obtained from the constructible universe by adjoining any number of mutually Cohen-generic reals.

1.7. Theorem. If (K), then every filter over generated by less than $2^a$ sets can be extended to a selective ultrafilter.

This result follows easily from the following proposition:

1.8. Proposition. If (K) holds and $F$ is a filter over $\omega$ generated by less than $2^a$ sets, and $\{X_i : i < \omega\}$ is a partitioning of $\omega$ so that for every $i < \omega$

$$\bigcup \{X_j : j > i\} \in F,$$

then there exists a set $X \subseteq \omega$ so that $\{X \cup F\} \in F$ has the finite intersection property and for every $i < \omega$: $|X \cap X_i| \leq 1$.

Proof. Suppose that no such $X$ exists. Let $\{C_i : i < \lambda\} \subseteq F$ so that

$$X \in F \Rightarrow \exists \lambda \leq \lambda : X \subseteq C_\lambda.$$

Let

$$T = \{f \in \omega^\omega : \forall i < \omega : f(i) \in X_i\}.$$ We can w.l.o.g. assume that $T$ is a perfect closed subset of $\omega$ in the usual product topology. Define for $x < \lambda$

$$T_x = \{f \in T : \text{range}(f) \cap C_x = 0\}.$$ Then

$$T = \bigcup \{T_x : x < \lambda\}.$$ But then, by (K), there exists a $x < \lambda$ so that the closure of $T_x$ contains an open set relative to $T_x$; i.e. there exists a $z < \omega$ and a function $f : \omega \rightarrow z$ so that $f(i) \in X_i$ for $i < n$ and if $n < m$ and $h : m \rightarrow z$ s.t. $h(i) \in X_i$ for $i < m$ and $z \geq h$, there exists a $g \in T_z$ with $g \geq h$. But this implies that

$$\bigcup \{\text{range}(f) : f \in T_x\} \supseteq \bigcup \{X_i : i > n\}$$

and therefore

$$C_x \cap (\bigcup \{X_i : i > n\}) = 0;$$

a contradiction. Q.E.D.

References


UNIVERSITY OF CALIFORNIA AT BERKELEY

Accepted for publication on 19. 8. 1974

On a method of constructing ANR-sets.

An application of inverse limits

by J. Krasinkiewicz (Warszawa)

Abstract. In the present paper we provide a method of constructing ANR-sets from a given ANR-sequence. We establish certain properties of the ANR-sets. Some applications are given. One of them is a simple proof of a theorem of H. Borel which says that for every natural number $a$ there exists an $(a+1)$-dimensional AR-set containing topologically every separable metric space of dimension $< a$. We prove that for every $n$-dimensional compactum $X$ there exists an $(n+1)$-dimensional infinite polyhedron $P$ disjoint from $X$ such that $X \cup P$ is an absolute retract. This result generalizes a theorem of Professor K. Borsuk.

1. A characterization of ANR-sets. By a compactum we mean a compact metric space, and a mapping is understood to mean a continuous function from a topological space to another one. A mapping $f$ from a metric space $X$ into a space $Y$ is called an $\varepsilon$-mapping provided that $\text{diam} f^{-1}(y) \leq \varepsilon$ for every $y \in f(X)$. If $f$ maps the space $X$ into itself and $g(x, f(x)) \leq \varepsilon$ for every $x \in X$, where $g$ is a metric in $X$, then we say that $X$ is an $\varepsilon$-mapping retract of $Y$. Clearly, an $\varepsilon$-mapping is an $2\varepsilon$-mapping. If $Y$ is a subset of $X$, then we say that $X$ is an $\varepsilon$-deformable into $Y$ provided there exists a mapping $\phi : X \times I \rightarrow X$ such that $\phi(x, 0) = x$, $\phi(x, 1) \in Y$ and $\text{diam} \phi(x, I) \leq \varepsilon$ for every $x \in X$. If moreover $\phi(y, 0) = y$ for every $(y, 0) \in X \times I$, then we say that $X$ is a strong $\varepsilon$-deformation retract of $Y$. Note that in this case each mapping $\phi : X \rightarrow Y$ given by the formula $\phi(x, t) = \phi(x, t)$ is an $\varepsilon$-push of $X$.

The aim of this section is to prove the following theorem:

1.1. Let $X$ be a compactum. Then it is an ANR-set if and only if for every $\varepsilon > 0$ there exists an ANR-set $Y \subseteq X$ such that $X$ is $\varepsilon$-deformable into $Y$.

The necessity of the condition is obvious. To prove its sufficiency we need a characterization of ANR-sets due to S. Lefschetz. Recall that a positive number $\eta$ is said to satisfy the condition of Lefschetz for a space $Y$ and for $\varepsilon > 0$ provided that for every polyhedron $W$, every triangulation $T$ of $W$, and every subpolyhedron $W'$ of this triangulation containing all vertices of $T$, every mapping $f' : W' \rightarrow Y$, such that $\text{diam} f'(\sigma \cap W') \leq \eta$ for each simplex $\sigma \in T$, has a continuous extension $f : W' \rightarrow Y$. Thus such that $\text{diam} f(\sigma) \leq \varepsilon$ for each simplex $\sigma \in T$. 

References


UNIVERSITY OF CALIFORNIA AT BERKELEY

Accepted for publication on 19. 8. 1974
1.2. A compactum \( Y \in \text{ANR} \) if and only if for every \( \varepsilon > 0 \) there exists a number \( \eta > 0 \) satisfying the condition of Lefschetz for \( Y \) and \( \varepsilon \) (see [1], p. 112).

 proof of the sufficiency of 1.1. According to 1.2 it suffices to show that for a given number \( \varepsilon > 0 \) there exists a number \( \eta > 0 \) such that

1. \( \eta \) satisfies the condition of Lefschetz for \( X \) and \( \varepsilon \).

Let \( \varepsilon' = \varepsilon \). By the assumptions there exist an ANR-set \( Y \subset X' \) and a homotopy \( \phi: X 	imes I \rightarrow X' \) satisfying the conditions: \( \phi(x, 0) = x \), \( \phi(x, 1) \in Y \) and

2. \( \operatorname{diam} \phi(x) \leq \varepsilon' \) for every \( x \in X \).

Hence 1.2 implies the existence of a number \( \eta' > 0 \) satisfying the condition of Lefschetz for \( Y' \) and \( \varepsilon' \). Let \( r: X \rightarrow Y' \) be defined by the formula \( r(x) = \phi(x, 1) \). Since \( r \) is uniformly continuous, there is a number \( \eta > 0 \) such that

3. \( \eta < \varepsilon' \).

4. \( A \subseteq X \wedge \operatorname{diam} A \leq \eta \Rightarrow \operatorname{diam} r(A) \leq \eta' \).

We shall show that the number \( \eta \) satisfies (1). In order to prove this consider a mapping \( g': W' \rightarrow Y \) of a subpolyhedron \( W' \) of \( W \) (in the triangulation \( T \)) satisfying the condition

5. \( \operatorname{diam} g'(\varepsilon \cap W') \leq \eta' \) for each simplex \( \varepsilon \in T \).

Setting \( f' = r \circ g': W' \rightarrow Y \) one gets a mapping such that \( \operatorname{diam} f'(\varepsilon \cap W') \leq \eta' \) for each simplex \( \varepsilon \in T \), by (4) and (5). Since \( \eta' \) satisfies the condition of Lefschetz for \( Y' \) and \( \varepsilon' \), there is a continuous extension \( f: W \rightarrow Y' \) of \( f' \) such that

6. \( \operatorname{diam} f(\varepsilon) \leq \varepsilon' \) for each simplex \( \varepsilon \in T \).

Consider the closed subset \( M = W' \times I \cup W \times \{1\} \) of the Cartesian product \( W 	imes I \). It is easy to see that there is a retraction \( k: W \times I \rightarrow M \) satisfying the condition

7. \( k(x, \varepsilon) = (x \cap W') \times I \cup x \times \{1\} \) for each simplex \( \varepsilon \in T \).

(compare the proof of Corollary 4, p. 117, in [8]). Let \( h: M \rightarrow X' \) be a mapping defined as follows:

\[
h(y, t) = \begin{cases} g'(y, t) & \text{for } (y, t) \in W' \times I, \\ f(y) & \text{for } (y, t) \in W \times \{1\}. \end{cases}
\]

This definition is correct because for \( (y, t) \in W' \times I \) we have \( g'(y, t) = r \circ g'(y) = f(y) \). Note also that for \( y \in W' \) we have \( h \circ k(y, 0) = h(y, 0) = g'(y, 0) = g'(y) \). Therefore setting

\[ g(y) = h \circ k(y, 0) \quad \text{for } y \in W', \]

one obtains a well-defined continuous extension \( g \) onto the polyhedron \( W \). Hence it remains to show that \( \operatorname{diam} g(\varepsilon) \leq \varepsilon \) for each simplex \( \varepsilon \in T \).

By (7) we have

\[ g(y) = h \circ k(y, 0) = (h(\varepsilon \cap W') \times I) \cup h(\varepsilon \times \{1\}) = \phi(g'(\varepsilon \cap W') \times I) \cup f(y). \]

Furthermore, by (7), (3) and (5), we obtain \( \operatorname{diam} \phi(g'(\varepsilon \cap W') \times I) \leq \varepsilon' \). Since the polyhedron \( W' \) contains all vertices of \( T \), the set \( \varepsilon \cap W' \) is not empty. Let \( y \) be a point of this set. Then \( \phi(g'(y, 1)) = f(y) \), and therefore the summands in the last union combine. Combining the above considerations with (6), we conclude that \( \operatorname{diam} g(\varepsilon) \leq 2\varepsilon' = e \), which completes the proof.

1.3. Corollary. Let \( X \) be a compactum. Then it is an AR-set if and only if for every number \( \varepsilon > 0 \) there exists an AR-set \( Y \subset X \) such that \( X \) is \( \varepsilon \)-deformable into \( Y \).

This corollary follows from 1.1 and the fact that an ANR-set contractible in itself is an AR-set (see [1], p. 96).

2. Quotient maps, decomposition of spaces and function spaces. A function \( f \) from a space \( X \) into a space \( Y \) is said to be a quotient map if \( f \) is onto and the following condition is satisfied: a set \( A \subset Y \) is open in \( Y \) iff the set \( f^{-1}(A) \) is open in \( X \). Hence each quotient map is a mapping. Each closed (open) mapping onto is a quotient map. The following results are almost evident; they are included here for future reference.

2.1. A mapping from a compactum onto a Hausdorff space is a quotient map.

2.2. Let \( p_i: X_i \rightarrow X_1 \), \( i = 1, 2 \) be quotient maps. Suppose \( f: X_1 \rightarrow X_2 \) is a mapping agreeing with \( p_1, p_2 \), i.e., for every \( x \in X_1 \) there exists a \( y \in X_2 \) such that \( f(x) = p_1^*(y) \). Then there exists a unique mapping \( f': X_1 \rightarrow X_2 \) such that \( f' = p_2 \).

As usual, we denote by \( 2^X \) the collection of all closed nonvoid subsets of a space \( X \). A class \( D \subset 2^X \) such that \( \bigcup D = X \) is called a decomposition of \( X \) if no two elements of \( D \) intersect. The decomposition is upper-semicontinuous if for every open subset \( U \) of \( X \) the union of elements of \( D \) which are contained in \( U \) is an open subset of \( X \). To every decomposition \( D \) corresponds a space \( \mathcal{B} \), called the space of \( D \), defined as follows: the points of \( \mathcal{B} \) are elements of \( D \), a set \( A \subset \mathcal{B} \) is open in \( D \) iff the union \( \bigcup A \) is an open subset of \( X \). Denote by \( D(x) \) the unique element of \( D \) which contains \( x \in X \). The function \( k: X \rightarrow \mathcal{B} \) given by the formula \( k(x) = D(x) \) is called the natural projection. The projection is a quotient map; if \( D \) is upper-semicontinuous, it is a closed map. The following result is well known:

2.3. The space of an upper-semicontinuous decomposition of a compactum is a compactum (see [5], p. 65).

If \( f \) maps a closed subset \( A \) of a compactum \( X \) into a compactum \( Y \), then the matching of \( X \) and \( Y \) by \( f \) is the space of the (upper-semicontinuous) decomposition, of the disjoint union of \( X \) and \( Y \), into individual points of the set \( X \cup Y \times f(A) \) and the sets \( (y) \cup f^{-1}(y) \) for \( y \in f(A) \). This space is denoted by \( X \cup Y \), and by 2.3 it is a compactum.

2.4. If \( X, A, Y \in \text{ANR} \), then \( X \cup Y \in \text{ANR} \) (see [1], p. 116).
Let $f$ be a mapping of a compactum $X$ into a compactum $Y$. If $A = X \times \{1\} \subseteq X \times I$ and $f' : A \to Y$ is defined by the formula $f'(x, 1) = f(x)$, then the matching of $X \times I$ and $Y$ by $f$ is denoted by $Z_f$, and is called the mapping cylinder of $f$ (see [8], p. 32). By 2.4 we have

2.5. If $X, Y \in \text{ANR}$, then $Z_f \in \text{ANR}.$

Let $Y^X$ be the set of all mappings from a space $X$ into a space $Y$. By the space $Y^X$ is meant the set $Y^X$ with the compact-open topology, i.e. the totality of sets $f(C) \subseteq H$, where $C \subseteq X$ is compact and $H \subseteq Y$ is open, is an open subbase of $Y^X$ (see [5], p. 76).

2.6. If $X$ is a compactum, and $f : I \to Y^X$ is a mapping, then $g : X \times I \to Y$ defined by the formula $g(x, t) = f(t)(x)$ is also a mapping (see [5], p. 86).

If $Y$ is a compactum, then we may also consider in the set $Y^X$ another topology called topology of uniform convergence defined by the metric:

$$|f-g| = \sup \{g(f(x), g(x)) : x \in X\}, \quad f, g \in Y^X,$$

where $g$ is a metric in $Y$ (see [5], p. 88). We have

2.7. If $X$ and $Y$ are compacta, then the compact-open topology of $Y^X$ coincides with its uniform convergence topology (see [5], p. 89).

3. Constructions and properties of spaces $\sigma X$, $\Sigma X$ and $\SigmaX$. Throughout this section $X$ denotes an inverse sequence of compacta, $X = \{X_n, f_{mn}\}$, with bonding maps $f_{mn} : X_n \to X_{m}$, $m \geq n$, satisfying the conditions: $f_{mm} = \text{id}_{X_m}$, $f_{mn} \circ f_{nm} = f_{nm}$; $X_0$ denotes the inverse limit of $X$, $X_0 = \text{invlim} X$, and $f_0 : X_0 \to X_0$ denotes the natural projection: $f_0(x) = x_n$ for $x = (x_1, x_2, \ldots) \in X_0$. We assume that all the sets $X_n, X_0$ are pairwise disjoint (this can always be achieved by taking the disjoint union of these sets).

**Definition 1.** The space $\sigma X$ is the set $X_0 \cup \bigcup_{n \geq 1} X_n$ with the topology defined by assuming that the totality of the following sets: open subsets of the spaces $X_n$, and sets of the form $f_{nm}^{-1}(U) \cup f_{nm}^{-1}(U)$ (where $U$ is an open subset of $X_n$, $n \geq 1$), is an open base of $\sigma X$. This space will be called the Freudenthal space of $X$. The construction is due to H. Freudenthal ([3]), p. 153, comp. also [7]). Recall the following result

3.1. The space $\sigma X$ is a compactum (see [3], p. 153–156).

3.2. The spaces $X_n, X_0$ are subspaces of $\sigma X$; the space $X_n$ is closed-open in $\sigma X$.

3.3. The function $p_n : \sigma X \to \sigma X$, $n \geq 1$, defined as follows:

$$p_n(x) = \begin{cases} x & \text{for } x \in X_n \text{ and } m \leq n, \\ f_{mn}(x) & \text{for } x \in X_n \text{ and } m > n, \\ f_0(x) & \text{for } x \in X_0, \end{cases}$$

is well defined and satisfies the following conditions:

1. $p_n$ is continuous, i.e. is a mapping,
2. $p_n$ is an $\varepsilon$-push, with $\lim_{n \to \infty} = 0$,
3. $p_n \circ p_m = p_{m}$ for $n \leq m$,
4. $p_n(x) = x$ for every $x \in \bigcup_{j=1}^{n} X_j$,
5. $p_n(\bigcup_{j=1}^{n} X_j \cup X_0) = X_n$.

**Proof.** It follows from our assumption on the sets $X_n, X_0$ and from Definition 1 that $p_n$ is well defined and continuous. Conditions 3, 4 and 5 are obvious, so it remains to prove 2.

We have to show that for every number $\varepsilon > 0$ there exists an integer $n_0$ such that $p_n$ is an $\varepsilon$-push for every $n \geq n_0$. Suppose, to the contrary, that this is not true. Then there exist an increasing sequence of integers $\{n_j\}$ and a sequence $\{y_j\}$ of points of the space $\sigma X$ such that for every $j$ we have

$$\varrho(y_j, p_n(y_j)) \leq \varepsilon,$$

where $\varrho$ is a metric in $\sigma X$. By 3.1 we may assume that $\{y_j\}$ converge to a point $y$. We claim that $y \in X_0$. Indeed, otherwise $y \in X_n$ for some $n$. Since $X_n$ is an open subset of $\sigma X$, there exists an integer $j$ such that $n_j \geq n$ and $y_j \notin X_n$. Consequently, by 4, we obtain $p_n(y_j) = y_j$, contrary to (1). Hence $y \in X_0$. Let $V$ be a neighborhood of $y$ in $\sigma X$ such that

$$\text{diam } V < \varepsilon.$$

By Definition 1 there exist an index $n$ and an open subset $U$ of $X_n$ such that $U_n = f_{nm}^{-1}(U) \cup f_{nm}^{-1}(U)$ is a neighborhood of $y$ in $\sigma X$ contained in $V$. Hence for some index $j$ we have $n_j \geq n$ and $y_j \in U_j$. It immediately follows from the definition of $p_n$, that in such a case $p_n(y_j) \in U_n$. Since $U_n$ is a subset of $V$, by (2) and (1) we obtain an absurdity. This completes the proof 3.3.

**Definition 2.** The space $\Sigma X$ is the following subspace of the Cartesian product $\sigma X \times I$:

$$\Sigma X = X_0 \times \{1\} \cup \bigcup_{n \geq 1} X_n \times I_n \times X_0 \times \{0\}, \quad \text{where } I_n = [(n, 1)/(n-1)].$$

Since $\Sigma X$ is a closed subset of the compactum $\sigma X \times I$ (see 3.1), we have

3.4. The space $\Sigma X$ is a compactum, and the mapping $h_1 : \sigma X \to \Sigma X$, defined by the formula

$$h_1(x) = \begin{cases} (x, 1/n) & \text{for } x \in X_n, n \geq 1, \\ (x, 0) & \text{for } x \in X_0, \end{cases}$$

is an embedding.
DEFINITION 3. The collection $D$ of subsets of $\Sigma X$ defined as follows: the single point sets $\{p\}$ for $p \in X \setminus \{X_i \times \{1\} \cup \bigcup_{n \geq 1} X_i \times (1/n, 1/(n-1))\}$, and the sets $(x, 1/n) \cup f^{n-1}_{\Sigma X} x \cup f^{-n+1}_{\Sigma X} x$ for $x \in X_n$, $n \geq 1$, is called the canonical decomposition of $\Sigma X$. The canonical decomposition is a decomposition in the usual sense (see lemma below); the space of the decomposition is called the space of the inverse sequence $X$ and is denoted by $\Sigma X$. The natural projection $k: \Sigma X \to X$ is called the canonical projection.

3.5. The canonical decomposition is an upper-semicontinuous one.

Proof. It is evident that the elements of $D$ are closed, monovex and disjoint. So we have to prove that every open set $U$ of $\Sigma X$ is the union of elements of $D$ which is a subset of $U$. Suppose $y \in D \subset U$ and consider four cases:

I. $y \in X_n \times (1/n, 1/(n-1))$ for some $n \geq 1$. Then $U = V \cap X_n \times (1/n, 1/(n-1))$ is the required neighbourhood of $y$.

II. $y \in X_n \times (1/n, 1/(n-1))$ for some $n \geq 1$. Let $y = (x, 1/n)$. Define $D = \{(x) \cup f^{n-1}_{\Sigma X}(x) \cup \{x, 1/n\}\}$. Since $U$ is open and $D \subset U$, there exists an open subset $W$ of $X_n$ such that $x \in W$ and $(W \cup f^{n-1}_{\Sigma X}(W)) \times (1/n, 1/(n-1)) \subset U$. Define a set $L_n$ as follows: if $n = 1$, then $L_n = \{1\}$; if $n > 1$, then $L_n = \{1/n, 2/n, \ldots, 1/(n-1)\}$, such that $W \cup L_n \subset U$. There exists an interval $I = (1/n, 1/(n-1))$, such that $(W \cup f^{n-1}_{\Sigma X}(W)) \times I \subset U$. It follows from the construction that $G = W \cup L_n \cup f^{n-1}_{\Sigma X}(W) \times I$ is the required neighbourhood of $y$.

III. $y = (x, 1/n) \in X_n \times (1/n, 1/(n-1))$ for some $n \geq 1$. In this case the point $y' = (x, 1/n)$ also belongs to $D$. Proceeding in case II, we obtain the neighbourhood $G$ of $y'$. This is also an appropriate neighbourhood of $y$.

IV. $y = (x, 0) \in X_0 \times \{0\}$. Since $y \in U$ and $U$ is an open subset of $\Sigma X \times \sigma X$, there exist an open neighbourhood $N$ of $x$ in $\sigma X$ and an interval $I = [a, b]$ such that $N \times I \subset \Sigma X \subset U$. Let $n \geq 1$ be an index such that $1/(n-1) < a$ or $b \leq 1/n$. There exists an index $n \geq N$ and an open subset $H$ of $X_n$ such that $H = f^{-n}_{\Sigma X}(x)$ and $f^{n-1}_{\Sigma X}(H) \cup f^{-n+1}_{\Sigma X}(H) \subset N \times M \subset \Sigma X \subset U$. By its definition the set $G$ is a neighbourhood of $y$. Since $G$ is the union of some elements of $D$, it is the required neighbourhood of $y$. This completes the proof of 3.5.

The following statement is obvious:

3.6. The mapping $\mu_1: \Sigma X \to I$ given by the formula

$$\mu_1(x, t) = t$$

agrees (in the sense given in §2) with the canonical projection $k$ and the identity mapping $1$.

3.7. Let $q: \{0\} \to N$, $N$ be the set of natural numbers, be defined as follows:

$$q(t) = \left\{ \begin{array}{ll}
1 & \text{for } t = 1, \\
n+1 & \text{for } 1/(n+1) < t < 1/n, \ n \geq 1.
\end{array} \right.$$ 

Let $\varphi_0$ denote the identity mapping of $\Sigma X$ and, for $t > 0$, let $\varphi_t: \Sigma X \to \Sigma X$ be given by the following formula:

$$\varphi_t(x) = \left\{ \begin{array}{ll}
\varphi_0(x) & \text{for } y \in \mu_1^{-1}(\{0, 1\}), \\
\varphi_0(x, t) & \text{for } y = (x, s) \in \mu_1^{-1}([0, 1]),
\end{array} \right.$$ 

Then the functions $\varphi_t$, $t \in I$, are continuous and agree with $k$, $\varphi_0$.

Proof. We may assume that $t > 0$. First we prove that $\varphi_t$ is well defined. Suppose $y \in C_{\mu_1^{-1}((0, 1))}$; then $\varphi_t(x) = (x, t)$. If $t = 1/n$, then $x \in X_n$ and $\varphi(t) = n$; hence $(\varphi_0(x, t), t) = y$, by 3.3.4. If $1/n < t < 1/(n-1)$, $n > 2$, then also $x \in X_n$ and $\varphi(t) = n$; hence $(\varphi_0(x, t), t) = y$ for the same reason as above. The continuity of $\varphi_t$ follows from 3.3.1. It remains to prove that $\varphi_t(D(y)) = D(\varphi_t(x))$ for every $y \in \Sigma X$. We may assume that $D(y)$ is nondegenerate, that is:

$$D(y) = ((x) \cup f^{n-1}_{\Sigma X}(x) \cup \{x, 1/n\}\} \times \{0\})$$

for some $x \in X_n$ and $n > 1$. We may also assume that $1/n < t < 1$, for otherwise $\varphi_t(y) = y'$ for every $y' \in D(y)$. But in such a case we have $D(y) = \mu_1^{-1}([0, 1])$ and $\varphi(t) = n + 1$; hence, by definition of $\varphi_t$ and 3.3, we obtain $\varphi_t(y) = (\varphi_0(x, t), t)$ for every $y' \in D(y)$.

Therefore $\varphi_t(D(y)) = D(\varphi_t(x)) = D(\varphi_t(x))$, which completes the proof.

We will consider $\sigma X \times I$ as a metric space with a metric $q$ given by the formula

$$q((x, s), (x', s')) = q(x, x') + |s - s'|,$$

where $q$ is a metric in $\sigma X$.

3.8. For every $t \in I$, define $U_t$ in the following way:

$$U_t = \left\{ \begin{array}{ll}
\{1/2, 1\} & \text{for } t = 1, \\
(1/(n+2), 1/n) & \text{for } t = 1/(n+1), \\
(1/(n+1), 1) & \text{for } 1/(n+1) < t < 1/n, \\
\{1/n, 0\} & \text{for } t = 0.
\end{array} \right.$$ 

Then for every $y \in \Sigma X$ we have

$$\varphi_t(D(y), D(\varphi_t(y))) = \left\{ \begin{array}{ll}
1_{\|x - \varphi_0(x)\| + |t|} & \text{for } t = 0 \text{ and } t' > 0, \\
1_{|t'|} & \text{for } t > 0 \text{ and } t' = 0.
\end{array} \right.$$ 

where $\varphi_0(A, B) = \inf \{q(a, b) : a \in A \text{ and } b \in B\}$.

Proof. We may assume $t \neq t'$. Let $y = (x, s)$. Suppose first $t = 0$ and $t' > 0$.

Then $\varphi_t(y) = y$ because $\varphi_0$ is the identity mapping of $\Sigma X$. The point $\varphi_t(y)$ is either $y$
or \((\rho_{\text{ref}}(x), r')\). In the latter case \(s \leq r\). Therefore \(\varphi_1(\varphi_s(x), \varphi_r(y)) \leq \varphi(\varphi_{s+1}(x), \varphi_{r+1}(y)) = |r' - r|\), which proves \((*)\) in the case of \(t = 0\).

Suppose now that \(t > 0\) and \(t \in U\). We have to consider several cases.

1. \(s < t = n / n\) for some \(n \geq 1\). Then \(q(t) = n = q(t') = n + 1\). If \(s < t\), then \(\varphi_0(x) = \varphi_s(x) = x\). If \(s = t\), then either \(x \in X_0\) or \(x \in X_{s+1}\). In the former case \(q_s(x) = \varphi_s(x)\) in the latter one \(\varphi_s(x) = y\) and \(\varphi_r(y) = \varphi_{s+1}(x)\), and therefore \(D(\varphi_r(y)) = D(y)\). If \(s < t < s\), then \(x \in X_{s+1}\). Hence \(D(\varphi_r(y)) = D(x, t) = \varphi_s(x) = y\). If \(s < t\), then \(D(\varphi_r(y)) = D(x, r) = \varphi_s(x) = y\). Hence in each case we obtain \((*)\).

II. \(t > t = n / n\).

Then \(q(t) = q(t') = n + 1\). If \(s > s',\) then \(\varphi_s(x) = \varphi_s(x)\). If \(s > s\), then \(\varphi_s(x) = y\) and \(\varphi_r(y) = \varphi_s(x)\). If \(s = t\) and \(x \in X_s\), then \(\varphi_s(x) = y\) and \(\varphi_r(y) = \varphi_s(x)\). If \(s < s\), then \(\varphi_s(x) = \varphi_s(x)\) and \(\varphi_r(y) = \varphi_s(x)\). Hence in each case we obtain \((*)\).

III. \(1/n + 1 < t < n\) for some \(n\). Then \(q(t) = q(t') = n + 1\), and by arguments similar to that used above we obtain \((*)\) in this case. The latter completes the proof.

The results which we have just proved will now be used to establish several properties of the space of the inverse sequence \(X\).

Notation. \(k(s, t) = [x, s]\).

By 3.4, 3.5 and 2.3 we obtain

3.9. The space \(SX\) is a compactum and the function \(h : eX \rightarrow SX\) defined by the formula

\[
h(x) = \begin{cases} [x, 1/n] & \text{for } x \in X_n \text{ and } n \geq 1, \\
[x, 0] & \text{for } x \in X_0
\end{cases}
\]

is an embedding.

By 3.9, 2.1 and 2.2 we obtain

3.10. The function \(\mu : SX \rightarrow eX\) given by the formula:

\[
\mu(x, t) = \begin{cases} \varphi(t) & \text{for } x \in \mu^{-1}([t, t]), \\
\varphi(x) & \text{for } x \in \mu^{-1}([t, t])
\end{cases}
\]

is a mapping. The set \(\mu^{-1}(t)\) is homeomorphic to \(X_0\). Moreover, \(\mu^{-1}(0) = h(X_0)\) and \(\mu^{-1}(1/n) = h(X_n)\) for every \(n \geq 1\). Finally, the set \(\mu^{-1}([1/n, 1/n])\) is homeomorphic to the mapping cylinder \(Z_{X_{1/n}}\), for \(1/n \leq t < 1\).

3.11. For every \(t \leq 1\) the function \(\psi : SX \rightarrow SX\) given by the formula:

\[
\psi(x) = \begin{cases} \varphi(t) & \text{for } z \in \mu^{-1}([t, t]), \\
\varphi_{\psi(x)}(x) & \text{for } z = (x, s) \in \mu^{-1}([t, t])
\end{cases}
\]

where \(\psi_0\) is defined by the first equality, is a mapping. The function \(\psi : SX \times SX\) defined as \(\psi(x, t) = \psi_t(x)\), is a mapping satisfying the following conditions:

1. \(\psi(x, 0) = x\) for every \(x \in SX\).
2. \(\psi(x, t) = x\) for \(x \in \mu^{-1}([t, t])\) and \(t < t'\).
3. \(\psi(x, t) \in \mu^{-1}([t, t])\) for \(x \in \mu^{-1}([t, t])\) and \(t < t'\).
4. \(\text{diam} \psi(x, [0, l]) \leq l, \text{for } l > 0\), with \(l = 0\).

Proof. The assertion about \(\psi\) easily follows from the corresponding properties of \(\varphi\). We shall show that \(\psi\) is continuous. Hence by 2.6 it suffices to show that the function \(f : I \times SX\rightarrow SX\), where \(f(t) = \psi(t)\), is continuous (the function space with the compact-open topology). Let \(d\) be a metric in \(SX\). By 2.7 we may assume that the function space has the uniform convergence topology defined by \(d\). Let \(t \in I\) and let \(\varepsilon > 0\) be a real number. We have to prove that \(|\psi(x) - \psi(y)| \leq \varepsilon\) for every \(y \in \psi(x)\) in a neighbourhood of \(t\). Since \(SX\) is a compactum, the canonical projection \(k\) is uniformly continuous, hence there is a number \(\eta > 0\) such that \(|k(x, y)| < \eta\) implies \(\varphi_n(x, y) \leq \varepsilon\). By 3.3.2 and 3.3.8, there is a neighbourhood \(U \subseteq I\) of \(t\) such that for \(t' \in U\), we have \(|t - t'| \leq \eta\) and \(|t - t'| \leq \eta\). Let \(t \in I\) be an arbitrary point of \(SX\). Then \(k^{-1}(\psi(x)) = D(\varphi_t(x))\) and \(k^{-1}(\psi(x)) = D(\varphi_r(x))\). So \(\psi_t(x) = \mu^{-1}([t, t])\) for every \(x \in I\), by 3.3 we obtain \(d(\psi(x), \psi(y)) \leq \varepsilon\). It follows that \(|\psi(x) - \psi(y)| \leq \varepsilon\). This proves the continuity of \(\psi\). Thus \(\psi\) is continuous.

The properties 1, 2 and 3 of \(\psi\) follow from 3.7. The property 4 follows again from 3.8 (*) by an argument similar to that used above.

The inverse sequence \(X\) is called an ANR-sequence provided every space \(X_n\) is an ANR-set.

3.12. If \(X\) is an ANR-sequence, then
1. \(\mu^{-1}([t, t]) \subseteq \text{ANR}\) for \(t, t' \in I\) and \(t < t' > 0\).
2. \(X_0 \subseteq \text{ANR} \Rightarrow \mu^{-1}([t, t]) \subseteq \text{ANR}\) for every \(t \in I\).

Proof. 1. By 3.10 the set \(\mu^{-1}([1/(n+1), 1/n])\) is homeomorphic to \(Z_{X_{1/n}}\), hence it is an ANR-set by 2.5. By 3.10 we have also

\[
\mu^{-1}([1/(n+1), 1/n]) \cap \mu^{-1}([1/(n+2), 1/(n+1)]) = h(X_{1/n}).
\]

The set on the right-hand-side of the equality is an ANR-set by 3.9. Since the union of two ANR-sets whose common part is an ANR-set is an ANR-set (see [1], p. 90), we have \(\mu^{-1}([1/(n+2), 1/n]) \subseteq \text{ANR}\). By an easy induction and by 3.10 we infer that \(\mu^{-1}([n+1], 1/n]) \subseteq \text{ANR}\) for every \(n < \eta < 1/n\). It is easy to see that if \(1/(n+1) < \eta < n/n\), then the set \(\mu^{-1}([n, n])\) is homeomorphic to \(X_{n+1} / \{x, x'\}\), and hence it is an ANR-set. The above two results, 3.10 and the quoted result on the union of two ANR-sets imply that \(\mu^{-1}([t, t]) \subseteq \text{ANR}\) for every \(t < n\). It remains to prove that \(\mu^{-1}([0, l]) \subseteq \text{ANR}\) for \(t > 0\). But 3.11 implies that for every \(t > 0\) there exists an ANR-set \(A \subseteq \mu^{-1}([0, l])\) (namely the set \(\mu^{-1}([0, l])\) for some \(0 < l < l\)) such that it is a strong \(s\)-deformation retract of \(\mu^{-1}([0, l])\). Hence \(\mu^{-1}([0, l]) \subseteq \text{ANR}\), by 1.1. This completes the proof of 1.

2. Let \(X_0 \subseteq \text{ANR}\). By the previous result \(SX = \mu^{-1}([0, l]) \subseteq \text{ANR}\). By 3.11 the set \(\mu^{-1}([0, l])\) is a strong deformation retract of \(SX\), and by 3.10 \(\mu^{-1}([0, l]) = h(X_0)\). Hence \(\mu^{-1}([0, l]) \subseteq \text{ANR}\). By 3.9 it follows that \(SX\) is an ANR-set contractible in itself, and therefore an AR-set (see [1], p. 96). By 3.11 the set \(\mu^{-1}([0, l])\) is a strong deformation retract of \(SX\). Hence it is an AR-set. This completes the proof.
4. Embedding of compacts into absolute retracts. The aim of this section is to prove the following theorem:

4.1. For every nondegenerate compactum \( X \) there exist an absolute retract \( M \) containing \( X \), a point \( v \in M \) and a mapping \( \mu : M \to I \) satisfying the following conditions:

1. \( \mu \) is an open mapping onto \( I \),
2. \( \mu^{-1}(0) = X \),
3. \( \mu^{-1}(1) = \{v\} \),
4. \( \mu^{-1}([t', t]) \) is ANR if \( t + t' > 0 \),
5. \( \mu^{-1}([t, 1]) \) is AR for every \( t \in I \),
6. \( \mu^{-1}(i) \) is a strong deformation retract of \( \mu^{-1}([t', t]) \) for every \( t \in I \) and \( t', t \).

Moreover, if \( \dim X = n \), then in addition
7. \( \mu^{-1}(i) \) is an \( n \)-dimensional polyhedron for every \( 0 < t < 1 \),
8. \( \mu^{-1}([t, 1]) \) is an \( (n+1) \)-dimensional compactum if \( t < t' \).

Proof. By a classical result of Freudenthal [3] there exists an inverse sequence of polyhedra \( (X_n, f_m) \) such that \( X = \injlim X_n \) and the bonding maps \( f_m : X_m \to X_n, n \leq m \), are mappings onto (comp. [6]). If \( \dim X = n \), then we may assume that \( \dim X_k = n \) for every \( k > 1 \). Without loss of generality we may also assume that \( X_1 \) is a single-point space. Put \( M = SX \). According to the results of Section 3 it is easy to check that the space \( M \) satisfies all the required conditions, because we can identify the compactum \( X \) with \( h(X_n) \) (comp. the figure).

An interesting application of 4.1 is a simple proof of the following result of H. Bothe:

4.2. COROLLARY [2]. For every natural number \( n \) there exists an \((n+1)\)-dimensional absolute retract containing topologically every \( k \)-dimensional metric separable space, with \( k \leq n \).

Proof. Let \( X_0 \) be an \( n \)-dimensional universal compactum (see [4], p. 64). Hence \( X_0 \) contains topologically every metric separable space of dimension \( \leq n \). Applying 4.1 we obtain an \((n+1)\)-dimensional AR-set containing \( X_0 \). Hence this absolute retract satisfies the conclusion of 4.2.

5. Remark on homotopy groups. The main result of this section is not of interest in this paper, but it will find an important application in a forthcoming paper of the author on the theory of continua. We begin with some lemmas. The following one is evident:

5.1. Let \( A \) be a subset of a space \( X \). If the inclusion map \( i : A \to X \) induces an epimorphism

\[
\pi_n(A, a) \to \pi_n(X, a)
\]

of the \( n \)-th homotopy groups for some point \( a \in A \), then it induces the epimorphism for every other point \( x \in A \) provided \( x \) belongs to the path-component of \( A \) which contains \( a \).

Let \( f \) be a mapping from a compactum \( X \) into a compactum \( Y \). Denote by \( [x, i] \) the point of the mapping cylinder \( Z_f \) which corresponds to the point \( (x, i) \in X \times I \) by the natural projection of \( X \times I \) onto \( Y \). Now, if \( x \in X \) and \( y \in Y \), we denote the point of \( Z_f \) which corresponds by this projection to the point \( p \in Y \). The mappings \( f : X \to Z_f \) and \( j : Y \to Z_f \) are given by the formula \((i, x) = [x, 0], j(y) = [y] \) are embeddings.

It is an easy exercise to prove the following lemma:

5.2. Let \( f : (X, x_0) \to (Y, y_0) \) and suppose that the induced homomorphism \( f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0) \) is an epimorphism. Then the induced homomorphism \( i_* : \pi_n(X, x_0) \to \pi_n(Z_f, [x_0, 0]) \) is also an epimorphism.

The main result of this section can be stated as follows:

5.3. Let \( A \) be a pointed compactum \((X, x_0)\) be the limit of an inverse pointed ANR-sequence \((X, x_0) = \{(X_n, x_n)_{f_m}\} \), i.e., \((X, x_0) = \injlim(X, x_0)\), where \( x_n = (x_1, x_2, \ldots) \in X \). Suppose that the bonding maps \( f_m : (X_n, x_n) \to (X_{m+1}, x_{m+1}) \) induce epimorphisms \( f_* : \pi_n(X, x_0) \to \pi_n(X_n, x_n) \) of the \( k \)-th homotopy groups, where \( m \geq n \). Then there exist an absolute retract \( M \) containing \( X \) and a decreasing sequence \( \{A_n\} \) of ANR-sets in \( M \) such that \( X = \bigcap A_n \) and the inclusion map \( i : (X, x_0) \to (A_n, x_0) \) induces an epimorphism of the corresponding \( k \)-th homotopy groups, for every \( n \geq 1 \).

Proof. Without loss of generality we may assume that \( X_1 \) is a single-point space. Let us adopt the notation of Section 3 and set \( M = SX \) and \( A_n = \mu^{-1}(n, 0, 0) \) for every \( n \geq 1 \). The Freudenthal space of \( X \) consists of the
spaces $X_0$ and $X$. Since $h$ is an embedding of the Freudenthal space into $M$, by 3.12 we have
\[ h(X) \subseteq M \subseteq \mathbb{R}^n. \]
(1)

\[ A_{n+1} \subseteq A_n \subseteq \mathbb{R}^n \text{ and } \bigcap A_n = h(X). \]

Now we shall prove that

(3) the induced homomorphism $(\iota_{I})_{h} : \pi_{n}(A_{n+1}, x_0) \rightarrow \pi_{n}(A_{n}, x_0)$ is an epimorphism.

By 3.10 we have $h(A_{n+1}) = \mu^{-1}(\{0,n+1\}) \subseteq A_{n+1}$; in particular $h(x_{n+1}) \in A_{n+1}$. By 3.3, 3.7, 3.9 and 3.11 the following diagram commutes:

\[ \begin{array}{ccc}
M & \xrightarrow{\mu^{-1}(\{0,n+1\})} & M \\
\downarrow{h} & & \downarrow{h} \\
X_{n+1} & \xrightarrow{\mu^{-1}(\{0,n+1\})} & X \\
\end{array} \]

Since $f_{n+1}(x_0) = x_{n+1}$, it follows from 3.11 that $\psi_{\mu^{-1}(\{0,n+1\})}(h(x_0)) = \psi_{\mu^{-1}(\{0,n+1\})}(h(x_0)) = \psi_{\mu^{-1}(\{0,n+1\})}$. Hence these points belong to the same path-component of $A_{n+1}$. Hence, by 5.1, in order to prove (3) we need only to show that

(4) $(\iota_{I})_{h} : \pi_{n}(A_{n+1}, h(x_0)) \rightarrow \pi_{n}(A_{n}, h(x_0))$ is an epimorphism.

Let $w : I \rightarrow I_{n+1} = \{0,n+1\} \cup \{1/n\}$ be a homeomorphism such that $w(0) = 1/(n+1)$ and $w(1) = 1/n$. It is easy to check that the function
\[ \alpha : Z_{f_{n+1}^{-1}(\{0,n+1\})} \rightarrow \mu^{-1}(\{1/(n+1), 1/n\}) = B \]

is a homeomorphism. Let us note that $\alpha(x_{n+1}, 0) = h(x_{n+1})$. Hence we have

(5) $(\iota_{I})_{h} : \pi_{n}(Z_{f_{n+1}^{-1}(\{0,n+1\})}, [x_{n+1}, 0]) \rightarrow \pi_{n}(B, h(x_{n+1}))$ is an epimorphism.

Consider the following diagram:

\[ \begin{array}{ccc}
(A_{n+1}, h(x_{n+1})) & \xrightarrow{u_{n}} & (A_{n+1}, h(x_{n+1})) \\
\downarrow{\iota_{I}} & & \downarrow{\iota_{I}} \\
(B, h(x_{n+1})) & \xrightarrow{\rho} & (X_{n+1}, x_{n+1}) \\
\end{array} \]

where $i$, $j$, and $k$ are inclusion maps, $\alpha' = j \circ \alpha$, $\beta(x) = h(x)$ and $\iota(x) = [x, 0]$. It is evident that the diagram commutes. It easily follows from 3.11 that $\beta(\mu^{-1}(\{0,n+1\})) = \mu^{-1}(\{1/(n+1)\})$ is a strong deformation retract of $A_{n+1}$, and $B$ is a strong deformation retract of $A_{n}$. Hence the homomorphisms $(\iota_{I})_{h} : \pi_{n}(A_{n+1}, h(x_0)) \rightarrow \pi_{n}(A_{n}, h(x_0))$ and $(\iota_{I})_{h} : \pi_{n}(B, h(x_0)) \rightarrow \pi_{n}(A_{n}, h(x_0))$ are isomorphisms (see [5]). By our assumption and by 5.2 the homomorphism $(\iota_{I})_{h}$ is an epimorphism. Combining these facts we see that $(\iota_{I})_{h}$ is an epimorphism, which proves (4) and therefore (3). Identifying $X$ with $h(X)$ and $x_0$ with $h(x_0)$ we obtain by (1), (2) and (3) the conclusion of 5.3. This completes the proof.

6. A generalization of Borsuk's theorem. Professor K. Borsuk proved the following theorem (see [1], p. 108):

"For every compactum $X$ there exists an infinite polyhedron $P$ with a null-triangulation such that $X \cup P$ is an absolute retract."

In this section we shall prove a strengthened version of this result. We first prove a list of lemmas which will be used in the proof of the theorem. We start with the following important fact established by J. H. C. Whitehead ([9], p. 259 and [10], p. 244):

6.1. Let $K$ and $L$ be finite complexes and let $f : |K| \rightarrow |L|$ be a mapping simplicial with respect to these complexes. Embed polyhedra $|K|$ and $|L|$ in the mapping cylinder $Z_{f}$ by the maps: $s \rightarrow [x, 0]$ and $y \rightarrow [x, 1]$ for $x \in |K|$ and $y \in |L|$. Let $\psi : \psi((x, t)) = t$ and $\psi((y, t)) = 1$. Then there exist a finite complex $P$, subcomplexes $K'$ and $L'$ of $P$, a homeomorphism $\phi : |P| \rightarrow Z_{f}$, and a mapping $\phi : |P| \rightarrow I$ simplicial with respect to $P$ and a triangulation $I'$ of $I$, satisfying the conditions: $\phi(|K'|) = |K|$, $\phi(|L'|) = |L|$, $\phi$ is a simplicial isomorphism between $K'(U)$ and $L'(U)$, respectively, and $\phi = \psi$. $I'$ is obtained by dividing $I$ at its middle point.

6.2. Let $\phi$ be a simplicial map from an $n$-simplex $\sigma^{n}$ onto a 1-simplex $\sigma^{1} = \langle x_{0}, x_{1} \rangle$. Let $\sigma_{i} = \phi^{-1}(\langle x_{i} \rangle)$, $i = 0, 1$. Hence $\sigma_{i}$ is the join of simplices $\sigma_{0} \cup \sigma_{1}$, i.e., $\sigma_{i} = \sigma_{0} \ast \sigma_{1}$. Suppose that $K_{0}$ is a subdivision of $\sigma_{0}$. Then there exists a subdivision $K$ of $\sigma$ such that $K_{0} \cup (\sigma_{0} \ast \sigma_{1}) \subseteq K$ and $\phi$ is simplicial with respect to $K$ and $\sigma$.

Proof. To prove the lemma it suffices to take as $K$ the subdivision of $\sigma_{i}$ composed of all simplices of the form $\tau \ast \sigma_{i}$, where $\tau \in K_{0}$, and all faces of these simplices, an immediate consequence of 6.2 is the following lemma:

6.3. Let $\phi$ be a simplicial map of a complex $K$ onto a 1-simplex $\sigma^{1} = \langle x_{0}, x_{1} \rangle$. Let $K_{1} = \phi^{-1}(\langle x_{1} \rangle)$, $i = 0, 1$, and suppose that $K_{0}$ is a subdivision of $K_{1}$. Then there exists a subdivision $K'$ of $K$ such that $K_{0} \cup K' \subseteq K'$ and $\phi$ is simplicial with respect to $K'$ and $\sigma^{1}$.

Let $\phi : \sigma^{1} \rightarrow \sigma^{1}$ be a simplicial map into $\sigma^{1} = \langle x_{0}, x_{1} \rangle$, and let $c = t_{0}a_{0} + t_{1}a_{1}$, $t_{0} + t_{1} = 1$, be an interior point of $\sigma^{1}$, i.e., $t \neq 0$. By a $(\phi, c)$-barycentre of $\sigma^{1}$ we understand a point $b_{\sigma_{i}} \in \sigma^{1}$ defined as follows: if $\phi(\sigma_{i}) = a_{i}$, then $b_{\sigma_{i}} = a_{i}$; if $\phi(\sigma_{i}) = a_{i}$, then $b_{\sigma_{i}} = b_{a_{i}}$. The usual barycentre of $\sigma_{i}$, if $\phi(\sigma_{i}) = a_{i}$, then we define $b_{a_{i}} = t_{0}a_{0} + t_{1}a_{1}$, where $t_{0} + t_{1} = 1$ — Fundamenta Mathematicae t. XCVII
$e_i = \varphi^{-1}(a_i)$ and $b_i$ is the barycentre of $\sigma_i$. Observe that $\varphi(b_i) = c_i$. For each sequence $\tau_0, \tau_1, \ldots$, $\tau_n$ of faces of $\sigma^*$ such that $\tau_i$ is a proper face of $\tau_{i+1}$, the sequence of corresponding $(\varphi, c_i)$-barycentres of $\tau_{i+1}$ span a simplex contained in $\sigma$. All simplices obtained in this manner form a subdivision of $\sigma^*$ denoted by $\sigma_{\text{sub}}$, and called a $(\varphi, c_i)$-subdivision of $\sigma^*$.

6.4. If $c$ is the barycentre of $\sigma^* = \langle a_0, a_1 \rangle$, then we have

$$\text{mesh}^2_{e_i} \leq \frac{2n+1}{2n+2} \text{ diam}^2 \sigma^*.$$

Proof. Let $e \in \sigma_{\text{sub}}$. Hence there exists a sequence $\langle \tau_i \rangle$, $0 \leq i \leq k$, $0 \leq k \leq n$, of faces of $\sigma^*$ such that $\tau_i$ is a proper face of $\tau_{i+1}$ and $e$ is the simplex spanned by the barycentres $b_{a_i}, b_{a_i}, \ldots, b_{a_i}$, i.e., $e = \langle b_{a_i}, b_{a_i}, \ldots, b_{a_i} \rangle$. We have to show that

$$\text{diam} \sigma^* \leq \frac{2n+1}{2n+2} \text{ diam} \sigma^*.$$

We have $\text{diam} \sigma^* = \langle |b_k - b_l| \rangle$ for some $i < j$. If $\tau_i = \varphi^{-1}(a_i)$, then also $\tau_j = \varphi^{-1}(a_j)$ and $b_{a_i}, \ldots, b_{a_j}$ is the barycentre of $\tau_i$. So we have

$$|b_k - b_l| = \frac{\text{dim} \tau_j}{\text{dim} \tau_j + 1} \text{ dim} \tau_j \leq \frac{n}{n+1} \text{ diam} \tau_j \leq \frac{2n+1}{2n+2} \text{ diam} \sigma^*.$$

Thus we obtain the conclusion of the lemma in this case. Suppose now that $\tau_i \neq \varphi^{-1}(a_i)$. It follows that $\varphi(\tau_j) = \varphi(a_j)$, and $\tau_j = \tau_j \cap \tau_i$, $\tau_j = \varphi^{-1}(a_j)$, and $\tau_i$, and $b_{a_j} = \langle h_{a_j} + h_{a_j} \rangle$ (because $e = \langle b_{a_i}, \ldots, b_{a_j} \rangle$). Since $\tau_i \cap \tau_j$ and $b_{a_j}$ belong to $\tau_j$, we have

$$b_{a_j} = h_{a_j}b_{a_j} + \cdots + h_{a_j}b_{a_j} \quad \text{ for } i = 1, \ldots, n.$$

It follows that $|b_k - b_l| \leq \max \{ |b_k - b_l| \}$. It remains to show that for every vertex $v_i$, $0 \leq m \leq r$, we have

$$|b_k - b_l| \leq \frac{2n+1}{2n+2} \text{ diam} \sigma^*.$$

(1)

Without loss of generality we may assume that $\tau_j = \langle p_0, \ldots, p_n \rangle$ and $\tau_j = \langle p_0, \ldots, p_n \rangle$. So we have

$$h_{a_j} = \frac{1}{2} \left( \sum_{i=0}^{n} \frac{1}{i+1} p_i + \sum_{i=0}^{r} \frac{1}{r-i} p_i \right).$$

We may also assume that $m \leq u$. Then we obtain

$$|b_k - b_l| \leq \frac{1}{2} \left( \sum_{i=0}^{u} \frac{1}{u+1} (p_i - p-u) + \sum_{i=0}^{r} \frac{1}{r-u} (p_i - p-u) \right) \leq \frac{1}{2} \left( \sum_{i=0}^{u} \frac{1}{u+1} |p_i - p-u| + \sum_{i=0}^{r} \frac{1}{r-u} |p_i - p-u| \right) \leq \frac{1}{2} \left( \frac{1}{u+1} \text{ diam} \sigma^* + \frac{1}{r-u} \text{ diam} \sigma^* \right) \leq \frac{1}{2} \left( \frac{u+1}{u+2} \text{ diam} \sigma^* + \frac{r+1}{r+2} \text{ diam} \sigma^* \right) \leq \frac{2n+1}{2n+2} \text{ diam} \sigma^*$$

because $u \leq r \leq n$. This proves condition (1) and completes the proof of 6.4.

If $\sigma$ is a simplicial map of a complex $K$ into a $1$-simplex $\sigma^1$, then we denote by $K_{\sigma}$ the complex $\sigma^* \cap K$. This complex is a subdivision of $K$ and is called $\sigma$-subdivision $\sigma^* \cap K$. It is evident that $\sigma$ is simplicial with respect to $K_{\sigma}$ and $\sigma_{\text{sub}}$ (the last symbol will be abbreviated to $\sigma^*$). From Lemma 6.4 we obtain the following corollary:

6.5. If $c$ is the barycentre of $\sigma^* = \langle a_0, a_1 \rangle$ and $0 \leq \text{dim} K \leq n$, then

$$\text{mesh} K_{\sigma^*} \leq \frac{2n+1}{2n+2} \text{ mesh } K.$$

6.6. Let $\varphi: \sigma^* \rightarrow \sigma^* = \langle a_0, a_1 \rangle$ be a simplicial map and let $e$ be a given positive number. There exists an $n > 0$ such that $|e| < n$, then $|e| \leq \text{max} \{ n, \text{ diam } \sigma^* \}$ for every $e \in \sigma^*$ such that $|e| \leq \text{mesh } (\varphi^{-1}(a_0), c_0)$.

Proof. Let $e = e_0 \cap c_0$, where $e_0 \cap c_0 = 1 + 1 > 0$. Let $K$ be the subcomplex of $\sigma^*$ such that $|K| \leq \text{mesh } (\varphi^{-1}(a_0), c_0)$. Let $K: K^0 \rightarrow e_0$ be a vertex map defined as follows. If $e$ is a vertex of $e_0$, then let $f(e) = e$. If $v = e_0$, then $v = e_0$. If $e = e_0 \cap c_0$, then $c_0$ is the vertex map $f(e) = e$. In this case we set $f(e) = e$, and observe that $f(e)$ may be extended linearly onto each simplex from $K$. Denote the extension by $f$ as well and note that $f(e) - e \leq \text{diam } e_0 = 1$. This observation easily leads to the existence of $n$.

As a corollary we obtain the following proposition.

6.7. If $\varphi: K^0 \rightarrow a_0, a_1$ is a simplicial map and $\varepsilon > 0$ is a given number, then there exists a subdivision $K'$ of $K$ and a point $c$ in the interior of an edge such that $\varphi_1$ is $\varepsilon$.
simplicial with respect to $K'$ and $\sigma_i$, $K_0 = \varphi^{-1}(a_0)$ is a subcomplex of $K'$ and
\[
\text{diam } \tau \leq \max\{\varepsilon, \text{mesh } K_0\} \quad \text{for each } \tau \in \varphi^{-1}(\langle a_0, c \rangle),
\]
where $\varphi_1$ is the map induced by $\varphi$.

6.8. If $\varphi: K \to L$ is a simplicial map, $\varepsilon > 0$ is a given number and $K_0 = \varphi^{-1}(a_0)$, then there exists subdivisions $K_{\varepsilon} \subseteq K$, $\varphi_1$ is a simplicial map $\varphi_1: K_{\varepsilon} \to L$. Finally, $\text{mesh } K_{\varepsilon} \leq \max\{\varepsilon, \text{mesh } K_0\}$ and $\text{mesh } \varphi^{-1}\langle a_0, c \rangle < \varepsilon$.

Proof. First choose $c \in \text{Int } a_0$, $\varphi_1$ and $K'$ as in 6.7. Let $K_0 = \varphi^{-1}(a_0)$, $K_{\varepsilon} = \varphi^{-1}(a_0)$, $M = \varphi_1^{-1}(\langle a_0, c \rangle)$ and $N = \varphi_1^{-1}(\langle c, a_0 \rangle)$. Applying 6.5 several times to the map $\varphi_1|N: N \to \langle c, a_0 \rangle$, we obtain subdivisions $N' \subseteq N$ with $N' \subseteq N$. Such that $\text{mesh } N' < \varepsilon$ and $\varphi_1$ is a simplicial map with respect to $N'$ and $\varphi_1$. The subdivision $N'$ induce a subdivision $K_{\varepsilon}' \subseteq K_{\varepsilon}$ by 6.3. The map $\varphi_1|N: N \to \langle c, a_0 \rangle$, we obtain a subdivision $M'$ of $M$ such that $K_{\varepsilon}' \cup K_{\varepsilon}$ is a subcomplex of $M'$ and $\varphi_1$ is a simplicial map with respect to $M'$. Such that $\text{mesh } M' < \varepsilon$, and $\varphi_1$. Put $K = M \cup N'$ and $L = \langle c, a_0 \rangle \cup L'$. Then $K$ is a subdivision of $K$, $L$ is a subdivision of $a_0$ and these subdivisions satisfy the conclusion of 6.8.

Now we need the following version of the Freudenthal theorem [3] (see also [9], p. 310).

6.9. If $X$ is a compactum, then there exist an inverse sequence $X = \{X_n, f_{n+1}\}$ and two sequences of finite complexes $K_1, K_2, \ldots, K_n, K_{n+1}$ satisfying the conditions:

(i) $X = \lim\text{inj } X_n = \{a\}$ is a single-point space,
(ii) if $\dim X = n$, then $\dim X_{n+1} = n$,
(iii) $X_0 = \{a\}$,
(iv) $K_n$ is a subdivision of $K_{n+1}$,
(v) $f_{n+1}: X_{n+1} \to X_n$ is simplicial with respect to $K_{n+1}$ and $K_n$.

The following theorem is the main result of the next section.

6.10. If $X$ is a compactum, then there exist an absolute retract $M$ containing $X$, a point $a \in M$, a mapping $\mu: M \to L$, an infinite countable complex $L$ with null-triangulation, and a triangulation $L$ of $\emptyset$, such that all conditions of 4.1 and the following ones are fulfilled:

1) $M \setminus X = \{a\}$
2) $\mu(M \setminus X)$ is simplicial with respect to $P$ and $L$
3) $\text{dim } X = n$, then $\dim M = n + 1$.

Proof. Since the details of the proof are technically complicated but easily verifiable, we limit ourselves to a sketch of the argument. Let $X, (K_n), (K_{n+1})$ be as in 6.9, let $M = SX$ and let $\mu$ be defined as in the proof of 4.1. Then $\mu^{-1}(\{a\})$ may be identified with the mapping cylinder $Z_{f_{n+1}}$, and $\mu^{-1}(\{a\})$ may be identified with $X$. Using these identifications, we have $X_n, X_{n+1} \subseteq Z_{f_{n+1}}$, and hence

4) $Z_{f_{n+1}} \cap Z_{f_{n+1+1}} = X_{n+1}$.

By 6.9 and 6.1 we may also assume that there exists a finite complex $R$ such that $Z_{f_{n+1}} = \{R\}$.

By 6.9 and 6.1 we may assume that there exists a finite complex $R$ such that $Z_{f_{n+1}} = \{R\}$. Then $K_n \subseteq R$.

and $\mu(Z_{f_{n+1}})$ is a simplicial map with respect to $R$ and $Q_n$, where $Q_n$ is a complex obtained by dividing the segment $[1/n, 1/(n+1)]$ at its midpoint $c_n$. According to 6.9, 6.3 and 4) we may subdivide $R_n$ in such a way that $K_n \subseteq R$ is a subcomplex of the subdivision, and $\mu(Z_{f_{n+1}})$ is still simplicial with respect to this subdivision and $Q_n$. Without loss of generality we may assume that already $a_n$ possesses these properties. In this way the collection $R = \bigcup R_n$ constitutes an infinite complex such that $|R| = |M \setminus X|$ and $\mu(M \setminus X)$ is simplicial with respect to $R$ and $Q_n$, where $Q_n$ denotes a triangulation of $\emptyset$ obtained by dividing each simplex of the form $[1/n, 1/(n+1)]$ at its midpoint $c_n$. Since $a_n = 1 > a_0 > a_2 > \cdots$ be the sequence of all vertices of $Q$ and let $a_n$, $n \geq 1$, be the subcomplex of $R$ such that $|a_n| = \mu^{-1}(c_n)$, where $c_n = a_{n-1}, a_n$. Let $R_n, n \geq 0$, be the subcomplex of $R$ such that $|a_n| = \mu^{-1}(c_n)$. Let $c_n$ be a decreasing sequence of positive numbers converging to zero. Let $a_n, a_{n+1}$ be the simplicial map induced by $\mu$. Since $R_0 = \{a\}$, by 6.8 there exist subdivisions $P_1$ of $A_1$ and $L_1$ of $\sigma_1$ such that mesh $P_1 < \varepsilon_1$, $\varepsilon_1$ is simplicial with respect to $P_1$ and $L_1$. Then mesh $P_1 \mid B_1 < \varepsilon_2$, where $P_1 \mid B_1$ is a subcomplex of $B_1$ induced by $P_1$. Applying 6.3 to $A_2$, we obtain a subcomplex $P_2$ of $A_2$ such that $P_1 \mid B_2 \subseteq P_2$ and $\varepsilon_2$ is simplicial with respect to $P_2$. Then $\varepsilon_2$ is simplicial with respect to $P_1$ and $L_1$. According to 6.8 there exist subdivisions $P_2$ of $P_1$ and $L_1$ of $\sigma_1$ such that $P_1 \mid B_2 \subseteq P_1$. Then mesh $P_2 < \varepsilon_3$, $\varepsilon_3$ is simplicial with respect to $P_1$ and $L_1$. Then mesh $P_2 \mid B_2 < \varepsilon_3$. Continuing this process, we obtain all the other $P_n$ and $L_n$. It is easy to see that the complexes $P = \bigcup P_n$ and $L = \bigcup L_n$ constitute the appropriate triangulations of $M \setminus X$ and $\emptyset$, respectively. This completes the proof.

I would like to thank Professor A. Kirkor for his help in the preparation of this paper.

References

Two model theoretic ideas in independence proofs

by

David Pincus (Cambridge, Mass.)

Abstract. Some new Fraenkel-Mostowski models are built on universal homogeneous structures. Also a connection is established between indiscernability theorems and models for the compactness theorem.

I. Introduction

This paper will illustrate the model theoretic ideas underlying some set theoretical independence proofs. The results include conceptual simplifications of known independence proofs, new independence proofs, and a new theorem in model theory.


In § II A we indicate what, besides the universality and homogeneity of the structure, is involved in proving the support intersection lemma of Mostowski [17]. These results are applied in the remainder of § II, § II B contains a conceptual proof of the combinatorial group-theoretic lemma of Läuchli [15]. The resulting Fraenkel-Mostowski model is then used to settle a question of Halpern [9]. In § II C we eliminate forcing from Gaunt’s solution ([7]) to Mostowski’s problem on the axiom of choice for finite sets. A by-product is that these results, and related ones of Truss [27], transfer automatically to ZF set theory (*). § II D is a brief mention of other applications. These are from the author’s thesis and are more fully expounded by Jech in [13].

(*) Our set theories incorporate classes when desirable. ZF is the usual Zermelo Fraenkel set theory, ZFA is the usual weakening (see [17]) of ZF to permit a set of atoms. E is Godel’s axiom of strong choice. ZFE is ZF+ E. ZFE is a conservative extension of ZF+ AC. We assume that our standard universe, Std, satisfies ZFE.