obtained from the constructible universe by adjoining any number of mutually Cohen-generic reals.

1.7. THEOREM. *If* (K), *then every filter over generated by less than* 2^ω *sets can be extended to a selective ultrafilter.*

This result follows easily from the following proposition:

1.8. PROPOSITION. *If* (K) *holds and* F *is a filter over* ω *generated by less than* 2^ω *sets, and* $\{X_i|\ i<\omega\}$ *is a partitioning of* ω *so that for every* $i<\omega$

$$\bigcup\{X_j|\ j>i\}\in F,$$

then there exists a set $X\subseteq\omega$ *so that* $\{X\}\cup F$ *has the finite intersection property and for every* $i<\omega$: $|X\cap X_i|\leqslant 1$.

Proof. Suppose that no such X exists. Let $\{C_\zeta|\ \zeta<\lambda<2^\omega\}\subseteq F$ so that

$$X\in F\leftrightarrow\exists\zeta<\lambda:\ X\supseteq C_\zeta.$$

Let

$$T=\{f\in{}^\omega\omega|\ \forall i<\omega:\ f(i)\in X_i\}.$$

We can w.l.o.g. assume that T is a perfect closed subset of ${}^\omega\omega$ in the usual product topology. Define for $\alpha<\lambda$

$$T_\alpha=\{f\in T|\ \operatorname{range}(f)\cap C_\alpha=0\}.$$

Then

$$T=\bigcup\{T_\alpha|\ \alpha<\lambda\}.$$

But then, by (K), there exists a $\alpha<\lambda$ so that the closure of T_α contains an open set relative to T; i.e. there exists a $n<\omega$ and a function $f\colon n\to\omega$ so that $f(i)\in X_i$ for $i<n$ and if $n<m$ and $h\colon m\to\omega$ s.t. $h(i)\in X_i$ for $i<m$ and $h\supseteq f$, there exists a $g\in T_\alpha$ with $g\supseteq h$. But this implies that

$$\bigcup\{\operatorname{range}(f)|\ f\in T_\alpha\}\supseteq\bigcup\{X_i|\ i>n\}$$

and therefore

$$C_\alpha\cap(\bigcup\{X_i|\ i>n\})=0;$$

a contradiction. ■

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Accepté par la Rédaction le 19. 8. 1974

On a method of constructing ANR-sets. An application of inverse limits

by

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Abstract. In the present paper we provide a method of constructing ANR-sets from a given ANR-sequence. We establish certain properties of the ANR-sets. Some applications are given. One of them is a simple proof of a theorem of H. Bothe which says that for every natural number n there exists an $(n+1)$-dimensional AR-set containing topologically every separable metric space of dimension $\leqslant n$. We prove that for every n-dimensional compactum X there exists an $(n+1)$-dimensional infinite polyhedron P disjoint from X such that $X\cup P$ is an absolute retract. This result generalizes a theorem of Professor K. Borsuk.

1. A characterization of ANR-sets. By a compactum we mean a compact metric space, and a mapping is understood to mean a continuous function from a topological space to another one. A mapping f from a metric space X into a space Y is called an *ε-mapping* provided that $\operatorname{diam} f^{-1}(y)\leqslant\varepsilon$ for every $y\in f(X)$. If f maps the space X into itself and $\varrho(x,f(x))\leqslant\varepsilon$ for every $x\in X$, where ϱ is a metric in X, then we say that it is an *ε-push* of X. Clearly, an ε-push is an 2ε-mapping. If Y is a subset of X, then we say that X is *ε-deformable* into Y provided there exists a mapping $\varphi\colon X\times I\to X$ such that $\varphi(x,0)=x$, $\varphi(x,1)\in Y$ and $\operatorname{diam}\varphi(\{x\}\times I)\leqslant\varepsilon$ for every $x\in X$. If moreover $\varphi(y,t)=y$ for every $(y,t)\in Y\times I$, then we say that Y is a *strong ε-deformation retract* of X. Note that in this case each mapping $\varphi_t\colon X\to X$ given by the formula $\varphi_t(x)=\varphi(x,t)$ is an ε-push of X.

The aim of this section is to prove the following theorem:

1.1. *Let X be a compactum. Then it is an* ANR-*set if and only if for every* $\varepsilon>0$ *there exists an* ANR-*set* $Y\subset X$ *such that* X *is ε-deformable into* Y.

The necessity of the condition is obvious. To prove its sufficiency we need a characterization of ANR-sets due to S. Lefschetz. Recall that a positive number η is said to satisfy the condition of Lefschetz for a space Y and for $\varepsilon>0$ provided that for every polyhedron W, every triangulation T of W, and every subpolyhedron W' of this triangulation containing all vertices of T, every mapping $f'\colon W'\to Y$, such that $\operatorname{diam} f'(\sigma\cap W')\leqslant\eta$ for each simplex $\sigma\in T$, has a continuous extension $f\colon W\to Y$ such that $\operatorname{diam} f(\sigma)\leqslant\varepsilon$ for each simplex $\sigma\in T$.

1.2. *A compactum* $Y \in \mathrm{ANR}$ *if and only if for every* $\varepsilon>0$ *there exists a number* $\eta>0$ *satisfying the condition of Lefschetz for* Y *and* ε (see [1], p. 112).

Proof of the sufficiency of 1.1. According to 1.2 it suffices to show that for a given number $\varepsilon>0$ there exists a number $\eta>0$ such that

(1) η satisfies the condition of Lefschetz for X and ε.

Let $\varepsilon' = \frac{1}{4}\varepsilon$. By the assumptions there exist an ANR-set $Y\subset X$ and a homotopy $\varphi\colon X\times I\to X$ satisfying the conditions: $\varphi(x,0)=x$, $\varphi(x,1)\in Y$ and

(2) $$\operatorname{diam}\varphi(\{x\}\times I)\leqslant\varepsilon' \quad \text{for every } x\in X.$$

Hence 1.2 implies the existence of a number $\eta'>0$ satisfying the condition of Lefschetz for Y and ε'. Let $r\colon X\to Y$ be defined by the formula $r(x)=\varphi(x,1)$. Since r is uniformly continuous, there is a number $\eta>0$ such that

(3) $$\eta\leqslant\varepsilon',$$

(4) $$A\subset X\wedge\operatorname{diam}A\leqslant\eta \Rightarrow \operatorname{diam}r(A)\leqslant\eta'.$$

We shall show that the number η satisfies (1). In order to prove this consider a mapping $g'\colon W'\to X$ of a subpolyhedron W' of W (in the triangulation T) satisfying the condition

(5) $$\operatorname{diam}g'(\sigma\cap W')\leqslant\eta \quad \text{for each simplex } \sigma\in T.$$

Setting $f'=r\circ g'\colon W'\to Y$ one gets a mapping such that $\operatorname{diam}f'(\sigma\cap W')\leqslant\eta'$ for each simplex $\sigma\in T$, by (4) and (5). Since η' satisfies the condition of Lefschetz for Y and ε', there is a continuous extension $f\colon W\to Y$ of f' such that

(6) $$\operatorname{diam}f(\sigma)\leqslant\varepsilon' \quad \text{for each simplex } \sigma\in T.$$

Consider the closed subset $M=W'\times I\cup W\times\{1\}$ of the Cartesian product $W\times I$. It is easy to see that there is a retraction $k\colon W\times I\to M$ satisfying the condition

(7) $$k(\sigma\times\{0\})\subset(\sigma\cap W')\times I\cup\sigma\times\{1\} \quad \text{for each simplex } \sigma\in T.$$

(compare the proof of Corollary 4, p. 117, in [8]). Let $h\colon M\to X$ be a mapping defined as follows:

$$h(y,t)=\begin{cases}\varphi(g'(y),t) & \text{for } (y,t)\in W'\times I,\\ f(y) & \text{for } (y,t)\in W\times\{1\}.\end{cases}$$

This definition is correct because for $(y,1)\in W'\times I$ we have $\varphi(g'(y),1)=r\circ g'(y)=f'(y)=f(y)$. Note also that for $y\in W'$ we have $h\circ k(y,0)=h(y,0)=\varphi(g'(y),0)=g'(y)$. Therefore setting

$$g(y)=h\circ k(y,0) \quad \text{for} \quad y\in W,$$

one obtains a well-defined continuous extension of g' onto the polyhedron W. Hence it remains to show that $\operatorname{diam}g(\sigma)\leqslant\varepsilon$ for each simplex $\sigma\in T$.

By (7) we have

$$g(\sigma)=h\circ k(\sigma\times\{0\})\subset h((\sigma\cap W')\times I)\cup h(\sigma\times\{1\})=\varphi(g'(\sigma\cap W')\times I)\cup f(\sigma).$$

Furthermore, by (2), (3) and (5), we obtain $\operatorname{diam}\varphi(g'(\sigma\cap W')\times I)\leqslant 3\varepsilon'$. Since the polyhedron W' contains all vertices of T, the set $\sigma\cap W'$ is not empty. Let y be a point of this set. Then $\varphi(g'(y),1)=f(y)$, and therefore the summands in the last union intersect. Combining the above considerations with (6), we conclude that $\operatorname{diam}g(\sigma)\leqslant 4\varepsilon'=\varepsilon$, which completes the proof.

1.3. Corollary. *Let X be a compactum. Then it is an* AR*-set if and only if for every number $\varepsilon>0$ there exists an* AR*-set $Y\subset X$ such that X is ε-deformable into Y.*

This corollary follows from 1.1 and the fact that an ANR-set contractible in itself is an AR-set (see [1], p. 96).

2. Quotient maps, decomposition of spaces and function spaces. A function f from a space X into a space Y is said to be a *quotient map* if f is onto and the following condition is satisfied: a set $A\subset Y$ is open in Y iff the set $f^{-1}(A)$ is open in X. Hence each quotient map is a mapping. Each closed (open) mapping onto is a quotient map. The following results are almost evident; they are included here for future reference.

2.1. *A mapping from a compact space onto a Hausdorff space is a quotient map.*

2.2. *Let $p_i\colon X_i\to X_i'$, $i=1,2$, be quotient maps. Suppose $f\colon X_1\to X_2$ is a mapping agreeing with p_1,p_2, i.e., for every $x\in X_1'$ there exists a $y\in X_2'$ such that $f(p_1^{-1}(x))\subset p_2^{-1}(y)$. Then there exists a unique mapping $f'\colon X_1'\to X_2'$ such that $f'\circ p_1=p_2\circ f$.*

As usual, we denote by 2^X the collection of all closed nonvoid subsets of a space X. A class $\boldsymbol{D}\subset 2^X$ such that $\bigcup\boldsymbol{D}=X$ is called a *decomposition* of X if no two elements of $\boldsymbol{D}$ intersect. The decomposition is *upper-semicontinuous* if for every open subset U of X the union of elements of $\boldsymbol{D}$ which are contained in U is an open subset of X. To every decomposition $\boldsymbol{D}$ corresponds a space $\hat{\boldsymbol{D}}$, called the *space* of $\boldsymbol{D}$, defined as follows: the points of $\hat{\boldsymbol{D}}$ are elements of $\boldsymbol{D}$, a set $A\subset\hat{\boldsymbol{D}}$ is open in $\hat{\boldsymbol{D}}$ iff the union $\bigcup A$ is an open subset of X. Denote by $D(x)$ the unique element of $\boldsymbol{D}$ which contains $x\in X$. The function $k\colon X\to\hat{\boldsymbol{D}}$ given by the formula $k(x)=D(x)$ is called the natural projection. The projection is a quotient map; if $\boldsymbol{D}$ is upper-semicontinuous, it is a closed mapping. The following result is well known:

2.3. *The space of an upper-semicontinuous decomposition of a compactum is a compactum* (see [5], p. 65).

If f maps a closed subset A of a compactum X into a compactum Y, then the matching of X and Y by f is the space of the (upper-semicontinuous) decomposition, of the disjoint union of X and Y, into individual points of the set $(X\setminus A)\cup(Y\setminus f(A))$ and the sets $\{y\}\cup f^{-1}(y)$ for $y\in f(A)$. This space is denoted by $X\overset{f}{\cup}Y$, and by 2.3 it is a compactum.

2.4. *If $X,A,Y\in\mathrm{ANR}$, then $X\overset{f}{\cup}Y\in\mathrm{ANR}$* (see [1], p. 116).

Let f be a mapping of a compactum X into a compactum Y. If $A = X\times\{1\} \subset X\times I$ and $f'\colon A\to Y$ is defined by the formula: $f'(x,1) = f(x)$, then the matching of $X\times I$ and Y by f' is denoted by Z_f and is called the *mapping cylinder* of f (see [8], p. 32). By 2.4 we have

2.5. *If $X, Y\in$ ANR, then $Z_f\in$ ANR.*

Let Y^X be the set of all mappings from a space X into a space Y. By the space Y^X is meant the set Y^X with the compact-open topology, i.e., the totality of sets $\Gamma(C, H) = \{f\in Y^X\colon f(C)\subset H\}$, where $C\subset X$ is compact and $H\subset Y$ is open, is an open subbase of Y^X (see [5], p. 76).

2.6. *If X is a compactum, and $f\colon I\to Y^X$ is a mapping, then $g\colon X\times I\to Y$ defined by the formula $g(x,t) = f(t)(x)$ is also a mapping* (see [5], p. 86).

If Y is a compactum, then we may also consider in the set Y^X another topology called topology of uniform convergence defined by the metric:

$$|f-g| = \sup\{\varrho(f(x), g(x))\colon x\in X\},\quad f, g\in Y^X,$$

where ϱ is a metric in Y (see [5], p. 88). We have

2.7. *If X and Y are compacta, then the compact-open topology of Y^X coincides with its uniform convergence topology* (see [5], p. 89).

3. Constructions and properties of spaces σX, ΣX and SX. Throughout this section $\mathbf{X}$ denotes an inverse sequence of compacta, $\mathbf{X} = \{X_n, f_{nm}\}$, with bonding maps $f_{nm}\colon X_m\to X_n$, $m\geqslant n$, satisfying the conditions: $f_{nn} = 1_{X_n}$, $f_{nm}\circ f_{mk} = f_{nk}$; X_∞ denotes the inverse limit of $\mathbf{X}$, $X_\infty = \operatorname{invlim}\mathbf{X}$, and $f_n\colon X_\infty\to X_n$ denotes the natural projection: $f_n(x) = x_n$ for $x = (x_1, x_2, \ldots)\in X_\infty$. We assume that all the sets X_n, X_∞ are pairwise disjoint (this can always be achieved by taking the disjoint union of these sets).

Definition 1. The space $\sigma\mathbf{X}$ is the set $X_\infty\cup\bigcup_{n\geqslant 1} X_n$ with the topology defined by assuming that the totality of the following sets: open subsets of the spaces X_n, and sets of the form $f_n^{-1}(U)\cup\bigcup_{m\geqslant n} f_{nm}^{-1}(U)$, where U is an open subset of X_n, $n\geqslant 1$, is an open base of $\sigma\mathbf{X}$.

This space will be called the *Freudenthal space* of $\mathbf{X}$. The construction is due to H. Freudenthal ([3], p. 153, comp. also [7]). Recall the following result

3.1. *The space $\sigma\mathbf{X}$ is a compactum* (see [3], p. 153–156).

3.2. *The spaces X_n, X_∞ are subspaces of $\sigma\mathbf{X}$; the space X_n is closed-open in $\sigma\mathbf{X}$.*

3.3. *The function $p_n\colon \sigma\mathbf{X}\to\sigma\mathbf{X}$, $n\geqslant 1$, defined as follows:*

$$p_n(x) = \begin{cases} x & \text{for}\quad x\in X_m \text{ and } m\leqslant n,\\ f_{nm}(x) & \text{for}\quad x\in X_m \text{ and } m\geqslant n,\\ f_n(x) & \text{for}\quad x\in X_\infty,\end{cases}$$

is well defined and satisfies the following conditions:

1. *p_n is continuous, i.e. is a mapping,*
2. *p_n is an ε_n-push, with $\lim_n \varepsilon_n = 0$,*
3. *$p_n\circ p_m = p_n$ for $n\leqslant m$,*
4. *$p_n(x) = x$ for every $x\in\bigcup_{j=1}^{n} X_j$,*
5. *$p_n(\bigcup_{m\geqslant n} X_m\cup X_\infty) = X_n$.*

Proof. It follows from our assumption on the sets X_n, X_∞ and from Definition 1 that p_n is well defined and continuous. Conditions 3, 4 and 5 are obvious, so it remains to prove 2.

We have to show that for every number $\varepsilon>0$ there exists an integer n_0 such that p_n is an ε-push for every $n\geqslant n_0$. Suppose, to the contrary, that this is not true. Then there exist an increasing sequence of integers $\{n_j\}$ and a sequence $\{y_j\}$ of points of the space $\sigma\mathbf{X}$ such that for every j we have

$$\varrho(y_j, p_{n_j}(y_j))\geqslant\varepsilon, \tag{1}$$

where ϱ is a metric in $\sigma\mathbf{X}$. By 3.1 we may assume that $\{y_j\}$ converge to a point y. We claim that $y\in X_\infty$. Indeed, otherwise $y\in X_n$ for some n. Since X_n is an open subset of $\sigma\mathbf{X}$, there exists an integer j such that $n_j\geqslant n$ and $y_j\in X_n$. Consequently, by 4, we obtain $p_{n_j}(y_j) = y_j$, contrary to (1). Hence $y\in X_\infty$. Let V be a neighbourhood of y in $\sigma\mathbf{X}$ such that

$$\operatorname{diam} V<\varepsilon. \tag{2}$$

By Definition 1 there exist an index n and an open subset U of X_n such that $U' = f_n^{-1}(U)\cup\bigcup_{m\geqslant n} f_{nm}^{-1}(U)$ is a neighbourhood of y in $\sigma\mathbf{X}$ contained in V. Hence for some index j we have $n_j\geqslant n$ and $y_j\in U'$. It immediately follows from the definition of p_{n_j} that in such a case $p_{n_j}(y_j)\in U'$. Since U' is a subset of V, by (2) and (1) we obtain an absurdity. This completes the proof 3.3.

Definition 2. The space $\Sigma\mathbf{X}$ is the following subspace of the Cartesian product $\sigma\mathbf{X}\times I$:

$$\Sigma\mathbf{X} = X_1\times\{1\}\cup\bigcup_{n>1} X_n\times I_n\cup X_\infty\times\{0\},\quad\text{where}\quad I_n = [1/n, 1/(n-1)].$$

Since $\Sigma\mathbf{X}$ is a closed subset of the compactum $\sigma\mathbf{X}\times I$ (see 3.1), we have

3.4. *The space $\Sigma\mathbf{X}$ is a compactum, and the mapping $h_1\colon \sigma\mathbf{X}\to\Sigma\mathbf{X}$, defined by the formula*

$$h_1(x) = \begin{cases}(x, 1/n) & \text{for}\quad x\in X_n,\ n\geqslant 1,\\ (x, 0) & \text{for}\quad x\in X_\infty,\end{cases}$$

is an embedding.

DEFINITION 3. The collection $\boldsymbol{D}$ of subsets of ΣX defined as follows: the single point sets $\{p\}$ for $p \in \Sigma X \setminus (X_1 \times \{1\} \cup \bigcup_{n>1} X_n \times \{1/n, 1/(n-1)\})$, and the sets $\{(x, 1/n)\} \cup \cup f_{n,n+1}^{-1}(x) \times \{1/n\}$ for $x \in X_n$, $n \geqslant 1$, is called the *canonical decomposition* of ΣX. The canonical decomposition is a decomposition in the usual sense (see lemma below); the space of the decomposition is called the *space of the inverse sequence* $\boldsymbol{X}$ and is denoted by SX. The natural projection $k\colon \Sigma X \to SX$ is called the *canonical projection*.

3.5. *The canonical decomposition is an upper-semicontinuous one.*

Proof. It is evident that the elements of $\boldsymbol{D}$ are closed, nonvoid and disjoint. So we have to prove that for every open subset U of ΣX the set $V = \bigcup \{D \in \boldsymbol{D}\colon D \subset U\}$ is an open subset of ΣX. Pick a point $y \in V$. To prove that V is open we need only to find a neighbourhood G of y such that G is the union of some elements of $\boldsymbol{D}$ each of which is a subset of U. Suppose $y \in D \subset U$ and consider four cases:

I. $y \in X_n \times (1/n, 1/(n-1))$ for some $n \geqslant 2$. Then $G = U \cap X_n \times (1/n, 1/(n-1))$ is the required neighbourhood of y.

II. $y \in X_n \times \{1/n\}$ for some $n \geqslant 1$. Let $y = (x, 1/n)$. Hence $D = (\{x\} \cup f_{n,n+1}^{-1}(x)) \times \times \{1/n\}$. Since U is open and $D \subset U$, there exists an open subset W of X_n such that $x \in W$ and $(\overline{W} \cup f_{n,n+1}^{-1}(\overline{W})) \times \{1/n\} \subset U$. Define a set L_n as follows: if $n = 1$, then $L_n = \{1\}$; if $n > 1$, then $L_n = [1/n, t_n)$, $1/n < t_n < 1/(n-1)$, is such that $W \times L_n \subset U$. There exists an interval $L_{n+1} = (t_{n+1}, 1/n]$, $1/(n+1) < t_{n+1} < 1/n$, such that $f_{n,n+1}^{-1}(W) \times L_{n+1} \subset U$. It follows from the construction that the set $G = W \times L_n \cup \cup f_{n,n+1}^{-1}(W) \times L_{n+1}$ is the required neighbourhood of y.

III. $y = (x, 1/n) \in X_{n+1} \times \{1/n\}$ for some $n \geqslant 1$. In this case the point $y' = (f_{n,n+1}(x), 1/n)$ also belongs to D. Proceeding as in case II, we obtain the neighbourhood G of y'. This is also an appropriate neighbourhood of y.

IV. $y = (x, 0) \in X_\infty \times \{0\}$. Since $y \in U$ and U is an open subset of $\Sigma X \subset \sigma X \times I$, there exist an open neighbourhood N of x in σX and an interval $M = [0, t_0]$ such that $N \times M \cap \Sigma X \subset U$. Let $n_0 > 1$ be an index such that $1/(n-1) < t_0$ for $n \geqslant n_0$. There exists an index $n \geqslant n_0$ and an open subset H of X_n such that $H' = \bigcup_{m \geqslant n} f_{nm}^{-1}(H) \cup \cup f_n^{-1}(H)$ is a neighbourhood of x in σX contained in N (see Definition 1). Let us note that

$$\begin{aligned} G &= H' \times [0, 1/(n-1)) \cap \Sigma X \\ &= H \times [1/n, 1/(n-1)) \cup \bigcup_{m>n} f_{nm}^{-1}(H) \times I_m \cup f_n^{-1}(H) \times \{0\} \subset N \times M \cap \Sigma X \subset U . \end{aligned}$$

By its definition the set G is a neighbourhood of y. Since G is the union of some elements of $\boldsymbol{D}$, it is the required neighbourhood of y. This completes the proof of 3.5.

The following statement is obvious:

3.6. *The mapping $\mu_1\colon \Sigma X \to I$ given by the formula*

$$\mu_1(x, t) = t$$

agrees (in the sense given in § 2) with the canonical projection k and the identity mapping 1_I.

3.7. *Let $q\colon I \setminus \{0\} \to N$, $N =$ the set of natural numbers, be defined as follows:*

$$q(t) = \begin{cases} 1 & \text{for} \quad t = 1 , \\ n+1 & \text{for} \quad 1/(n+1) \leqslant t < 1/n, \quad n \geqslant 1 . \end{cases}$$

Let φ_0 denote the identity mapping of ΣX and, for $t > 0$, let $\varphi_t\colon \Sigma X \to \Sigma X$ be given by the following formula:

$$\varphi_t(y) = \begin{cases} y & \text{for} \quad y \in \overline{\mu_1^{-1}((t, 1])} , \\ (p_{q(t)}(x), t) & \text{for} \quad y = (x, s) \in \mu_1^{-1}([0, t]) . \end{cases}$$

Then the functions φ_t, $t \in I$, are continuous and agree with k, k.

Proof. We may assume that $t > 0$. First we prove that φ_t is well defined. Suppose $y \in \mathrm{Cl}\,\mu_1^{-1}((t, 1]) \cap \mu_1^{-1}([0, t])$. Then $y = (x, t)$. If $t = 1/n$, then $x \in X_n$ and $q(t) = n$; hence $(p_{q(t)}(x), t) = y$, by 3.3,4. If $1/n < t < 1/(n-1)$, $n \geqslant 2$, then also $x \in X_n$ and $q(t) = n$; hence $(p_{q(t)}(x), t) = y$ for the same reason as above. The continuity of φ_t follows from 3.3,1. It remains to prove that $\varphi_t(D(y)) \subset D(\varphi_t(y))$ for every $y \in \Sigma X$. We may assume that $D(y)$ is nondegenerate, that is:

$$D(y) = (\{x\} \cup f_{n,n+1}^{-1}(x)) \times \{1/n\}$$

for some $x \in X_n$ and $n \geqslant 1$. We may also assume that $1/n \leqslant t$, for otherwise $\varphi_t(y') = y'$ for every $y' \in D(y)$. But in such a case we have $D(y) \subset \mu_1^{-1}([0, t])$ and $q(t) \leqslant n < n+1$; hence, by definition of φ_t and 3.3, we obtain $\varphi_t(y') = (p_{q(t)}(x), t)$ for every $y' \in D(y)$. Therefore $\varphi_t(D(y)) \subset D(p_{q(t)}(x), t) = D(\varphi_t(y))$, which completes the proof.

We will consider $\sigma X \times I$ as a metric space with a metric ϱ_1 given by the formula

$$\varrho_1((x, s), (x', s')) = \varrho(x, x') + |s - s'| ,$$

where ϱ is a metric in σX.

3.8. *For every $t \in I$ define U_t in the following way:*

$$U_t = \begin{cases} (1/2, 1] & \text{for} \quad t = 1 , \\ (1/(n+2), 1/n) & \text{for} \quad t = 1/(n+1) , \\ (1/(n+1), 1/n) & \text{for} \quad 1/(n+1) < t < 1/n , \\ I & \text{for} \quad t = 0 . \end{cases}$$

Then for every $y \in \Sigma X$ we have

$$(*) \qquad \varrho_1\big(D(\varphi_t(y)), D(\varphi_{t'}(y))\big) \leqslant \begin{cases} \|1_{\sigma X} - p_{q(t')}\| + |t'| & \text{for} \quad t = 0 \text{ and } t' > 0 , \\ |t - t'| & \text{for} \quad t > 0 \text{ and } t' \in U_t , \end{cases}$$

where $\varrho_1(A, B) = \inf\{\varrho_1(a, b)\colon a \in A \text{ and } b \in B\}$.

Proof. We may assume $t \neq t'$. Let $y = (x, s)$. Suppose first $t = 0$ and $t' > 0$. Then $\varphi_t(y) = y$ because φ_0 is the identity mapping of ΣX. The point $\varphi_{t'}(y)$ is either y

or $(p_{q(t')}(x), t')$. In the latter case $s \leqslant t'$. Therefore $\varrho_1(\varphi_t(y), \varphi_{t'}(y)) \leqslant \varrho(x, p_{q(t')}(x)) + |t'|$, which proves (*) in the case of $t = 0$.

Suppose now that $t>0$ and $t' \in U_t$. We have to consider several cases.

I. $t'<t = 1/n$ for some $n \geqslant 1$. Then $q(t) = n$ and $q(t') = n+1$. If $s>t$, then $\varphi_t(y) = \varphi_{t'}(y) = y$. If $s = t$, then either $x \in X_n$ or $x \in X_{n+1}$. In the former case $\varphi_t(y) = y = \varphi_{t'}(y)$, in the latter one $\varphi_{t'}(y) = y$ and $\varphi_t(y) = (f_{n,n+1}(x), t)$, and therefore $D(\varphi_t(y)) = D(y)$. If $t' \leqslant s < t$, then $x \in X_{n+1}$. Hence $D(\varphi_t(y)) = D(f_{n,n+1}(x), t) = D(x, t)$ and $\varphi_{t'}(y) = y$. If $s<t'$, then $D(\varphi_t(y)) = D(p_n(x), t) = D(p_{n+1}(x), t)$ and $\varphi_{t'}(y) = (p_{n+1}(x), t')$. Hence in each case we obtain (*).

II. $t'>t = 1/n$. Then $q(t) = n = q(t')$. If $s \geqslant t'$, then $\varphi_t(y) = \varphi_{t'}(y)$. If $t<s<t'$, then $\varphi_t(y) = y$ and $\varphi_{t'}(y) = (x, t')$. If $s = t$ and $x \in X_n$, then $\varphi_t(y) = y$ and $\varphi_{t'}(y) = (x, t')$. If $s = t$ and $x \in X_{n+1}$, then $\varphi_t(y) = (p_n(x), t)$ and $\varphi_{t'}(y) = (p_n(x), t')$. If $s<t$, then $\varphi_t(y) = (p_n(x), t)$ and $\varphi_{t'}(y) = (p_n(x), t')$. Hence in each case we obtain (*).

III. $1/(n+1)<t<1/n$ for some n. Then $q(t) = q(t') = n+1$, and by arguments similar to that used above we obtain (*) in this case. This completes the proof.

The results which we have just proved will now be used to establish several properties of the space of the inverse sequence $\boldsymbol{X}$.

Notation. $k(x, s) = [x, s]$.

By 3.4, 3.5 and 2.3 we obtain

3.9. *The space SX is a compactum and the function h: $\sigma X \to SX$ defined by the formula*

$$h(x) = \begin{cases} [x, 1/n] & \text{for} \quad x \in X_n \text{ and } n \geqslant 1, \\ [x, 0] & \text{for} \quad x \in X_\infty \end{cases}$$

is an embedding.

By 3.9, 2.1 and 2.2 we obtain

3.10. *The function μ: $SX \to I$ given by the formula*

$$\mu([x, t]) = t$$

is a mapping. The set $\mu^{-1}(t)$ is homeomorphic to $X_{q(t)}$. Moreover, $\mu^{-1}(0) = h(X_\infty)$ and $\mu^{-1}(1/n) = h(X_n)$ for every $n \geqslant 1$. Finally, the set $\mu^{-1}([1/n, t])$ is homeomorphic to the mapping cylinder $Z_{f_{n,n+1}}$ for $1/n > t \geqslant 1/(n+1)$.

3.11. *For every $t \in I$ the function ψ_t: $SX \to SX$ given by the formula:*

$$\psi_t(z) = \begin{cases} z & \text{for} \quad z \in \mu^{-1}([t, 1]), \\ [p_{q(t)}(x), t] & \text{for} \quad z = (x, s) \in \mu^{-1}([0, t]), \end{cases}$$

where ψ_0 is defined by the first equality, is a mapping. The function ψ: $SX \times I \to SX$, defined as $\psi(z, t) = \psi_t(z)$, is a mapping satisfying the following conditions:

1. *$\psi(z, 0) = z$ for every $z \in SX$,*
2. *$\psi(z, t) = z$ for $z \in \mu^{-1}([t', 1])$ and $t \leqslant t'$,*
3. *$\psi(z, t) \in \mu^{-1}(t)$ for $z \in \mu^{-1}(t')$ and $t \geqslant t'$,*
4. *$\operatorname{diam} \psi(\{z\} \times [0, t]) \leqslant \varepsilon(t)$, with $\lim\limits_{t \to 0} \varepsilon(t) = 0$,*

Proof. The assertion about ψ_t easily follows from the corresponding properties of φ_t. We shall show that ψ is continuous. Hence by 2.6 it suffices to show that the function f: $I \to SX^{SX}$, where $f(t) = \psi_t$, is continuous (the function space with the compact-open topology). Let d be a metric in SX. By 2.7 we may assume that the function space has the uniform convergence topology defined by d. Let $t \in I$ and let $\varepsilon>0$ be a real number. We have to prove that $|\psi_t - \psi_{t'}| \leqslant \varepsilon$ for every t' in a neighbourhood of t. Since SX is a compactum, the canonical projection k is uniformly continuous, hence there is a number $\eta>0$ such that condition $\varrho_1(y, y') \leqslant \eta$ implies $d([y], [y']) \leqslant \varepsilon$. By 3.3,2 and 3.8, there is a neighbourhood $U \subset U_t$ of t such that for $t' \in U$ we have $|t-t'| \leqslant \eta$ and if $t = 0$, then $|1_{\sigma X} - p_{q(t')}| + |t'| \leqslant \eta$. Let $z = [y]$ be an arbitrary point of SX. Then $k^{-1}(\psi_t(z)) = D(\varphi_t(y))$ and $k^{-1}(\psi_{t'}(z)) = D(\varphi_{t'}(y))$. Since $\psi_s \circ k = k \circ \varphi_s$ for every $s \in I$, by 3.8 we obtain $d(\psi_t(z), \psi_{t'}(z)) \leqslant \varepsilon$. It follows that $|\psi_t - \psi_{t'}| \leqslant \varepsilon$. This proves the continuity of f. Thus ψ is continuous.

The properties 1,2 and 3 of ψ follow from 3.7. The property 4 follows again from 3.8 (*) by an argument similar to that used above.

The inverse sequence $\boldsymbol{X}$ is called an ANR-sequence provided every space X_n is an ANR-set.

3.12. *If $\boldsymbol{X}$ is an ANR-sequence, then*

1. *$\mu^{-1}([t, t']) \in$ ANR for $t, t' \in I$ and $t+t'>0$,*
2. *$X_1 \in$ AR $\Rightarrow \mu^{-1}([t, 1]) \in$ AR for every $t \in I$.*

Proof. 1. By 3.10 the set $\mu^{-1}([1/(n+1), 1/n])$ is homeomorphic to $Z_{f_{nn+1}}$; hence it is an ANR-set by 2.5. By 3.10 we have also

$$\mu^{-1}([1/(n+1), 1/n]) \cap \mu^{-1}([1/(n+2), 1/(n+1)]) = h(X_{n+1}).$$

The set on the right-hand side of the equality is an ANR-set by 3.9. Since the union of two ANR-sets whose common part is an ANR-set is an ANR-set (see [1], p. 90), we have $\mu^{-1}([1/(n+2), 1/n]) \in$ ANR. By an easy induction and by 3.10 we infer that $\mu^{-1}([t, 1/n]) \in$ ANR for every $0<t \leqslant 1/n$. It is easy to see that if $1/(n+1) \leqslant s \leqslant s' < 1/n$, then the set $\mu^{-1}([s, s'])$ is homeomorphic to $X_{n+1} \times [s, s']$, and hence it is an ANR-set. The above two results, 3.10 and the quoted result on the union of two ANR-sets imply that $\mu^{-1}([t, t']) \in$ ANR for every $0<t \leqslant t'$. It remains to prove that $\mu^{-1}([0, t]) \in$ ANR for $t>0$. But 3.11 implies that for every $\varepsilon>0$ there exists an ANR-set $A \subset \mu^{-1}([0, t])$ (namely the set $\mu^{-1}([t', t])$ for some $0<t' \leqslant t$) such that it is a strong ε-deformation retract of $\mu^{-1}([0, t])$. Hence $\mu^{-1}([0, t]) \in$ ANR, by 1.1. This completes the proof of 1.

2. Let $X_1 \in$ AR. By the previous result $SX = \mu^{-1}([0, 1]) \in$ ANR. By 3.11 the set $\mu^{-1}(1)$ is a strong deformation retract of SX, and by 3.10 $\mu^{-1}(1) = h(X_1)$. Hence $\mu^{-1}(1) \in$ AR, by 3.9. It follows that SX is an ANR-set contractible in itself, and therefore an AR-set (see [1], p. 96). By 3.11 the set $\mu^{-1}([t, 1])$ is a (strong deformation) retract of SX. Hence it is an AR-set. This completes the proof.

4. Embedding of compacta into absolute retracts. The aim of this section is to prove the following theorem:

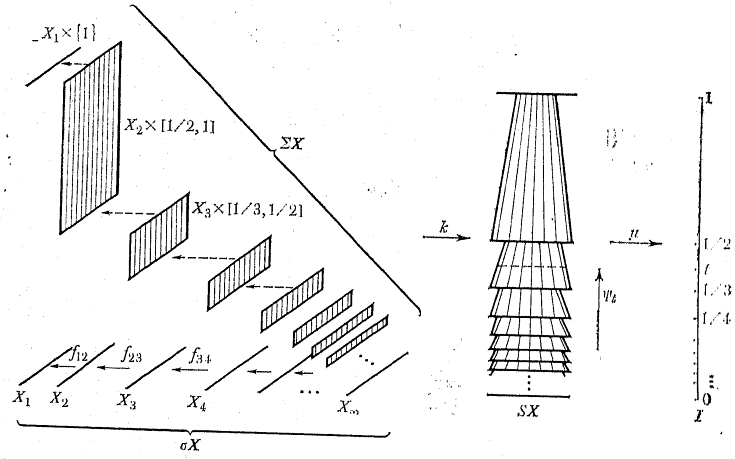
4.1. *For every nondegenerate compactum X there exist an absolute retract M containing X, a point $v \in M$ and a mapping μ: $M \to I$ satisfying the following conditions*:

1. *μ is an open mapping onto I,*
2. $\mu^{-1}(0) = X$,
3. $\mu^{-1}(1) = \{v\}$,
4. $\mu^{-1}([t, t']) \in \mathrm{ANR}$ *if* $t+t'>0$,
5. $\mu^{-1}([t, 1]) \in \mathrm{AR}$ *for every* $t \in I$,
6. *$\mu^{-1}(t)$ is a strong deformation retract of $\mu^{-1}([t', t])$ for every $t \in I$ and $t' \leqslant t$.*

Moreover, if $\dim X = n$, *then in addition*

7. *$\mu^{-1}(t)$ is an n-dimensional polyhedron for every $0<t<1$,*
8. *$\mu^{-1}([t, t'])$ is an $(n+1)$-dimensional compactum if $t<t'$.*

Proof. By a classical result of Freudenthal [3] there exists an inverse sequence of polyhedra $\mathbf{X} = \{X_n, f_{nm}\}$ such that $X = \mathrm{invlim}\,\mathbf{X}$, and the bonding maps $f_{nm}\colon X_m \to X_n$, $n \leqslant m$, are mappings onto (comp. [6]). If $\dim X = n$, then we may assume that $\dim X_k = n$ for every $k>1$. Without loss of generality we may also assume that X_1 is a single-point space. Put $M = SX$. According to the results of Section 3 it is easy to check that the space M satisfies all the required conditions, because we can identify the compactum X with $h(X_\infty)$ (comp. the figure).



The constructions

An interesting application of 4.1 is a simple proof of the following result of H. Bothe:

4.2. COROLLARY [2]. *For every natural number n there exists an $(n+1)$-dimensional absolute retract containing topologically every k-dimensional metric separable space, with $k \leqslant n$.*

Proof. Let X_0 be an n-dimensional universal compactum (see [4], p. 64). Hence X_0 contains topologically every metric separable space of dimension $\leqslant n$. Applying 4.1 we obtain an $(n+1)$-dimensional AR-set containing X_0. Hence this absolute retract satisfies the conclusion of 4.2.

5. Remark on homotopy groups. The main result of this section is not of interest in this paper, but it will find an important application in a forthcoming paper of the author on the theory of continua. We begin with some lemmas. The following one is evident:

5.1. *Let A be a subset of a space X. If the inclusion map i: $A \to X$ induces an epimorphism*

$$(i)_{\#}\colon \pi_n(A, a) \to \pi_n(X, a)$$

of the n-th homotopy groups for some point $a \in A$, then it induces the epimorphism for every other point $x \in A$ provided x belongs to the path-component of A which contains a.

Let f be a mapping from a compactum X into a compactum Y. Denote by $[x, t]$ the point of the mapping cylinder Z_f which corresponds to the point $(x, t) \in X \times I$ by the natural projection of $X \times I \cup Y$ into Z_f and, likewise, by $[y]$ we denote the point of Z_f which corresponds by this projection to the point $y \in Y$. The mappings $i\colon X \to Z_f$ and $j\colon Y \to Z_f$ given by the formula $i(x) = [x, 0]$, $j(y) = [y]$ are embeddings.

It is an easy exercise to prove the following lemma:

5.2. *Let f: $(X, x_0) \to (Y, y_0)$ and suppose that the induced homomorphism $f_{\#}$: $\pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an epimorphism. Then the induced homomorphism $i_{\#}$: $\pi_n(X, x_0) \to \pi_n(Z_f, [x_0, 0])$ is also an epimorphism.*

The main result of this section can be stated as follows:

5.3. *Let a pointed compactum (X, x_0) be the limit of an inverse pointed* ANR*-sequence* $(\mathbf{X}, x_0) = \{(X_n, x_n), f_{nm}\}$, *i.e.*, $(X, x_0) = \mathrm{invlim}(\mathbf{X}, x_0)$, *where* $x_0 = (x_1, x_2, \ldots) \in X$. *Suppose that the bonding maps f_{nm}: $(X_m, x_m) \to (X_n, x_n)$ induce epimorphisms $(f_{nm})_{\#}$: $\pi_k(X_m, x_m) \to \pi_k(X_n, x_n)$ of the k-th homotopy groups, where $m \geqslant n$. Then there exist an absolute retract M containing X and a decreasing sequence $\{A_n\}$ of* ANR*-sets in M such that $X = \bigcap A_n$ and the inclusion map i_n: $(A_{n+1}, x_0) \to (A_n, x_0)$ induces an epimorphism of the corresponding k-th homotopy groups, for every $n \geqslant 1$.*

Proof. Without loss of generality we may assume that X_1 is a single-point space. Let us adopt the notation of Section 3 and let $M = SX$ and $A_n = \mu^{-1}([0, 1/n])$ for every $n \geqslant 1$. The Freudenthal space of X consists of the spaces X_n and X. Since h is an embedding of the Freudenthal space into M, by 3.12 we have

(1) $$h(X)\subset M\in \mathrm{AR},$$

(2) $$A_{n+1}\subset A_n\in \mathrm{ANR} \quad \text{and} \quad \bigcap A_n = h(X).$$

Now we shall prove that

(3) the induced homomorphism $(i_n)_{\#}\colon \pi_k(A_{n+1}, x_0)\to\pi_k(A_n, x_0)$ is an epimorphism.

By 3.10 we have $h(X_{n+1}) = \mu^{-1}(1/(n+1))\subset A_{n+1}$; in particular $h(x_{n+1})\in A_{n+1}$. By 3.3, 3.7, 3.9 and 3.11 the following diagram commutes:

$$\begin{tikzcd} M & M \arrow[l, "\psi_{1/(n+1)}"'] \\ X_{n+1} \arrow[u, "h|X_{n+1}"] & X \arrow[l, "f_{n+1}"] \arrow[u, "h|X"'] \end{tikzcd}$$

Since $f_{n+1}(x_0) = x_{n+1}$, it follows from 3.11 that $\psi_{1/(n+1)}(\{h(x_0)\}\times[0, 1/(n+1)])$ is an arc in A_{n+1} joining $h(x_0)$ with $h(x_{n+1})$. Hence these points belong to the same path-component of A_{n+1}. Hence, by 5.1, in order to prove (3) we need only to show that

(4) $(i_n)_{\#}\colon \pi_k(A_{n+1}, h(x_{n+1}))\to\pi_k(A_n, h(x_{n+1}))$ is an epimorphism.

Let $u\colon I\to I_{n+1} = [1/(n+1), 1/n]$ be a homeomorphism such that $u(0) = 1/(n+1)$ and $u(1) = 1/n$. It is easy to check that the function

$$\alpha\colon Z_{f_{n,n+1}}\to\mu^{-1}([1/(n+1), 1/n]) = B$$

given by the formula

$$\alpha(y) = \begin{cases} [x, u(t)] & \text{for} \quad y = [x, t]\in Z_{f_{n,n+1}} \quad \text{and } x\in X_{n+1}, \\ [x, 1/n] & \text{for} \quad y = [x]\in Z_{f_{n,n+1}} \quad \text{and } x\in X_n \end{cases}$$

is a homeomorphism. Let us note that $\alpha([x_{n+1}, 0]) = h(x_{n+1})$. Hence we have

(5) $\alpha_{\#}\colon \pi_k(Z_{f_{n,n+1}}, [x_{n+1}, 0])\to\pi_k(B, h(x_{n+1}))$ is an epimorphism.

Consider the following diagram:

$$\begin{tikzcd} (A_n, h(x_{n+1})) & & (A_{n+1}, h(x_{n+1})) \arrow[ll, "i_n"'] \\ & (B, h(x_{n+1})) \arrow[ul, "j"] & \\ (Z_{f_{n,n+1}}, [x_{n+1}, 0]) \arrow[uu, "\alpha'"] \arrow[ur, "\alpha"] & & (X_{n+1}, x_{n+1}), \arrow[ll, "i"] \arrow[uu, "\beta"'] \end{tikzcd}$$

where i_n and j are inclusion maps, $\alpha' = j\circ\alpha$, $\beta(x) = h(x)$ and $i(x) = [x, 0]$. It is evident that the diagram commutes. It easily follows from 3.11 that $\beta(X_{n+1}) = \mu^{-1}(1/(n+1))$ is a strong deformation retract of A_{n+1}, and B is a strong deformation retract of A_n. Hence the homomorphisms $(\alpha')_{\#}$ and $(\beta)_{\#}$ are isomorphisms (see [5]). By our assumption and by 5.2 the homomorphism $i_{\#}$ is an epimorphism. Combining these facts we see that $(i_n)_{\#}$ is an epimorphism, which proves (4) and therefore (3). Identifying X with $h(X)$ and x_0 with $h(x_0)$ we obtain by (1), (2) and (3) the conclusion of 5.3. This completes the proof.

6. A generalization of Borsuk's theorem. Professor K. Borsuk proved the following theorem (see [1], p. 108):

For every compactum X there exists an infinite polyhedron P with a null-triangulation such that $X\cup P$ is an absolute retract.

In this section we shall prove a strengthened version of this result. We first prove a list of lemmas which will be used in the proof of the theorem. We start with the following important fact established by J. H. C. Whitehead ([9], p. 259 and [10], p. 244):

6.1. *Let K and L be finite complexes and let $f\colon |K|\to|L|$ be a mapping simplicial with respect to these complexes. Embed polyhedra $|K|$ and $|L|$ in the mapping cylinder Z_f by the maps: $x\to[x, 0]$ and $y\to[y]$ for $x\in|K|$ and $y\in|L|$. Let $\psi\colon Z_f\to I$ be the natural projection, i.e., $\psi([x, t]) = t$ and $\psi([y]) = 1$. Then there exist a finite complex P, subcomplexes K' and L' of P, a homeomorphism $\tau\colon |P|\to Z_f$ onto Z_f, and a mapping $\varphi\colon |P|\to I$ simplicial with respect to P and a triangulation I' of I, satisfying the conditions: $\tau(|K'|) = |K|$, $\tau(|L'|) = |L|$, τ is a simplicial isomorphism between $K'(L')$ and K (L, respectively), and $\varphi = \psi\circ\tau$. (I' is obtained by dividing I at its middle point.)*

6.2. *Let φ be a simplicial map from an n-simplex σ^n onto a 1-simplex $\sigma^1 = \langle a_0, a_1\rangle$. Let $\sigma_i = \varphi^{-1}(a_i)$, $i = 0, 1$. Hence simplex σ^n is the join of simplexes σ_0 and σ_1, i.e., $\sigma^n = \sigma_0 * \sigma_1$. Suppose that K_0 is a subdivision of σ_0. Then there exists a subdivision K of σ^n such that $K_0\cup(\sigma_1)\subset K$ and φ is simplicial with respect to K and σ^1.*

Proof. To prove the lemma it suffices to take as K the subdivision of σ^n composed of all simplexes of the form $\tau * \sigma_1$, where $\tau\in K_0$, and all faces of these simplexes.

An immediate consequence of 6.2 is the following lemma:

6.3. *Let φ be a simplicial map of a complex K onto a 1-simplex $\sigma^1 = \langle a_0, a_1\rangle$. Let $K_i = \varphi^{-1}(a_i)$, $i = 0, 1$, and suppose that K_0' is a subdivision of K_0. Then there exists a subdivision K' of K such that $K_0'\cup K_1\subset K'$ and φ is simplicial with respect to K' and σ^1.*

Let $\varphi\colon \sigma^n\to\sigma^1$ be a simplicial map into $\sigma^1 = \langle a_0, a_1\rangle$ and let $c = t_0 a_0 + t_1 a_1$, $t_0 + t_1 = 1$, be an interior point of σ^1, i.e., $t_i > 0$. By a (φ, c)-barycentre of σ^n we understand a point $b_{\sigma^n}^c\in\sigma^n$ defined as follows: if $\varphi(\sigma^n) = a_i$, then $b_{\sigma^n}^c = b_{\sigma^n}$ is the usual barycentre of σ^n; if $\varphi(\sigma^n) = \sigma^1$, then we define $b_{\sigma^n}^c = t_0 b_{\sigma_0} + t_1 b_{\sigma_1}$, where

$\sigma_i = \varphi^{-1}(a_i)$ and b_{σ_i} is the barycentre of σ_i. Observe that $\varphi(b^c_{\sigma^n}) = c$. For each sequence $\tau_0, \tau_1, \ldots, \tau_k$ of faces of σ^n such that τ_i is a proper face of τ_{i+1} the sequence of corresponding (φ, c)-barycentres of τ's span a simplex contained in τ_k. All simplexes obtained in this manner form a subdivision of σ^n denoted by $\sigma^n_{\varphi,c}$ and called a *(φ, c)-subdivision* of σ^n.

6.4. *If c is the barycentre of $\sigma^1 = \langle a_0, a_1\rangle$, then we have*

$$\operatorname{mesh}\sigma^n_{\varphi,c} \leqslant \frac{2n+1}{2n+2}\operatorname{diam}\sigma^n .$$

Proof. Let $\sigma \in \sigma^n_{\varphi,c}$. Hence there exists a sequence $\{\tau_i\}$, $0 \leqslant i \leqslant k$, $0 \leqslant k \leqslant n$, of faces of σ^n such that τ_i is a proper face of τ_{i+1} and σ is the simplex spanned by the barycentres $b^c_{\tau_0}, b^c_{\tau_1}, \ldots, b^c_{\tau_k}$, i.e., $\sigma = \langle b^c_{\tau_0}, \ldots, b^c_{\tau_k}\rangle$. We have to show that

$$\operatorname{diam}\sigma \leqslant \frac{2n+1}{2n+2}\operatorname{diam}\sigma^n .$$

We have $\operatorname{diam}\sigma = |b^c_{\tau_i} - b^c_{\tau_j}|$ for some $i<j$. If $\tau_j \subset \varphi^{-1}(a_i)$, then also $\tau_i \subset \varphi^{-1}(a_i)$ and $b^c_{\tau_l}$, $l = i, j$, is the barycentre of τ_l. So we have

$$|b^c_{\tau_i} - b^c_{\tau_j}| = |b_{\tau_i} - b_{\tau_j}| \leqslant \frac{\dim\tau_j}{\dim\tau_j + 1}\operatorname{diam}\tau_j \leqslant \frac{n}{n+1}\operatorname{diam}\sigma^n < \frac{2n+1}{2n+2}\operatorname{diam}\sigma^n .$$

Thus we obtain the conclusion of the lemma in this case. Suppose now that $\tau_j \not\subset \varphi^{-1}(\{a_0, a_1\})$. It follows that $\varphi(\tau_j) = \sigma^1$, and $\tau_j = \tau' * \tau''$, where $\tau' = \varphi^{-1}(a_0) \cap \cap \tau_j$, $\tau'' = \varphi^{-1}(a_1) \cap \tau_j$, and $b^c_{\tau_j} = \frac{1}{2}(b_{\tau'} + b_{\tau''})$ (because $c = \frac{1}{2}(a_0 + a_1)$). Let $\tau_j = \langle p_0, p_1, \ldots, p_r\rangle$. Since $\tau_i < \tau_j$ and $b^c_{\tau_i}$ belongs to τ_j, we have

$$b^c_{\tau_i} = \alpha_0 p_{i_0} + \ldots + \alpha_s p_{i_s}, \qquad \sum \alpha_l = 1,\ \alpha_l \geqslant 0 .$$

It follows that $|b^c_{\tau_j} - b^c_{\tau_i}| \leqslant \max\limits_{0 \leqslant l \leqslant s} |b^c_{\tau_j} - p_{i_l}|$. It remains to show that for every vertex p_m, $0 \leqslant m \leqslant r$, we have

$$|b^c_{\tau_j} - p_m| \leqslant \frac{2n+1}{2n+2}\operatorname{diam}\sigma^n. \tag{1}$$

Without loss of generality we may assume that $\tau' = \langle p_0, \ldots, p_u\rangle$ and $\tau'' = \langle p_{u+1}, \ldots, p_r\rangle$. So we have

$$b^c_{\tau_j} = \frac{1}{2}\left(\sum_{v=0}^{u} \frac{1}{u+1} p_v + \sum_{v=u+1}^{r} \frac{1}{r-u} p_v\right).$$

We may also assume that $m \leqslant u$. Then we obtain

$$\begin{aligned}
|b^c_{\tau_j} - p_m| &= \frac{1}{2}\left|\sum_{v=0}^{u} \frac{1}{u+1}(p_v - p_m) + \sum_{v=u+1}^{r} \frac{1}{r-u}(p_v - p_m)\right| \\
&\leqslant \frac{1}{2}\left(\sum_{v=0}^{u} \frac{1}{u+1}|p_v - p_m| + \sum_{v=u+1}^{r} \frac{1}{r-u}|p_v - p_m|\right. \\
&\leqslant \frac{1}{2}\left(\frac{u}{u+1}\operatorname{diam}\sigma^n + \operatorname{diam}\sigma^n\right. = \frac{2u+1}{2u+2}\operatorname{diam}\sigma^n \leqslant \frac{2n+1}{2n+2}\operatorname{diam}\sigma^n
\end{aligned}$$

because $u < r \leqslant n$. This proves condition (1) and completes the proof of 6.4.

If φ is a simplicial map of a complex K into a 1-simplex σ^1, then we denote by $K_{\varphi,c}$ the complex $\bigcup\limits_{\sigma^n \in K} \sigma^n_{\varphi,c}$. This complex is a subdivision of K and is called *(φ, c)-subdivision* of K. It is evident that φ is simplicial with respect to $K_{\varphi,c}$ and $\sigma^1_{1_{\sigma^1},c}$ (the last symbol will be abbreviated to σ^1_c). From Lemma 6.4 we obtain the following corollary:

6.5. *If c is the barycentre of σ^1 and $0 < \dim K \leqslant n$, then*

$$\operatorname{mesh} K_{\varphi,c} \leqslant \frac{2n+1}{2n+2}\operatorname{mesh} K .$$

Let $\varphi\colon \sigma^n \to \sigma^1 = \langle a_0, a_1\rangle$ be a simplicial map onto. Let $\sigma_i = \varphi^{-1}(a_i)$, $i = 0, 1$, and let c be an interior point of σ^1. For each face σ of σ^n which is mapped onto σ^1 take the point b^c_σ and define a subdivision $c\sigma^n$ in the following way: for the vertices of $c\sigma^n$ we take all vertices of σ^n and the points b^c_σ; a simplex belongs to $c\sigma^n$ iff it is a face of a simplex spanned by the vertices $v_1, v_2, \ldots, v_k, b^c_{\tau_1}, b^c_{\tau_2}, \ldots, b^c_{\tau_l}$, where $\sigma = \langle v_1, \ldots, v_k\rangle$ is a simplex in either σ_0 or σ_1, $\sigma < \tau_1 < \ldots < \tau_l$ and $\varphi(\tau_1) = \sigma^1$. Observe that $\sigma_{\varphi,c}$ and $c\sigma^n$ induce on $\varphi^{-1}(c)$ the same subdivisions.

6.6. *Let $\varphi_1\colon c\sigma^n \to \sigma^1_c$ be the simplicial map defined by φ and let ε be a given positive number. There exists an $\eta > 0$ such that if $|a_0 - c| < \eta$, then $\operatorname{diam}\sigma \leqslant \max\{\varepsilon, \operatorname{diam}\sigma_0\}$ for every $\sigma \in c\sigma^n$ such that $|\sigma| \subset \varphi_1^{-1}(\langle a_0, c\rangle)$.*

Proof. Let $c = t_0 a_0 + t_1 a_1$, where $t_0 + t_1 = 1$ and $t_i > 0$. Let K be the subcomplex of $c\sigma^n$ such that $|K| = \varphi_1^{-1}(\langle a_0, c\rangle)$. Let $f\colon K^{(0)} \to \sigma^n$ be a vertex map defined as follows. If v is a vertex of σ_0, then let $f(v) = v$. If $v = b^c_\sigma$, then $v = t_0 b_{\sigma'} + t_1 b_{\sigma''}$, where $\sigma' = \sigma \cap \sigma_0$ and $\sigma'' = \sigma \cap \sigma_1$; in this case we set $f(v) = b_{\sigma'}$ and observe that f may be extended linearly onto each simplex from K. Denote the extension by f as well and note that $|f(v) - v| \leqslant t_1 |b_{\sigma''} - b_{\sigma'}| \leqslant t_1 \operatorname{diam}\sigma^n$. This observation easily leads to the existence of η.

As a corollary we obtain the following proposition

6.7. *If $\varphi\colon K \to \sigma^1 = \langle a_0, a_1\rangle$ is a simplicial map and $\varepsilon > 0$ is a given number, then there exists a subdivision K' of K and a point c in the interior of σ^1 such that φ_1 is* *simplicial with respect to* K' *and* σ_c^1, $K_0 = \varphi^{-1}(a_0)$ *is a subcomplex of* K' *and*

$$\operatorname{diam}\tau \leqslant \max\{\varepsilon, \operatorname{mesh} K_0\} \quad \textit{for each } \tau \in \varphi_1^{-1}(\langle a_0, c\rangle),$$

where φ_1 *is the map induced by* φ.

6.8. *If* $\varphi\colon K \to \sigma^1 = \langle a_0, a_1\rangle$ *is a simplicial map,* $\varepsilon > 0$ *is a given number and* $K_0 = \varphi^{-1}(a_0)$, *then there exist subdivisions* $\tilde{K}$ *of* K *and* L *of* σ^1 *such that* $K_0 \subset \tilde{K}$, φ *induces a simplicial map* $\tilde{\varphi}\colon \tilde{K} \to L$, $\operatorname{mesh}\tilde{K} \leqslant \max\{\varepsilon, \operatorname{mesh} K_0\}$ *and* $\operatorname{mesh}\tilde{\varphi}^{-1}(a_1) < \varepsilon$.

Proof. First choose $c \in \operatorname{int}\sigma^1$, φ_1 and K' as in 6.7. Let $K_0 = \varphi^{-1}(a_0)$, $K_c = \varphi_1^{-1}(c)$, $M = \varphi_1^{-1}(\langle a_0, c\rangle)$ and $N = \varphi_1^{-1}(\langle c, a_1\rangle)$. Applying 6.5 several times to the map $\varphi_1|N\colon N \to \langle c, a_1\rangle$, we obtain subdivisions N', L' of N, $\langle c, a_1\rangle$, respectively, such that $\operatorname{mesh} N' < \varepsilon$ and φ_1 is simplicial with respect to N' and L'. The subdivision N' induce a subdivision K_c' of K_c. Applying 6.3 to the map $\varphi_1|M\colon M \to \langle c, a_0\rangle$, we obtain a subdivision M' of M such that $K_c' \cup K_0$ is a subcomplex of M' and φ_1 is simplicial with respect to M' and $\langle c, a_0\rangle$. Put $\tilde{K} = M' \cup N'$ and $L = \{\langle c, a_0\rangle\} \cup L'$. Then $\tilde{K}$ is a subdivision of K, L is a subdivision of σ^1 and these subdivisions satisfy the conclusion of 6.8.

Now we need the following version of the Freudenthal theorem [3] (see also [0], p. 310).

6.9. *If* X *is a compactum, then there exist an inverse sequence* $\mathbf{X} = \{X_n, f_{nm}\}$ *and two sequences of finite complexes* $K_1, K_2, \ldots, K_1', K_2', \ldots$ *satisfying the conditions*:

(i) $X = \operatorname{invlim}\mathbf{X}$, $X_1 = \{v\}$ *is a single-point space,*

(ii) *if* $\dim X = k$, *then* $\dim X_{n+1} = k$,

(iii) $X_n = |K_n|$,

(iv) K_n' *is a subdivision of* K_n,

(v) $f_{n,n+1}\colon X_{n+1} \to X_n$ *is simplicial with respect to* K_{n+1} *and* K_n'.

The following theorem is the main result of this section:

6.10. *If* X *is a compactum, then there exist an absolute retract* M *containing* X, *a point* $v \in M$, *a mapping* $\mu\colon M \to I$, *an infinite countable complex* P *with null-triangulation, and a triangulation* L *of* $I \setminus \{0\}$ *such that all conditions of* 4.1 *and the following ones are fulfilled*:

$$M \setminus X = |P|, \tag{1}$$

2) $\mu|(M\setminus X)$ *is simplicial with respect to* P *and* L

$$\textit{if} \quad \dim X = n, \quad \textit{then} \quad \dim M = n+1. \tag{3}$$

Proof. Since the details of the proof are technically complicated but easily verifable, we limit ourselves to a sketch of the argument. Let X, $\{K_n\}$, $\{K_n'\}$ be as in 6.9, let $M = SX$ and let μ be defined as in the proof of 4.1. Then $\mu^{-1}([1/n, 1/(n+1)])$ may be identified with the mapping cylinder $Z_{f_{n,n+1}}$ and $\mu^{-1}(1/n)$ may be identified with X_n. Using these identifications, we have $X_n, X_{n+1} \subset Z_{f_{n,n+1}}$ and

$$Z_{f_{n,n+1}} \cap Z_{f_{n+1,n+2}} = X_{n+1}. \tag{4}$$

By 6.9 and 6.1 we may also assume that there exists a finite complex R_n such that $Z_{f_{n,n+1}} = |R_n|$,

$$K_{n+1}, K_n' \subset R_n,$$

and $\mu|Z_{f_{n,n+1}}$ is a simplicial map with respect to R_n and Q_n, where Q_n is a complex obtained by dividing the segment $[1/n, 1/(n+1)]$ at its middle point c_n. According to 6.9, 6.3 and (4) we may subdivide R_n in such a way that $K_n' \cup K_{n+1}'$ is a subcomplex of the subdivision, and $\mu|Z_{f_{n,n+1}}$ is still simplicial with respect to this subdivision and Q_n. Without loss of generality we may assume that already, R_n possesses these properties. In this way the collection $R = \bigcup_n R_n$ constitutes an infinite complex such that $|R| = M \setminus X$ and $\mu|(M\setminus X)$ is simplicial with respect to R and Q, where Q denotes a triangulation of $I\setminus\{0\}$ obtained by dividing each simplex of the form $[1/n, 1/(n+1)]$ at its middle point c_n. Let $a_0 = 1 > a_1 > a_2 > \ldots$ be the sequence of all vertices of Q and let A_n, $n \geqslant 1$, be the subcomplex of R such that $|A_n| = \mu^{-1}(\sigma_n^1)$, where $\sigma_n^1 = \langle a_{n-1}, a_n\rangle$. Let B_n, $n \geqslant 0$, be the subcomplex of R such that $|B_n| = \mu^{-1}(a_n)$. Let $\{\varepsilon_n\}$ be a decreasing sequence of positive numbers converging to zero. Let $\varphi_n\colon A_n \to \sigma_n^1$ be the simplicial map induced by μ. Since $B_0 = \{v\}$, by 6.8 there exist subdivisions P_1 of A_1 and L_1 of σ_1^1 such that $\operatorname{mesh} P_1 < \varepsilon_1$, φ_1 is simplicial with respect to P_1 and L_1 and $\operatorname{mesh} P_1|B_1 < \varepsilon_2$, where $P_1|B_1$ denotes the subdivision of B_1 induced by P_1. Applying 6.3 to A_2, we obtain a subdivision P_2' of A_2 such that $P_1|B_1, B_2 \subset P_2'$ and φ_2 is simplicial with respect to P_2' and σ_2^1. According to 6.8 there exist subdivisions P_2 of P_2' and L_2 of σ_2^1 such that $P_1|B_1 \subset P_2$, $\operatorname{mesh} P_2 < \varepsilon_2$, φ_2 is simplicial with respect to P_2 and L_2 and $\operatorname{mesh} P_2|B_2 < \varepsilon_3$. Continuing this process, we obtain all the other P_n and L_n. It is easy to see that the complexes $P = \bigcup_n P_n$ and $L = \bigcup_n L_n$ constitute the appropriate triangulations of $M \setminus X$ and $I\setminus\{0\}$, respectively. This completes the proof.

I would like to thank Professor A. Kirkor for his help in the preparation of this paper.

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Accepté par la Rédaction le 8. 7. 1974

Two model theoretic ideas in independence proofs

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Abstract. Some new Fraenkel-Mostowski models are built on universal homogeneous structures. Also a connection is established between indiscernability theorems and models for the compactness theorem.

I. Introduction

This paper will illustrate the model theoretic ideas underlying some set theoretical independence proofs. The results include conceptual simplifications of known independence proofs, new independence proofs, and a new theorem in model theory.

In § II we discuss Fraenkel-Mostowski models built on universal homogeneous structures. The idea dates back to Mostowski's proof, [17], of the independence of the axiom of choice, (AC), from the ordering principle. Mathias [16] reawakened interest in the idea with his proof of the independence of the order extension principle from the ordering principle. Others followed, notably Plotkin ([23] and [24]), and Felgner ([3] and [4]) as well as the author. Except for [17] the work cited above is set up in the language of forcing. Arguments here and in [13] demonstrate that only Fraenkel-Mostowski ideas are involved.

In § IIA we indicate what, besides the universality and homogeneity of the structure, is involved in proving the support intersection lemma of Mostowski [17]. These results are applied in the remainder of § II. § IIB contains a conceptual proof of the combinatorial group-theoretic lemma of Läuchli [15]. The resulting Fraenkel-Mostowski model is then used to settle a question of Halpern [9]. In § IIC we eliminate forcing from Gauntt's solution ([7]) to Mostowski's problem on the axiom of choice for finite sets. A by-product is that these results, and related ones of Truss [27], transfer automatically to ZF set theory ([1]). § IID is a brief mention of other applications. These are from the author's thesis and are more fully exposited by Jech in [13].

([1]) Our set theories incorporate classes when desirable. ZF is the usual Zermelo Fraenkel set theory. ZFA is the usual weakening (see [17]) of ZF to permit a set of atoms. E is Godel's axiom of strong choice. ZFE is ZF+E. ZFE is a conservative extension of ZF+AC. We assume that our standard universe, Std, satisfies ZFE.