $\mathscr{C}$ homorphism from B to A. The first assumption is a natural one.

a If $A \in \mathscr{C}$ the intersection of a family of substructures of A is a substructure of A. If A is isomorphic to a structure of $\mathscr{C}$ then $A \in \mathscr{C}$.

Assumption a implies that if A is a structure and $X \subset A$ (the underlying set of A) then there is a unique substructure $[X]$, of A generated by X. The next 3 assumptions on are reasonable in view of [14].

b There is a unique (up to isomorphism) structure which can be embedded in every other structure.

A and B are said to have *coherent intersection* if $A \cap B$ is a substructure of both A and B and if the operations and relations of A agree with those of B on $A \cap B$.

c If A and B have coherent intersection then both are substructures of a common structure C.

d The direct union of structures is a structure.

We now come to the ugly assumption. Fix the finitely generated structures B_0, B_1, B_2, and B with $B_0 = B_1 \cap B_2$ and $B = [B_1 \cup B_2]$. A pair, (Ω, π) is *disposed* when Ω is a structure and $\Omega = [B \cup C]$ for a structure C such that π is an isomorphism from B to C which fixes B_0 pointwise. Two disposed pairs (Ω, π) and (Ω', π') are *singly linked* if for some $m \in \{1, 2\}$, $\mathrm{Id}\, B_m \cup \pi' \pi^{-1}$ extends to a well defined isomorphism, φ_m: $[B_m \cup C] \to [B_m \cup C']$. The single linking relation is reflexive and symmetric. The *linking* relation is the transitive closure of the single linking relation. It is an equivalence relation. The assumption is:

e Let (Ω, π) and (Ω', π') be disposed pairs with $B \cap C = B \cap C' = B_0$. Then (Ω, π) and (Ω', π') are linked.

(²) This result is abstracted in [22].

In the present context a *universal homogeneous* (UH) structure, A is one which satisfies:

1) Every finitely generated B can be embedded in A.

2) Every isomorphism of finitely generated substructures of A extends to an automorphism of A.

Such a structure consequently also satisfies:

3) If B is finitely generated and C is a common substructure of B and A then there is an embedding of B into A which is the identity of C.

IIA2. Lemma ([14]). *Assumptions* a, b, c, *and* d *of* IIA1 *guarantee that* $\mathscr{C}$ *contains a* UH *structure.*

IIA3. Lemma. *Under the assumptions of* IIA1, *if* B_0, B_1, B_2, *and* B *are fixed* (*as before* IIA1e) *then any two disposed pairs are linked.*

Proof. One must show that any disposed (Ω, π) is linked to a disposed (Ω', π') with $B \cap C' = B_0$. Let ϱ: $[B_1 \cup C]_\Omega \to Z$ be an isomorphism to a structure Z with $Z \cap B = B_1$ and let ϱ be the identity of B_1. This is easily arranged via assumption a of IIA1. Z and B have the coherent intersection B_1 so let $\Omega^* = [B, Z]$ be a common extension of B and Z from assumptions c and a of IIA1. Let $\pi^* = \varrho\pi$. ϱ is the required extension of $\mathrm{Id} B_1$ and $\pi^* \pi^{-1}$ so (Ω, π) and (Ω^*, π^*) are linked. $B \cap C^* = B_1$.

We now do a similar process to eliminate intersection with $B_1 - B_0$. μ: $[B_2 \cup C^*]_{\Omega^*} \to \Sigma$ is an isomorphism where $\Sigma \cap B = B_0$ and μ is the identity on B_0. This is possible since $C^* \cap B_2 = B_0$. $\Omega' = [B, \Sigma]$ and $\pi' = \mu\pi^*$ from the required disposed pair linked to (Ω, π) and satisfying $B \cap C' = B_0$.

IIA4. Mostowski's intersection lemma generalized. *Under the assumptions of* IIA1, *let* $A \in \mathscr{C}$ *be a* UH *structure. If* B *is a substructure let* $\Gamma(B)$ *denote the group of automorphisms of* A *which are the identity on* B. *Then for arbitrary finitely generated substructures* B_1 *and* B_2 *of* A,

$$\Gamma(B_1 \cap B_2) = [\Gamma(B_1) \cup \Gamma(B_2)] .$$

Proof. Evidently $[\Gamma(B_1) \cup \Gamma(B_2)] \subset \Gamma(B_1 \cap B_2)$. For the converse fix $\sigma \in \Gamma(B_1 \cap B_2)$. Set $B_0 = B_1 \cap B_2$, $\Omega = B = [B_1 \cup B_2]_A$, $\pi = \mathrm{Id}\, B$, $\Omega' = [B \cup \sigma[B]]$, and $\pi' = \sigma \mathrm{Id} B$. By Lemma IIA3 (Ω, π) and (Ω', π') are linked. This means there is a singly linked sequence $(\Omega, \pi) = (\Omega_0, \pi_0)$, (Ω_1, π_1), ..., $(\Omega_n, \pi_n) = (\Omega', \pi')$. We modify this sequence so that each $\Omega_i \subset A$.

Let $\Omega_i \not\subset A$. Use property 3) of UH structures (see Lemma IIA2) to give an embedding $\varrho\colon \Omega_i \to A$ which is the identity on $\boldsymbol{B}$. Let $\Omega_i^* = \varrho[\Omega_i]$ and $\pi_i^* = \varrho\pi_i\varrho^{-1}$. We establish that (Ω_i^*, π_i^*) is singly linked with $(\Omega_{i-1}, \pi_{i-1})$ and $(\Omega_{i+1}, \pi_{i+1})$. For some $m, k \in \{1, 2\}$ there are maps μ and ν which extend $\operatorname{Id}\boldsymbol{B}_m \cup \pi_i\pi_{i-1}^{-1}$, and $\operatorname{Id}\boldsymbol{B}_k \cup \pi_{i+1}\pi_i^{-1}$ respectively. $\varrho\mu$ and $\nu\varrho^{-1}$ respectively extend $\operatorname{Id}\boldsymbol{B}_m \cup \pi_i^*\pi_{i-1}^{-1}$ and $\operatorname{Id}\boldsymbol{B}_k \cup \pi_{i+1}\pi_i^{*-1}$. This is because ϱ is the identity on $\boldsymbol{B}$ hence $\varrho\mu$ and $\nu\varrho$ are identities on $\boldsymbol{B}_m$ and $\boldsymbol{B}_k$ respectively. On $\boldsymbol{C}_{i-1}$ $\varrho\mu$ extends $\varrho\pi_i\pi_{i-1}^{-1} = \varrho\pi_i\varrho^{-1}\pi_{i-1}^{-1} = \pi_i^*\pi_{i-1}^{-1}$, again because ϱ is the identity on $\boldsymbol{B}$. Similarly $\nu\varrho$ extends $\pi_{i+1}\pi_i^{*-1}$ on $\boldsymbol{C}_i^*$. The new sequence obtained by replacing each (Ω_i, π_i), $\Omega_i \not\subset \boldsymbol{A}$, with (Ω_i^*, π_i^*) is as desired.

For $i = 2, \ldots, n$ let $m_i \in \{1, 2\}$ and μ_i be maps such that μ_i extends $\pi_i\pi_{i-1}^{-1}$ and $\operatorname{Id}\boldsymbol{B}_{m_i}$. Let $\bar{\mu}_i$ be an extension of μ_i to an automorphism of A. Evidently $\bar{\mu}_i \in \Gamma(\boldsymbol{B}_{m_i})$ so $\bar{\mu}_n\bar{\mu}_{n-1} \ldots \bar{\mu}_2 \in [\Gamma(\boldsymbol{B}_1) \cup \Gamma(\boldsymbol{B}_2)]$. $\bar{\mu}_n \ldots \bar{\mu}_2$ extends

$$(\pi_n\pi_{n-1}^{-1})(\pi_{n-1}\pi_{n-2}^{-1}) \ldots (\pi_2\pi_1^{-1})$$

which reduces to $\pi_n\pi_1^{-1} = \sigma \operatorname{Id}\boldsymbol{B} = \sigma$ on $\boldsymbol{B}$. Therefore $(\mu_n \ldots \mu_2)^{-1}\sigma$ is the identity on $\boldsymbol{B}$ so $(\mu_n \ldots \mu_2)^{-1}\sigma \in \Gamma(\boldsymbol{B}_1)$ and $\sigma \in [\Gamma(\boldsymbol{B}_1) \cup \Gamma(\boldsymbol{B}_2)]$.

IIB. On a question of Halpern. Halpern [9] answered a question of Tarski by showing that SPI ("Every infinite set has a nonprincipal prime ideal in its power set.") is strictly weaker than PI, the prime ideal theorem for Boolean algebras. His results show that SPI is really a statement about sets of small (Dedekind finite or countable) cardinality. He then introduces the global form S, of SPI which follows:

> "There is a function which associates every infinite set with a nonprincipal prime ideal on its power set".

S has considerably more strength than SPI. Halpern proves, using regularity, that S implies the axiom of choice for families of finite sets. Howard and the author have independently eliminated the use of regularity.

IIB1. THEOREM (ZF without regularity). S *implies the axiom of choice for families of finite sets.*

Proof. For each finite set A let $X_A = \omega \times A$. Let θ_A be the designated prime ideal in $\mathscr{C}(X_A)$. Choose from A the unique $a \in A$ such that $\{a\} \times \omega \in \theta_A$.

Halpern asks whether $\mathrm{S} \to \mathrm{PI}$. We state on the contrary:

IIB2. THEOREM. *The ordering principle is independent of* S *in* ZF.

We give in [21] Postscript IIIa proof that S transfers from a Fraenkel-Mostowski model to ZF in conjunction with class II statements from [20]. Thus it suffices to replace ZF by ZFA in Theorem IIB2 and use a Fraenkel-Mostowski model. This is done below. Along the way we reprove:

IIB3. LEMMA (Läuchli [15]). *There is a group, Γ, acting on a set A such that*

a *If $G \subset A$ is finite there are $\pi_1, \pi_2 \in \Gamma(G)$ and $a, b, c, \in A$ such that*

$$a \xrightarrow{\pi_1} b \xrightarrow{\pi_1} c \qquad b \xrightarrow{\pi_2} c \xrightarrow{\pi_2} a\,(^3).$$

b *If $G \subset A$ is finite and $\pi \in \Gamma$ permutes G then $\pi \in \Gamma(G)$.*

c *If G_1 and G_2 are finite subsets of A then $\Gamma(G_1 \cap G_2) = [\Gamma(G_1) \cup \Gamma(G_2)]$.*

IIB4. The theory of finite choice operators. FC is a first order theory with a partial function symbol f_n for each $n \in \omega$, $n \geqslant 1$. The axioms of FC state that $f_n(x_1, \ldots, x_n)$ is an n-ary choice operator. This means that f_n is defined when $\bigwedge_{i \neq j} x_i \neq x_j$, $f_n(x_1, \ldots, x_n) = f_n(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for any permutation π on $\{1, \ldots, n\}$, and when $f_n(x_1, \ldots, x_n)$ is defined then $f(x_1, \ldots, x_n)$ is some x_i. These axioms are easily given first order formulation. In the context of FC one can forget n and talk of $f\{x_1, \ldots, x_n\}$.

IIB5. LEMMA. *Let $\mathscr{C}$ be the category of models of* FC. *$\mathscr{C}$ satisfies the assumptions of* IIA1.

Proof. The assumptions a, b, c, and d of IIA1 are familiar and their proofs are left to the reader. Fix $\boldsymbol{B}_0$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, and $\boldsymbol{B}$ as in assumption e and let (Ω, π), (Ω', π') be disposed pairs satisfying $\boldsymbol{B} \cap \boldsymbol{C} = \boldsymbol{B} \cap \boldsymbol{C}' = \boldsymbol{B}_0$. Since every subset of a model of FC is a model of FC we have $\Omega = \boldsymbol{B} \cup \boldsymbol{C}$, $\Omega' = \boldsymbol{B} \cup \boldsymbol{C}'$. Since $\boldsymbol{B} \cap \boldsymbol{C} = \boldsymbol{B} \cap \boldsymbol{C}'$, $|\Omega| = |\Omega'|$. $\boldsymbol{C} \simeq \boldsymbol{C}'$ via the map $\pi'\pi^{-1}$. Thus it is no loss of generality to assume that $\boldsymbol{C} = \boldsymbol{C}'$, $\pi = \pi'$, Ω and Ω' have the same underlying set, and that the $(f_n)_\Omega$ and $(f_n)_{\Omega'}$ disagree only on n-tuples which intersect both $\boldsymbol{B} - \boldsymbol{B}_0$ and $\boldsymbol{C} - \boldsymbol{B}_0$.

Let Ω^* have the same underlying set as Ω and Ω'. Let $\pi^* = \pi = \pi'$. For distinct $a_1, \ldots, a_n$ let $(f_n)_{\Omega^*}(a_1, \ldots, a_n) = (f_n)_\Omega(a_1, \ldots, a_n)$ if each $a_i \in \boldsymbol{B}_1 \cup \boldsymbol{C}$ and let $(f_n)_{\Omega^*}(a_1, \ldots, a_n) = (f_n)_{\Omega'}(a_1, \ldots, a_n)$ if some $a_i \in \boldsymbol{B}_2 \cup \boldsymbol{C}$. $(f_n)_{\Omega^*}$ is well defined since $\boldsymbol{B}_2 \cap \boldsymbol{C} = \boldsymbol{B}_0$. Our construction guarantees that $[\boldsymbol{B}_1 \cup \boldsymbol{C}]_\Omega = [\boldsymbol{B}_1 \cup \boldsymbol{C}]_{\Omega^*}$ and $[\boldsymbol{B}_2 \cup \boldsymbol{C}]_{\Omega^*} = [\boldsymbol{B}_2 \cup \boldsymbol{C}]_{\Omega'}$. Therefore (Ω^*, π^*) is singly linked with (Ω, π) and (Ω', π') so (Ω, π) and (Ω', π') are linked.

IIB6. Proof of Läuchli's Lemma (IIB3).

Let A be a UH *model of* FC *with underlying set A. Let Γ be the group of automorphisms of A. Γ and A satisfy the properties of* IIB3.

Proof. Every $G \subset A$ is a submodel of A so property IIB3c follows from the generalized intersection lemma (IIA4). Property IIB3b is clear since if $|G| = n$ and $\pi \in \Gamma$, π preserves $f_n(G), f_{n-1}(G - f_n(G)), \ldots$ etc.

To establish property IIB3a fix a finite $G \subset A$ and let $\boldsymbol{B} \subset A$ have underlying set G. Let $\boldsymbol{C}$ have underlying set $G \cup \{a, b, c\}$ where $\{a, b, c\} \cap G = \varnothing$. It is easy to define operations $(f)_{\boldsymbol{C}}$ on $G \cup \{a, b, c\}$ which satisfy:

$$(f)_{\boldsymbol{C}}(G^*) = (f)_{\boldsymbol{B}}(G^* \cap G) \quad \text{when} \quad |G^* \cap G| \geqslant 1,$$

$$f_{\boldsymbol{C}}\{a, b\} = a, \quad f_{\boldsymbol{C}}\{b, c\} = b, \quad f_{\boldsymbol{C}}\{c, a\} = c.$$

(3) $\Gamma(G)$ is the subgroup of Γ consisting of elements which are the identity on G.

Property 3) of UH structures (preceding II A2) makes it possible to embed C in A by a map which fixed B. We thus assume $C \subset A$.

Let $\pi_1: G \cup \{a, b\} \to G \cup \{b, c\}$ be the identity on G, map a to b, and map b to c. By Property 2) of UH structures π_1 extends to an automorphism of A. Similarly let π_2 be an automorphism of A which is the identity on G, maps b to c, and maps c to a. π_1 and π_2 are as desired in II B3a.

IIB7. The Fraenkel-Mostowski model. We briefly review a special case of Mostowski's construction [17]. Let M be a universe satisfying ZFA+E with Std as its subuniverse of well founded sets. Let I be the set of atoms of M and let Γ be a group of permutations on I. (i.e. automorphisms of M). If $G \subset I$ is finite x is *supported by* G if $\pi(x) = x$ for all $\pi \in \Gamma(G)$ (see footnote (3)). The Fraenkel-Mostowski model, V, is the class of those sets which are hereditarily supported. V-classes are supported subclasses of V. V is a model of ZFA. As remarked in [19] there are $\emptyset$-supported V classes $\mathcal{V}$ and T satisfying:

a $G\mathcal{V}X \leftrightarrow G$ supports x. We let $\mathcal{V}G = \{x: G\mathcal{V}x\}$. If x is V-definable from parameters supported by G then $x \in \mathcal{V}G$.

b $T(H, \alpha)$ is a function where H is an ordering of a finite $G \subset I$ and α is an ordinal. For fixed such H $T(H, \alpha)$ is a 1:1 map from the class of ordinals, On, onto $\mathcal{V}G$.

Theorem II B2 is proved by endowing I with the structure of a UH model, A, of FC. Γ is taken to be the group of automorphisms of A. The sequence $\{(f_n)_A\}_{n \geq 1}$ is seen to be in $\mathcal{V}_\emptyset$, hence $A \in \mathcal{V}_\emptyset$.

IIB8. THEOREM. *S and the axiom of choice for families of well orderable sets hold in* V. *The ordering theorem is false in* V.

Proof. That the axiom of choice for families of well orderable sets holds in V and that the ordering theorem is false in V is exactly as in Läuchli [15]. It remains to prove S in V.

For finite $G \subset I$ let $H(G)$ denote a fixed canonical ordering of G. The function $H(G)$ can be defined in V using choice for families of well orderable sets. Let $T^*(G, \beta)$ denote $T(H(G), \alpha)$ for the βth α such that G is the minimal G' with $T(H(G), \alpha) \in \mathcal{V}G'$. It is standard that $T^*: \mathscr{P}_{<\omega}(I) \times On \to$ V is 1:1, onto, and defined in V.

FC_1 denotes the theory with unary predicates D and E as well as choice operators $f_1, f_2, \ldots$ The axioms of FC_1 are those of FC together with $(\exists x) \sim \mathrm{E}(x)$. A *type* is the type of a finite model of FC_1. The types are absolutely coded in the integers. For each $n \in \omega$ there are finitely many models of FC_1 with cardinal n. If τ is a fixed type and $G \subset I$ is finite define in V,

$$y(\tau, G) = \{F \in \mathscr{P}_{<\omega}(I): (F \cup G, F, G, (f_1)_A, (f_2)_A, \ldots) \text{ has type } \tau\}.$$

For $x \in$ V let G_x be the minimal G with $x \in \mathcal{V}G$.

CLAIM. *If $x \in \mathscr{P}_{<\omega}(I)$ is infinite there is at least one nonempty type τ such that $y(\tau, G_x) \subset x$.*

Proof of Claim. Since x is infinite there is an $F_0 \in x - \mathscr{P}(G_x)$. Let τ be the type of $(F_0 \cup G_x, F_0, G_x, (f_1)_A, \ldots)$. τ is the desired type. Suppose $F_1 \in y(\tau, G_x)$. $(F_1 \cup G_x, F_1, G_x, (f_1)_A, \ldots)$ has the same type as $(F_0 \cup G_x, F_0, G_x, (f_1)_A, \ldots)$. The isomorphism π, of these two systems is unique, mapping $f_A(F_1 \cup G_x)$ to $f_A(F_0 \cup G_x)$, $f_A(F_1 \cup G_x - \{f_A(F_1 \cup G_x)\})$ to $f_A(F_0 \cup G_x - \{f_A(F_0 \cup G_x)\})$, etc. π is the identity on G_x and, by the UH property of A, extends to an automorphism, π^*, of V. π takes F_1 to F_0 and x to x so $F_1 \in x$. Therefore $y(\tau, G_x) \subset x$.

Now let τ and λ be two types such that models B and C of CF_1 with types τ and λ respectively have isomorphic CF subsystems $(E_B, (f_1)_B, \ldots)$ and $(E_C, (f_1)_C, \ldots)$. The combined type $\tau < \lambda$ is that of a structure $B < C$ which is formed from systems B and C as above with $(E_B, (f_1)_B, \ldots) = (E_C, (f_1)_C, \ldots)$ and $B \cap C = E_B$. The underlying set of $B<C$ is $B \cup C$, $D_{B<C} = D_B \cup D_C$, $E_{B<C} = E_B = E_C$ and if K is a finite subset of $B \cup C$ $f_{B<C}(K) = f_B(K \cap B)$ if this is nonempty and otherwise $f_{B<C}(K) = f_C(K)$. $B \leqslant C$ extends both B and C because of the assumptions on $E_B = E_C$. Intuitively $B<C$ is obtained by choosing from B first if possible. It is clear that $\tau < \lambda$ is uniquely defined irrespective of the choice of B and C.

We now define a canonical prime ideal $\mathscr{J}(\tau, G)$ in $\mathscr{P}(y(\tau, G))$. We actually define an ultrafilter $U(\tau, G)$. $\mathscr{J}(\tau, G)$ is the set of complements to the members of $U(\tau, G)$. We say $z \in U(\tau, G)$ exactly when there is a $G_1 \in \mathscr{P}_{<\omega}(I)$ and a type λ such that

$$G_1 \supset G,$$
$$(G_1 \cup G, \emptyset, G, f_A, \ldots) \text{ has type } \lambda,$$
$$\{F \in y(\tau, G): (F \cup G_1, F, G) \text{ has type } \tau < \lambda\} \subset z.$$

$U(\tau, G)$ must be shown to be an ultrafilter. $U(\tau, G)$ is closed under superset since its definition asserts that z has a subset of a certain type. The UH property of A guarantees that for any G_1 and λ as above $\{F \in y(\tau, G): (F \cup G_1, F, G)$ has type $\tau < \lambda\} \neq \emptyset$ so $U(\tau, G)$ does not contain the empty set. We now assert that for an arbitrary $z \subset y(\tau, G)$, if $(G_z \cup G, \emptyset, G, f_A, \ldots)$ has type λ then

$$\{F \in y(\tau, G): (F \cup G_z, F, G, f_A, \ldots) \text{ has type } \tau < \lambda\}$$

is a subset of either z or $y(\tau, G) - z$. This implies that either z or $y(\tau, G) - z$ is in $U(\tau, G)$. Indeed let F_0 and F_1 be such that both $(F_0 \cup G_z, F_0, G, f_A, \ldots)$ and $(F_1 \cup G_z, F_1, G, f_A, \ldots)$ have type $\tau < \lambda$. The two systems are isomorphic by a map, Π, which is quickly seen to be the identity on G_z and to map F_0 to F_1. Thus the UH property together with the symmetry of V, yields that $F_0 \in z \leftrightarrow F_1 \in z$.

The last requirement $U(\tau, G)$ must satisfy is that $z_1, z_2 \in U(\tau, G) \to z_1 \cap z_2 \in U(\tau, G)$. Accordingly for $i \in \{1, 2\}$ let $(G_i, \emptyset, G, f_A, \ldots)$ have type λ_i and $D_i = \{F \in y(\tau, G): (F \cup G_{z_i}, F, G, f_A, \ldots)$ has type $\tau < \lambda_i\} \subset z_i$. Let λ be the type of $(G_1 \cup G_2, \emptyset, G, f_A, \ldots)$. We claim that

$$D = \{F \in y(\tau, G): (F \cup G_1 \cup G_2, F, G, f_A, \ldots) \text{ has type } \tau < \lambda\} \subset z_1 \cap z_2.$$

In fact of course $D \subset D_1 \cap D_2$. If $F \in D$ then for any $K \subset F \cup G_1 \cup G_2$, $f_A(K) = f_A(K \cap F)$ when $K \cap F \neq \emptyset$. In particular this holds when $K \subset F \cup G_i$ which is what is needed to show $(F \cup G_i, F, G, f_A, \dots)$ has type $\tau < \lambda_i$, i.e $F \in D_i$.

A canonical prime ideal $\mathscr{J}(x)$ is now defined in an arbitrary $\mathscr{P}(x)$. For $x = \emptyset$ the definition is clear. Otherwise let $\alpha(x)$ be the least α with $T^*[\mathscr{P}_{<\omega}(I) \times \{\alpha\}] \neq \emptyset$. Let $x_1 = \{F \in \mathscr{P}_{<\omega}(I)\colon T^*(F, \alpha(x)) \in x\}$. x_1 is canonically embedded in x so it suffices to define $\mathscr{J}(x_1)$ i.e. to assume $x \subset \mathscr{P}_{<\omega}(I)$. Now let τ be the first (in the integer code) type with $y(\tau, G_x) \subset x$. $\mathscr{J}_x = \{z \subset x\colon z \cap y(\tau, G_x) \in \mathscr{J}(\tau, G_x)\}$ is a canonically defined prime ideal in $\mathscr{P}(x)$.

Finally we define a canonical prime ideal $\mathscr{J}^*(x)$ in $\mathscr{P}(x)$ such that $\mathscr{J}^*(x)$ is nonprincipal for infinite x. Fix a non-principal prime $\mathscr{J}(\omega)$ on $\mathscr{P}(\omega)$ ($\omega \in \mathrm{Std}$). Define the sequence $\mathscr{J}^n(x)$ of ideals in $\mathscr{P}(x)$ via $\mathscr{J}^0(x) = \mathscr{J}(x)$, $\mathscr{J}^{n+1}(x) = \mathscr{J}((x - \{a_0, \dots, a_n\})$ where $\mathscr{J}^0(x), \dots, \mathscr{J}^n(x)$ are principal primes generated by $a_0, \dots, a_n$ respectively, and $\mathscr{J}^{n+1}(x) = \mathscr{J}^n(x)$ if $\mathscr{J}^n(x)$ is nonprincipal. Let $\mathscr{J}^*(x) = \mathscr{J}^n(x)$ for some n with $\mathscr{J}^n(x)$ nonprincipal where possible. In the remaining case $\{a_0, a_1, \dots\}$ is an enumerated infinite subset of x. $\mathscr{J}(\omega)$ induces a nonprincipal prime on $\{a_0, a_1, \dots\}$, hence on x. This is taken to be $\mathscr{J}^*(x)$. S is now established in V.

IIC. Gauntt's results on the axiom of choice for families of finite sets. For $n \in \omega$ C^n denotes the statement:

"Every collection of n element sets has a choice function."

If $Z = \{n_1, \dots, n_k\}$ C_Z denotes the statement $C^{n_1} \wedge \dots \wedge C^{n_k}$. Mostowski [18] originally posed the question: "For what Z and n does $C_Z \to C^n$?" He also gave a sufficient condition, $D(Z, n)$, for this. Gauntt [7] answered the question by proving $D(Z, n)$ necessary (4).

Gauntt raised a technical question in [7]. His model of ZFA uses forcing rather than the Mostowski construction. He conjectures that the Mostowski construction is insufficient for a necessity proof of $D(Z, n)$. A related question arises from [7]. Gauntt states as a separate theorem that there is a ZF model for the necessity of $D(Z, n)$. If a Fraenkel-Mostowski model for the necessity of $D(Z, n)$ existed then the ZF necessity would be automatic by results of [20]. In this section we prove:

IIC1. Theorem.

a *If $D(Z, n)$ fails there is a Fraenkel-Mostowski model for $C_Z \nrightarrow C^n$.*

b *Gauntt's model for $C_Z \nrightarrow C^n$ is an inner model of the Fraenkel-Mostowski model of part* a.

(4) The exact statement of $D(Z, n)$ is:

"For every fixed point free permutation group G on n elements there is a subgroup $H \subset G$ and subgroups $H_i \subset H$, $i = 1, \dots, m$, such that

$$\sum_{i=1}^{m} |H_i|/|H| \in Z."$$

We do not refer to $D(Z, n)$ here but use instead Lemma 5 of [7].

The import of Theorem IIC1 is that Gauntt's results for ZFA do not even really use forcing, let alone require it. Similar considerations apply to Truss' extension of Gauntt's work (5).

IIC2. The category $\mathscr{C}_Z$. Let $Z = \{n_1, \dots, n_k\} \subset \omega$. $\mathscr{C}_Z$ is the category of structures,

$$\boldsymbol{B} = \{B; D, \in^*, \{\ \}^*, \emptyset^*, f_{n_1}, \dots, f_{n_k}\}$$

where:

a $\in^*$ is a well founded relation.

b $\emptyset^*$ is a constant.

c $D \cup \{\emptyset^*\}$ is the set of $\in^*$-minimal elements.

d $\in^*$ is extensional on $B - D$.

e $\{\ \}^*$ is an unordered n-tuple function with respect to $\in^*$. Strictly it is an ω-sequence of functions $\{\ \}^*_n$.

f B is the closure of $D \cup \{\emptyset^*\}$ under $\{\ \}^*$.

g f_{n_i} is an n_i-ary choice operator (see IIB4) on the set D^Z. D^Z consists of all elements with the form $(i^*, J)^*$ where the $*$-elements of J are $*$-orderings of a finite subset of D and $i \leq \max(n_1, \dots, n_k)$. Note. $*$ indicates the relativization of a set-theoretic concept to $\in^*, \emptyset^*, \{\ \}^*$.

IIC3. Lemma. *$\mathscr{C}_Z$ satisfies the assumptions of* IIA1.

Proof. Much of IIA1 becomes clear once we make the following observation in ZFA. Let $\boldsymbol{B} \in \mathscr{C}$ and let $\varphi\colon D_{\boldsymbol{B}} \to G$ where G is a set of individuals and φ is 1:1 and onto. Requirements a-f of IIC2 are exactly what is needed to show that φ extends uniquely to an isomorphism of $(B; \in^*, \emptyset^*, \{\ \}^*)$ with $(\mathrm{HF}(G), \in, \emptyset, \{\ \})$ where $\mathrm{HF}(G)$ is the collection of hereditarily finite sets over G. There is thus a unique set of choice operators $\{f'_{n_1}, \dots, f'_{n_k}\}$ on $\mathrm{HF}(G)^Z$ such that φ is an isomorphism of $\boldsymbol{B}$ and $(\mathrm{HF}(G), \in, \emptyset, \{\ \}, f'_{n_1}, \dots, f'_{n_k})$.

The reader should have no trouble verifying assumptions a, b, c, and d of IIA1 for $\boldsymbol{B}$'s of the form $(\mathrm{HF}(G), G, \in, \emptyset, \{\ \}, f_{n_1}, \dots, f_{n_k}\})$ and translating the results to arbitrary $\boldsymbol{B}$'s. In verifying IIA1e one can assume that Ω and Ω' both have the indicated form where D_Ω and $D_{\Omega'}$, are finite sets of individuals. As in Lemma IIB5 it develops that $|D_\Omega| = |D_{\Omega'}|$ so that one can actually assume $D_\Omega = D_{\Omega'}$ and $\pi = \pi'$. In this situation the only difference between Ω and Ω' is in the definition of the f_{n_i}. Again as in IIB5 $D^Z_{\boldsymbol{B}_1} \cap D^Z_{\boldsymbol{B}} = D^Z_{\boldsymbol{B}_2} \cap D^Z_{\boldsymbol{B}} = D^Z_{\boldsymbol{B}_0}$. Thus the same 2-step linking as in IIB5 (introduce an Ω^* with hybrid definitions of the f_{n_i}) establishes assumption e.

(5) Truss considers a statement C^0_Z which does not seem to have an automatic ZF transfer. The methods of [20] do give an automatic transfer for the statement:

"Every linear ordering of a family whose members have cardinality in Z is associated with a choice function on the family."

This and the work of [27], suffice to transfer all the results of [27] to ZF.

IIC4. The Fraenkel-Mostowski model V_Z. We use the Mostowski construction as reviewed in IIB7. Let M be a universe of ZFA+E with Std as its subuniverse of well founded sets. Let I be the set of individuals in M and let

$$A^* = (I;\, D, \in^*, \emptyset^*, \{\ \}^*, f^*_{n_1}, \ldots, f^*_{n_k})$$

be a UH structure in $\mathscr{C}_Z$ with ground set I. Let Γ be the group of automorphisms of A^* and let V_Z be the model obtained from Γ be the Mostowski construction. As in IIB7 $A^* \in V_0$.

As is essentially argued in IIC3 there is a unique system

$$A = (\mathrm{HF}(D),\, D, \in, \emptyset, \{\ \}, f_{n_1}, \ldots, f_{n_k})$$

such that the identity map on D extends to an isomorphism from A^* to A. The isomorphism is definable in V_Z. From the standpoint of M one can state that a permutation π of D extends to an automorphism of A^* if any only if it extends to an automorphism of A.

We remark that if $G \subset I$ is finite and $G^* = D \cap \mathrm{TC}^*(G)$ ($\mathrm{TC}^*(G)$ is the $*$-transitive closure of G) then G^* is finite and $\nabla G \subset \nabla G^*$. We also note that for $G \subset D$, $\Gamma(G)$, is $\Gamma([G])$, the group of automorphisms of A^* which fix $[G]$. Mostowski's intersection lemma (IIA6) thus gives for $G_1, G_2 \subset D$:

$$\Gamma(G_1 \cap G_2) = \Gamma([G_1 \cap G_2]) = [\Gamma[G_1] \cup \Gamma[G_2]] = [\Gamma(G_1) \cup \Gamma(G_2)]\,.$$

From this, as in [17], $\nabla(G_1 \cap G_2) = \nabla G_1 \cap \nabla G_2$ for $G_1, G_2 \subset D$.

IIC5. Proof of Theorem IIC1 Part a.

C^n holds in V_Z if and only if $D(Z, n)$ is true.

Proof. Except for the absence of forcing the argument is as in [7]. We first prove C^n for $n \in Z$. The result of Mostowski [18] then concludes C^n when $D(Z, n)$ is true.

We handle the case $n \in Z$ by associating to each x with $|x| = n$ a 1:1 map $\Psi_x\colon x \to D^Z$. One can then choose from x the element $\Psi_x^{-1}(f_n(\Psi_x[x]))$. To define Ψ_x first let $G_x^* = \bigcup_{y\in x} G_y$ where G_y is the unique minimal $G \subset D$ with $y \in \nabla G$. For $y \in x$ let α_y be the least ordinal α such that for some ordering H of G_x^*, $y = T(H, \alpha)$. Let J_y be the set of those orderings, H, of G_x^* such that $y = T(H, \alpha_y)$. $\{\alpha_y\colon y \in x\}$ is a set of ordinals with cardinal $\leqslant \max\{n_1, \ldots, n_k\}$. Let i_y be such that α_y is the i_yth member of $\{\alpha_y\colon y \in x\}$. $\Psi_x(y) = (i_y, J_y)$ is the desired map. It is clearly 1:1 and into D^Z.

If $D(Z, n)$ fails we must show that C^n fails in V_Z. In fact the collection of n-element subsets of D has no choice function in V_Z. If such a function, g, existed $g \in \nabla G_0$ would hold for a finite $G_0 \subset D$. We now cite:

IIC6. Lemma in ZFA (Gauntt [7], Lemma 5). Let $D(Z, n)$ be false. Let $G \supset G_0$ be a finite set of atoms with $|G - G_0| = n$. Let $f_{n_1}, \ldots, f_{n_k}$ be choice operators on G_0^Z. There are extensions of the f_{n_1} to G^Z such that if $\boldsymbol{B}$ is the corresponding object in $\mathscr{C}_Z$ and $a \in G - G_0$ some automorphism of $\boldsymbol{B}$ fixes G_0 pointwise and moves a.

The argument is quickly concluded. By the UH property of A there is a $G^* \subset D$ such that $[G^*]_A$ is isomorphic to the $\boldsymbol{B}$ of IIC6 via an isomorphism fixing G_0. It can thus be assumed that $G = G^*$. Assume $g(G - G_0) = a$ and apply the automorphism, π, of IIC6 to obtain $\pi g(\pi G - \pi G_0) = \pi(a) \neq a$. Since $\pi(G) = G$, π fixes G_0 pointwise, and $g \in \nabla G_0$ the equation becomes $g(G - G_0) \neq a$. This contradiction completes the proof.

IIC7. Proof of Theorem IIC1 Part b.

Gauntt's model can be described in V_Z *as* $(\mathrm{Std}[D])[f]$ *where* $f = \bigcup_{n\in Z} f_n$ *and* $U[x]$ *denotes, in general, the smallest transitive subuniverse of* M *containing* U *and* x.

Proof. Gauntt describes his model as $(\mathrm{Std}[D])[f']$ where f' is a generic choice function on those subsets of D^Z with cardinal in Z. In defining "generic" Gauntt uses the formulation of Cohen [1] which involves countable models and complete sequences. The following reformulation of Gauntt's definition is possible from the reformulation of "generic" in [25].

P denotes the inclusion ordered set of finite choice functions which are defined on those subsets of some G^Z with cardinal in Z. $Q \subset P$ is dense if every $\varphi \in P$ has some extension in Q. Let f' be a choice function on those subsets of D^Z with cardinal in Z. f' is *generic* if for every dense $Q \in \mathrm{Std}[D]$ there is some $\varphi \in Q$ such that f' extends φ.

$(\mathrm{Std}[D])(f)$ can be identified with Gauntt's model once it is shown that f is generic. Accordingly let $Q \in \mathrm{Std}[D]$ be dense. $Q \in \nabla^{\mathrm{Std}[D]}G$ for some finite $G \subset D$. Let φ be the restriction of f to G^Z. Let $\varphi^* \in Q$ be an extension of φ. It is a well known property of $\mathrm{Std}[D]$ (See e.g. [17]) that *any* permutation (in M) of D extends to an automorphism of $\mathrm{Std}[D]$. If φ^* is defined on G_1^Z then by the UH property of A there is a 1:1 map $\pi\colon G_1 \to D$ such that π is the identity on G and $\pi(\varphi^*)$ is a restriction of f. Extend this π to an automorphism of $\mathrm{Std}[D]$. $\varphi^* \in Q \to \pi(\varphi^*) \in \pi Q = Q$ since π is the identity on G_1. Thus $\pi(\varphi^*)$ is a restriction of f in Q so f is generic.

IID. Other uses of the intersection lemma. We originally proved the intersection lemma for the categories $\mathscr{C}_{n,m}$ whose members are structures with linear orderings $<_1, \ldots, <_n$ and partial orderings $\prec_1, \ldots, \prec_m$. This proof is given in Jech [13].

The case $n = 1$, $m = 0$ gives the result of Mostowski [17]. The case $n = 1$, $m = 1$, gives an alternate route to the result of Mathias [16]. The case $n = 0$, $m = 1$ was originally used to give an independence proof of "Every infinite set contains an infinite $\subset$-chain." from "Every infinite set is the disjoint union of infinite sets." This result is mentioned in [20]. Truss [26] proved this and a number of related results by different methods ([6]). Jech and Felgner (see [5]) use our model to establish the ZFA independence of Kurepa's antichain principle from "Every linearly orderable set is well orderable."

([6]) His methods are not so very different. He uses Gauntt's techniques to produce, essentially, our model of § IIB.

III. Indiscernability and models of the compactness theorem

We recall the notion of model theoretic indiscernability from Ehrenfeucht and Mostowski [2]. Let $A = (A; <, R_1, R_2, \ldots)$ be a first order structure where $<$ is a linear ordering. $U \subset \text{Domain} <$ is a set of indiscernables if for any formula $\Phi(x_1, \ldots, x_n)$ in the language of A (⁷) and any $a_1 < \ldots < a_n$, $a'_1 < \ldots < a'_n$, $\Phi(a_1, \ldots, a_n) \leftrightarrow \Phi(a'_1, \ldots, a'_n)$ holds in A.

III1. THEOREM (Ehrenfeucht and Mostowski [2]). *Let $B = (B, <, R_1, \ldots)$ be a first order structure where Domain $<_B$ is infinite. Let λ be a linear order type. $\dot{B}$ is elementarily equivalent to a structure A where Domain $<A$ contains a set of indiscernable of type λ.*

Theorem III1 is shown to be essentially equivalent to the following independence result.

III2. THEOREM (Halpern [8]). *Let* M *satisfy* ZFA+E *and let* $<$ *be a* UH *linear ordering of the set I of individuals of* M. *Let Γ be the automorphism group of $(I, <)$ and let* V *be the model obtained by the Mostowski construction* (IIB7) *from Γ. The following statement, essentially the compactness theorem, holds in* V.

If $\tau \in V G$ is a first order theory then some structure $A \in V G$ is a model for τ.

III3. Proof of a special case of Theorem III2 from Theorem III1. In V the following statement holds.

Let τ be a first order theory satisfying:

a *τ is consistent* (⁸).
b *Every nonconstant function or predicate of τ is in $V\emptyset$.*
c $\tau \in V\phi$.
d *Every constant of τ is in $V\phi \cup I$.*
e *If $a, b \in I$ and $a \neq b$ then "$a \neq b$" is an axiom of τ.*

Then τ has a model in $V\phi$.

Proof. The UH property of $<$ guarantees that I is a set of indiscernables for V, i.e. if $\Phi(x_1, \ldots, x_n)$ is a formula of set theory with parameters in $V\phi$ and $a_1 < \ldots < a_n$, $b_1 < \ldots < b_n$ are in I then $\Phi(a_1, \ldots, a_n) \leftrightarrow \Phi(b_1, \ldots, b_n)$ holds in V. This property holds relative to V so the remainder of the argument takes place within V.

(⁷) In set theory without choice or foundation a first order language can have arbitrary objects (sets or individuals) among its operation and relation symbols. This is subject to the conditions that these objects are disjoint from standard integer representations of the quantifiers, connectives, and variables. The objects must also admit a well defined function which determines the number of places in the operation or relation and distinguishes operations from relations.

A theory may be Gödel numbered in the integers only in the presence of an enumeration of the operation and relation symbols. This need not exist in the absence of choice. More to the point, such an enumeration may exist in one model of ZFA but fail to exist in a submodel. These points should be borne in mind during the ensuing arguments.

By convention when $\Phi(x_1, \ldots, x_n)$ is written it is assumed that its free variables are exactly $x_1, \ldots, x_n$.

(⁸) τ is *consistent* when every finite subset of τ has a model.

Theorem III1 (the EM Theorem) is first applied (in Std where AC is true) to build a prototype, A'', for A. A is then constructed from A''. More specifically let τ' be the following theory. The language of τ' is formed from that of τ by deleting the constants in I and adding a single binary relation symbol $W \in V\phi$. The axioms of τ' are of the following sorts.

1. A single axiom states that W linearly orders its domain.

2. A list of axioms Ψ_n state that the domain of W has at least n elements.

3. For every axiom $\Phi(a_1, \ldots, a_n)$ of τ with exactly the $a_1 < \ldots < a_n$ in I an axiom Z_Φ of τ' states $(\forall x_1, \ldots, x_n)[x_1 W \ldots W x_n \rightarrow \Phi(x_1, \ldots, x_n)]$.

We show that τ' is consistent. Let σ' be a finite subtheory of τ'. Let n_0 be the largest n such that Ψ_n occurs in σ'. Let $a_1 < \ldots < a_{n_0}$ in I. Let σ be the subtheory of τ with the following axioms:

a) all $a_i \neq a_j$ $l \leqslant i \neq j \leqslant n_0$,

b) all $\Phi(b_1, \ldots, b_n)$ where Φ is a formula of σ', $b_1 < \ldots < b_n$, $\{b_1, \ldots, b_n\} \subset \{a_1, \ldots, a_{n_0}\}$.

The axioms of type b) are in τ since some $\Phi(c_1, \ldots, c_m)$, $c_1 < \ldots < c_m \in I$, is in τ and I is a set of indiscernibles for V. Therefore σ has a model B. Build a model B' of σ' by deleting as constants the $(a_i)_B$ and letting $(a_i)_B (W)_{B'} (a_j)_{B'} \leftrightarrow a_i < a_j$. Thus τ' is consistent.

All functions and relations of τ' are in $V\phi$ so τ' can be assumed to be coded in Std. Let τ'' be the closure of τ' by Skolem functions. τ'' is also consistent and coded in Std so it has a model, C. $(<)_C$ has infinite domain because the axioms Ψ_n are in τ''. Thus the EM Theorem is applied in Std to give a model A'' with a set, $\mathscr{I}$, of $<-$ indiscernibles with the order type of the rational numbers. Since τ'' is closed under Skolem functions we may assume that every $a'' \in A''$ is $f_{A''}(r_1, \ldots, r_n)$ for some function f of τ'' and some $r_1 (W)_{A''} r_2 \ldots (W)_{A''} r_n$ in $\mathscr{I}$. We also remark that if $\{r_1, \ldots, r_n\} \subset \{s_1, \ldots, s_k\} \subset \mathscr{I}$ then for some function g of τ'', $f_{A''}(r_1, \ldots, r_n) = g_{A''}(s_1, \ldots, s_k)$.

The model A of τ is built by "locally approximating" A''. This is to say that the substructure of A generated by $a_1 < \ldots < a_n$ in I is isomorphic to the substructure of A'' generated by $r_1 (W)_{A''} \ldots (W)_{A''} r_n$ in $\mathscr{I}$. It does not matter which $r_1, \ldots, r_n$ is chosen since $\mathscr{I}$ is a set of indiscernibles.

Once stated the idea is routine to execute. Start with the set:

$$D = \{(G, f)\colon (\exists n \in \omega)[G \subset I \wedge |G| = n \wedge f \text{ is an } n\text{-ary function of } \tau'']\}.$$

For each predicate $p(x_1, \ldots, x_n)$ of τ (or τ'') and n tuple $(G_1, f_1), \ldots, (G_n, f_n)$ from D we define the "truth value" of $p((G_1, f_1), \ldots, (G_n, f_n))$. Let $G = G_1 \cup \ldots \cup G_n$. Let φ be an order preserving embedding of G into $\mathscr{I}$. $\varphi\langle G_i \rangle$ denotes the tuple of the elements of $\varphi[G_i]$ arranged in $(W)_{A''}$ order. The truth value of $p((G_1, f_1), \ldots, (G_n, f_n))$

is defined as that of $p_{A''}((f_1)_{A''}\varphi\langle G_1\rangle, \ldots, (f_n)_{A''}\varphi\langle G_n\rangle)$ in A''. The indiscernibility of $\mathscr{J}$ guarantees that this definition is independent of the choice of φ.

Let A be the set of equivalence classes of D under the relation $d_1 \sim d_2$ if and only if the truth value of $d_1 = d_2$ is T. For each predicate p of τ say $p_{A^*}([d_1], \ldots, [d_n])$ is true when the truth value of $p(d_1, \ldots, d_n)$ is T. This definition is independent of the representative of the $[d_i]$ chosen. For each n-ary function symbol f of τ'' and $[d_1], \ldots, [d_n]$ in A we define the function f_{A^*} at $([d_1], \ldots, [d_n])$. Since τ'' is Sholem closed we remarked earlier that there is a single $G \subset I$ and function symbols $f_1, \ldots, f_n$ of τ'' such that (G, f_i) represents $[d_i]$. If $|G| = m$ let h be the τ'' function symbol of m variables defined by

$$h(x_1, \ldots, x_m) = f(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)).$$

$f_{A^*}([d_1], \ldots, [d_n])$ is defined to be $[(G, h)]$. This definition is independent of the choice of $G, f_1, \ldots, f_n$.

It is now straightforward to show that if $\Phi(x_1, \ldots, x_n)$ is a formula of τ'', $[G_1, f_1], \ldots, [G_n, f_n] \in A$, $G = G_1 \cup \ldots \cup G_n$, and φ is an order-preserving map of G into $\mathscr{J}$ then $\Phi_{A^*}([G_1, f_1], \ldots, [G_n, f_n])$ is true if and only if $\Phi_{A''}(f_1\varphi\langle G_1\rangle, \ldots, f_n\varphi\langle G_n\rangle)$ is true. Thus in particular A^* and A'' are elementarily equivalent and A^* is a model of τ''.

A is just A^* except that $<$ is "forgotten" and a_A is interpreted as $[\{a\}, \text{Identity}]$. Let $\Phi(a_1, \ldots, a_n)$ be an axiom of τ where $a_1 < \ldots < a_n$. Let $\varphi\colon \{a_1, \ldots, a_n\} \to \mathscr{J}$ be order preserving. The truth value of $\Phi_A([a_1, \text{Id}])$ is that of $\Phi_{A''}(\text{Id}(\varphi(a_1)), \ldots, \text{Id}(\varphi(a_n)) = \Phi_{A''}(\varphi(a_1), \ldots, \varphi(a_n))$. The statement

$$Z_\Phi \leftrightarrow (\forall x_1, \ldots, x_n)[x_1 < \ldots < x_n \to \Phi(x_1, \ldots, x_n)]$$

is an axiom of τ'' so $\Phi_{A''}(\varphi(a_1), \ldots, \varphi(a_n))$ is true. $\Phi(a_1, \ldots, a_n)$ is therefore true in A and A is a model for τ. A is defined from τ and A'' so $A \in V\phi$.

III4. Proof of Theorem III2 from the special case. Let $\tau_1 \in VG$ be a consistent theory in V. We will produce theories $\tau_2, \ldots, \tau_5$ in V such that τ_i satisfies the first i special properties of III3 and such that a model for τ_{i-1} is definable from τ_{i-1} and a model for τ_i. Since a model for τ_5 exists in $V\phi$ (III3) a model for τ_1 exists in VG.

Every nonconstant function and predicate of τ_2 must be in $V\phi$. This is accomplished by condensing the n-ary predicates (operations) of τ_1 into a single $(n+1)$-ary predicate P_{n+1} (operation F_{n+1}) of τ_2. The countably many relations and functions can easily be coded in $V\phi$. We also add a unary predicate E (for "true element") and include the functions and relations of τ_1 as constants of τ_2. If Φ is a formula of τ_1 let Z_Φ be the formula of τ_2 obtained by the following inductive process.

1) Replace terms $ft_1 \ldots t_n$ of τ_1 with $f_{n+1}ft_1 \ldots t_n$ of τ_2.

2) Replace formulae $pt_1 \ldots t_n$ of τ_1 with $p_{n+1}pt_1 \ldots t_n$ of τ_2.

3) Relativize all variables to E.

The axioms of τ_2 are the formulae of the following forms: $\sim Ep$ for each predicate p of τ_1. $\sim Ef$ for every nonconstant function of τ_1, $(\forall x_1 \ldots x_n)[Ex_1 \wedge \ldots \wedge Ex_n \to EF_n fx_1 \ldots x_n]$ for every n-ary function f of τ_1, and Z_Φ for every axiom Φ of τ_1. It is easy to show that τ_2 is consistent and to obtain a model for τ_1 from one of τ_2 (look at the extension of E, interpret $fx_1 \ldots x_n = F_{n+1}fx_1 \ldots x_n$ and $px_1 \ldots x_n \leftrightarrow P_{n+1}px_1 \ldots x_n$).

τ_3 must be in $V\phi$ and satisfy the properties of τ_1 and τ_2. Let R_α be the least rank (over I) containing τ_2. Let $U \in V\phi$ be the set of *all* consistent theories $\tau \in R_\alpha$ with the given set of predicates and nonconstant functions. τ_3 is rigged so that a model of τ_3 contains a U-indexed sum of models of the members of U. When this is done the τ_2 component of the model is a model of τ_2.

τ_3 includes the given functions and predicates (of τ_2), a binary predicate $p(x, y)$, a new "true element" predicate E^*, and each $\tau \in U$ as a constant. For $\tau \in U$ let E^*_τ denote the defined predicate $E^*_\tau x \leftrightarrow E^*x \wedge p\tau x$. The axioms of τ_3 assert the disjointness of the E^*_τ for $\tau \in U$, the closedness of E^*_τ under the operations, and the relativization, $\Phi^{E^*_\tau}$, of each axiom Φ of τ. τ_3 clearly satisfies a, b and c of III3 and $E^*_{\tau_2}$ gives a model of τ_2.

τ_4 must satisfy the requirements on τ_3 and have constants only in $V\phi \cup I$. τ_5 must have in addition all statements "$a \neq b$" for $a \neq b$ in I. We actually have $\tau_4 = \tau_5$. Recall ([15] or IIB8) that there is a 1:1 function T^* from $\mathscr{P}_{<\omega}(I) \times On$ onto V. The language of τ_5 is obtained by deleting all constants of τ_3, adding the elements of I as new constants, adding a new "true element" predicate E^{**}, and adding a new n-ary function symbol $f_{n\alpha}$ (coded in $V\phi$) for each constant of τ_3 with the form $T^*(G, \alpha)$ where $|G| = n$. The axioms of τ_5 consist of the "$a \neq b$" for $a \neq b$ in I, $E^{**}f_{n\alpha}(a_1, \ldots, a_n)$ when $a_1 < \ldots < a_n$ and $T^*(\{a_1, \ldots, a_n\}, \alpha)$ is a constant of τ_3, and each $\Phi^{E^*}(f_{n_1\alpha_1}\langle G_1\rangle, \ldots, f_{n_k\alpha_k}\langle G_k\rangle)$ where $\Phi(T^*(G_1, \alpha_1), \ldots, T^*(G_k, \alpha_k))$ is an axiom of τ_3 (n_i denotes $|G_i|$ and $\langle G_i\rangle$ denotes the tuple of G_i arranged in increasing $<$ – order). The consistency of τ_5 and the existence of a model of τ_3 are clear because of the 1:1 correspondence between the constants $T^*(G, \alpha)$ of τ_3 and the "true element terms" $f_{n\alpha}\langle G\rangle$ of τ_5.

The following argument expresses the heuristic equivalence of Theorems III1 and III2.

III5. Derivation of the EM Theorem from Halpern's Theorem (III2).

Proof. Let λ be the given order type. Let μ be a UH order type in which an ordering of type λ can be embedded. The existence of μ is established using ultrapowers (giving a much stronger result) or by a direct construction. Extend the standard universe, Std, so that it becomes the well founded sets of a universe, M, of ZFA+E in which I admits an ordering, $<$, of type μ. Let V be the model obtained from the Mostowski construction using the group of $<$-automorphisms ([17]).

Now let $\tau \in \text{Std}$ be the given theory with the ordering W. Let $\tau' \in$ V be the theory with the elements of I added as constants and with "aWb" added as axioms for those pairs (a, b) with $a < b$. τ' is consistent since W has an infinite domain in some model of τ. $\tau' \in V\phi$ since I, $<$, and τ are the parameters in its definition.

Let $A \in V\phi$ be a model of τ (Theorem III2). The mapping $a \to a_A$ is necessarily 1:1 so we may assume $a_A = a$. Let $a_1 < \ldots < a_n$ and $b_1 < \ldots < b_n$. Since I is a set of <-indiscernibles for V we may say there "$(a_1, \ldots, a_n)$ is true in A" if and only if "$\Phi(b_1, \ldots, b_n)$ is true in A". Thus $\Phi(a_1, \ldots, a_n) \leftrightarrow \Phi(b_1, \ldots, b_n)$ is true in A relative to V. By the absoluteness of the satisfaction relation I is a set of indiscernibles for A relative to M. Since E holds in M A is isomorphic to a structure, A^* in the well founded sets, Std. A^* is a model of τ and has a set of indiscernibles with order type μ, hence it has one with order type λ.

It is natural to ask at this point whether other instances of the equivalence between indiscernibility theorems and models for the compactness theorem exist. There is another well known model of the compactness theorem due to Halpern and Levy [12]. One could ask in particular whether an indiscernibility theorem underlies the proof of the compactness theorem in this model. We exhibit such an indiscernibility theorem below. It is similar to the EM Theorem but the EM Theorem seems neither to generalize nor specialize it. An unfortunate feature of our theorem is the restriction to countable τ. We do not know the situation for uncountable τ.

The theorem is phrased in the following context. Let τ be a fixed theory and let $\wp$ be a fixed infinite set of unary predicates of τ. Let A be a model of τ. $\wp$ leads to a natural topology on A as follows. For every finite function $f \subset \wp \times \{T, F\}$ let

$$U_f = \{a \in A\colon (\forall p \in \mathrm{Domain} f)[\text{Truth Value in } A \text{ of } p(a) = f(p)]\}.$$

The U_f are a basis for the topology we call the $\wp$-topology on A. The most common example of this situation is when $A = 2^\omega$, $\wp = \{p_i\colon i \in \omega\}$, and $p_i(a) \leftrightarrow a(i) = 1$. The topology is then the usual topology on 2^ω.

Let $\Phi(x_1, \ldots, x_n)$ be a formula of τ, let $J \subset A$, and let $a_1, \ldots, a_n \in \tau$ be distinct members of A. Φ *holds locally* on J at $a_1, \ldots, a_n$ if there are disjoint neighborhoods $U_1, \ldots, U_n$ of $a_1, \ldots, a_n$ such that whenever $b_i \in U_i \cap J$, $i = 1, \ldots, n$, then $\Phi(b_1, \ldots, b_n)$ holds. $J \subset A$ is a set of *local indiscernibles* if whenever $\Phi(a_1, \ldots, a_n)$ holds at distinct $a_1, \ldots, a_n \in J$ then $\Phi(x_1 \ldots x_n)$ holds locally on J at $a_1, \ldots, a_n$.

$\wp$ is said to be independent if every $U_f \neq \emptyset$ (⁹). The motivation for this is that the truth value of $p_1(a), \ldots, p_n(a)$ does not determine that of $p_{n+1}(a)$. The predicates $p_i(a) \leftrightarrow a(i) = 1$ in 2^ω are independent.

III 6. THEOREM (see footnote (⁹)). *Let τ be a countable theory and let $\wp$ be an independent set of unary predicates of τ. τ has a model with a dense set of local indiscernibles.*

Proof. Extend Std to obtain the Halpern-Levy model, V of [12]. V = Std$[I]$ where I is an independent set of generic members of 2^ω. From within V this translates, as in [12], to:

(⁹) One could replace this with the weaker:

"If $U_f \neq \emptyset$ and $U_g \neq \emptyset$ then either $U_f \cap U_g \neq \emptyset$ or f and g are incompatible".

(1) $I \subset 2^\omega$, V = Std$[I]$.

(2) The predicates $p_i(a) \leftrightarrow a \in I \wedge a(i) = 1$ are independent.

(3) If $\Phi(x_1, \ldots, x_n)$ has parameters in Std $\cup \{I\}$ and $\Phi(a_1, \ldots, a_n)$ holds for distinct $a_1, \ldots, a_n$ in I then $\Phi(x_1, \ldots, x_n)$ holds locally on I at $a_1, \ldots, a_n$.

3) says that I is a set of local indiscernibles for V (together with $\in$, I, and constants for the elements of Std). We remark that V and T are definable in V from I and satisfy the standard properties (see [19] and II B 7). The main theorem of [12] is that a theory in VG has a model in VG.

Let $\tau \in$ Std be the given theory. Let $\{p_i\colon i \in \omega\}$ be an enumeration, in Std, of $\wp$. Let τ' be the extension of τ with new constants from I and exactly the axioms $p_i(a)$ where $a \in I$ and $a(i) = 1$. τ' is consistent since the p_i are independent. $\tau' \in V\phi$ so τ' has a model $A \in V\phi$. Arguing as in III5 we can assume $I \subset A$ and assert that I is a set of local indiscernibles for A relative to V.

We do not see how to proceed from here without invoking the countability of τ. Begin the proof over again starting from the countable transitive model Std* in which τ is absolute and in which enough of ZFE holds in order to carry out the specific arguments above. Extend Std* to V* via complete sequences. The absoluteness of the satisfaction relation establishes that I is a set of local indiscernibles relative to the true universe, Std.

III7. Final remarks.

a. Using Skolem functions one can refine Theorem III6 to insure that the $\wp$ topology is Hausdorff on J.

b. Using the notion of forcing in model theory and the combinatorial theorem of Halpern and Läuchli, [11] one can give a proof of Theorem III6 without using set theory. This proof does not evade the countability of τ.

c. Ramsey's combinatorial theorem entered into the proof of the EM Theorem and Halpern's theorem (III2). Gaifman [6] eliminates it from the EM Theorem. This and III4 eliminates it also from Halpern's theorem.

d. Theorem III6 is equivalent to the theorem of Halpern and Levy [12] in the sense that one can derive the combinatorial theorem of Halpern and Läuchli [11], from Theorem III6. Halpern [10] gives some idea of how such an argument might go. This was part of the inspiration for Theorem III6. We hope to say more on this subject at a later date.

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Accepté par la Rédaction le 2. 9. 1974

Equitable partitions of the continuum

by

W. Charles Holland (Bowling Green, Ohio)

Abstract. The real line contains no dense homogeneous subset which is isomorphic to its complement.

A partition of the real line R into two subsets S and $R\setminus S$ is *equitable* if S is dense in R, S is homogeneous (in the sense of order-preserving permutations) and $S\approx R\setminus S$ (as ordered sets). It became of interest to find such a partition during the study of automorphisms of certain ordered permutation groups [2]. However, it seems to be of independent set-theoretic interest whether such a partition exists. It will be shown in this paper that no such partition exists.

Suppose then that S is such a subset. First note that $R\setminus S$ must also be a dense subset of R. For consider a maximal real interval J contained in S; J cannot be open, for then its end points are adjacent points of $R\setminus S$, which implies $R\setminus S$, and hence also S, is discrete, an obvious impossibility. Thus, J must contain at least one of its end points. Suppose that J is not just a single point. Then since S is homogeneous, there must exist an order-preserving permutation f of S which maps an end point of J into the interior of J. Because of the density of S in R, f has a unique extension to an order-preserving permutation of R. Then $J\cup Jf^{-1}$ is a real interval contained in S, denying the maximality of J. Hence J is just a single point and $R\setminus S$ is dense in R.

Let $\varphi\colon S\to R\setminus S$ denote the assumed order isomorphism and observe that also $\varphi\colon R\setminus S\to S$. Because of the density of S and $R\setminus S$, φ has a unique extension to an order-preserving permutation of R. The same is true of any order-preserving permutation of S or of $R\setminus S$. No notational distinction will be made between a map and its extension.

It is now useful to study the group $A(S)$ of all order-preserving permutations of S. A convex subset $B\subseteq S$ is an *o-block* if for each $g\in A(S)$, $Bg=B$ or $Bg\cap B$ is empty. $A(S)$ is *o-primitive* if there are no non-trivial o-blocks. It is known [1] that for an o-primitive $A(S)$, there are just these two possibilities: (i) $A(S)$ is *o-2-transitive*; for each $x<y$, $z<w$, $x, y, z, w\in S$, there exists $h\in A(S)$ such that $xh=z$ and