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On the existence of P-points in the Stone-Čech compactification of integers

by

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Abstract. The object of this paper is to derive set-theoretic criteria for the existence of P-points in the Stone-Čech compactification of integers.

0. Introduction. Our notation and terminology conforms to that used in the most recent set-theoretic literature. For example, cardinals are initial ordinals. Ordinals are denoted by small greek letters $\alpha, \beta, \ldots$ The cardinality of the set X is denoted by $|X|$. We shall now state our fundamental definitions:

0.1. DEFINITION. An ultrafilter D over ω is a *P-point* if and only if for every partitioning $\{X_i|\ i<\omega\}$ of ω into ω pieces there exists a $X\in D$ so that for every $i<\omega$ $X\cap X_i$ is finite.

0.2. DEFINITION. An ultrafilter D over ω is *selective* if and only if for every partitioning $\{X_i|\ i<\omega\}$ of ω into ω pieces there exists a $X\in D$ so that for every $i<\omega$ $|X\cap X_i|\leqslant 1$.

The notion of a P-point was first defined by W. Rudin in [1], where he proved that the continuum hypothesis implies the existence of P-points. It is not known whether the existence of P-points can be proved directly from the axioms of set-theory. K. Kunen has shown, however, that it is consistent to assume the non-existence of selective ultrafilters. The purpose of this paper is to establish more general criteria for the existence of P-points and selective ultrafilters.

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1. Some existence criteria. We shall show that the structure of P-points is intimately interconnected with the structure of functions $\omega\to\omega$.

1.1. DEFINITION. (a) If $x, y\subseteq\omega$, then we say x is *contained in y modulo finite sets*, in symbols, $x\subseteq y$(mf) iff there exists a $k<\omega$ so that $x-k\subseteq y$;

(b) If f, g are functions $\omega\to\omega$, then g dominates f; in symbols: $f\leqslant g$, iff there exists a $k<\omega$ so that for all $i\geqslant k$ $f(i)\leqslant g(i)$.

Now, let (H) be the following statement:

(H): No family of functions from ω to ω of power less than 2^ω dominates all functions $\omega\to\omega$; i.e. if $\lambda<2^\omega$ and $\{f_\alpha|\ \alpha<\lambda\}$ is a family of functions $\omega\to\omega$, then there is a f: $\omega\to\omega$ so that for every $\alpha<\lambda$: $f\not\leqslant f_\alpha$.

It is easy to see that the continuum hypothesis implies (H) and that it holds in any model obtained from the constructible universe by adjoining any number of mutually Cohen-generic reals. The following theorem is then our fundamental result:

1.2. THEOREM. *If* (H) *holds, then there are P-points. As a matter of fact,* (H) *holds if and only if every filter over* ω *generated by less than* 2^ω *sets can be extended to a P-point.*

One direction of this statement easily follows by transfinite induction from the following proposition.

1.3. PROPOSITION. *Given a filter F over* ω *generated by less than* 2^ω *elements, and a sequence* $\{A_i|\ i<\omega\}$ *of elements of F, there exists a set* $A\subseteq\omega$ *so that* $F\cup\{A\}$ *has the finite intersection property and for every* $i<\omega$, $A\subseteq A_i$(mf).

Proof. W. l.o.g. assume that $A_1\supseteq A_2\supseteq\ldots$, and let $\lambda<2^\omega$, $\{C_\zeta|\ \zeta<\lambda\}\subseteq F$ so that

$$X\in F\leftrightarrow\exists\zeta<\lambda:\ C_\zeta\subseteq F\,.$$

For any function f: $\omega\to\omega$, let

$$X(f) = \bigcup\{A_i\cap\{j|\ j\leqslant f(i)\}|\ i\leqslant\omega\}\,.$$

Then for every function f: $\omega\to\omega$ we have: For every $i<\omega$, $X(f)\subseteq A_i$(mf).

Now, for $\zeta<\lambda$, let

$$g_\zeta(i) = \text{least element of } A_i\cap C_\zeta\,.$$

Then, by (H), there exists a function f: $\omega\to\omega$ so that for every $\zeta<\lambda$, the set $\{i|\ g_\zeta(i)<f(i)\}$ is infinite and therefore $X(f)\cap C_\zeta\neq 0$. It follows that the set $A = X(f)$ then satisfies our requirements. ∎

The proof of the "only if" direction in Theorem 1.2 is also immediate: For suppose that (H) is false; i.e. there exists a $\lambda<c$ with a dominating family $\{f_\alpha|\ \alpha<\lambda\}$ of functions $\omega\to\omega$ indexed by λ. Let $\{C_i|\ i<\omega\}$ be a partitioning of ω into ω pieces of power ω. Let

$$X_\alpha = -\bigcup_{i<\omega} C_i\cap\{j|\ j\leqslant f_\alpha(i)\}\,.$$

Then $\{-C_i|\ i<\omega\}\cup\{X_\alpha|\ \alpha<\lambda\}$ generates a filter which cannot be extended to a P-point. ∎

As an immediate consequence we get:

1.4. PROPOSITION. *If* (H), *then every ultrafilter generated by less than* 2^ω *elements is a P-point.*

However, we have:

1.5. PROPOSITION. *If* (H), *then no ultrafilter generated by less than* 2^ω *elements is selective.*

Proof. For suppose that $\{A_\alpha|\ \alpha<\lambda\}$ generates a selective ultrafilter D so that

$$X\in D\leftrightarrow\exists\alpha<\lambda:\ A_\alpha\subseteq X\,.$$

Let θ_α: $\omega\to\omega$ enumerate A_α in order for $\alpha<\lambda$. Then the θ_α's dominate every function $\omega\to\omega$: For let f: $\omega\to\omega$ be a strictly increasing function. Let

$$C_i = \{j|\ f(i)\leqslant j<f(i+1)\}\,.$$

By selectivity, we can find a set $S\in D$ and a strictly increasing function θ: $\omega\to\omega$ so that $\{\theta(i)\} = S\cap C_i$ and range $(\theta) = S$. Pick $\alpha<\lambda$ so that $A_\alpha\subseteq S$. Then for every $i<\omega$:

$$\theta_\alpha(i)\geqslant\theta(i)\geqslant f(i)\,. \ ∎$$

We have a "converse" to Proposition 1.4:

1.6. PROPOSITION. *If not* (H), *then if there exists an ultrafilter generated by less than* 2^ω *sets, there exists an ultrafilter generated by less than* 2^ω *sets which is not a P-point.*

Proof. Suppose that not (H) holds and that $\lambda<2^\omega$ so that there exists an ultrafilter D and a family $\{A_\alpha|\ \alpha<\lambda\}\subseteq D$ so that

$$X\in D\leftrightarrow\exists\alpha<\lambda:\ A_\alpha\subseteq X\,.$$

W. l.o.g. assume that D is a P-point. We claim that $D\times D$ satisfies our requirements. Here $D\times D$ is the product filter over $\omega\times\omega$ defined by: For $X\subseteq\omega\times\omega$

$$X\in D\times D\leftrightarrow\{i|\ \{j|\ (i,j)\in X\}\in D\}\in D\,.$$

By not (H), there exists a family $\{f_\alpha|\ \alpha<\mu\}$ of functions $\omega\to\omega$ dominating every function so that $\mu<2^\omega$. Then it is easy to see that the sets of the form

$$(A_\alpha\times A_\alpha)\cap\{(i,j)|\ j>f_\beta(i),\ i>k\}$$

where $\alpha<\lambda$, $\beta<\mu$, $k<\omega$ generate $D\times D$. It is obvious that $D\times D$ is not a P-point. ∎

The situation described in Proposition 1.6 occurs, for example, in the model obtained by adjoining ω_2 mutually generic Sacks-reals to the constructible universe: K. Kunen has shown that every selective ultrafilter in the ground-model generates a selective ultrafilter in the extension.

We can also derive a criterion for the existence of selective ultrafilters similar to Theorem 1.2: Let (K) stand for the following statement:

(K): The continuum is not the union of less than 2^ω sets of first category.

It is easy to see that (K) implies (H) and that (K) also implies that there are no ultrafilters generated by less than 2^ω sets. (K) holds, for example, in any model obtained from the constructible universe by adjoining any number of mutually Cohen-generic reals.

1.7. THEOREM. *If* (K), *then every filter over generated by less than* 2^ω *sets can be extended to a selective ultrafilter.*

This result follows easily from the following proposition:

1.8. PROPOSITION. *If* (K) *holds and* F *is a filter over* ω *generated by less than* 2^ω *sets, and* $\{X_i|\ i<\omega\}$ *is a partitioning of* ω *so that for every* $i<\omega$

$$\bigcup\{X_j|\ j>i\}\in F\,,$$

then there exists a set $X\subseteq\omega$ *so that* $\{X\}\cup F$ *has the finite intersection property and for every* $i<\omega$: $|X\cap X_i|\leqslant 1$.

Proof. Suppose that no such X exists. Let $\{C_\zeta|\ \zeta<\lambda<2^\omega\}\subseteq F$ so that

$$X\in F\leftrightarrow\exists\zeta<\lambda:\ X\supseteq C_\zeta\,.$$

Let

$$T=\{f\in{}^\omega\omega|\ \forall i<\omega:\ f(i)\in X_i\}\,.$$

We can w.l.o.g. assume that T is a perfect closed subset of ${}^\omega\omega$ in the usual product topology. Define for $\alpha<\lambda$

$$T_\alpha=\{f\in T|\ \mathrm{range}(f)\cap C_\alpha=0\}\,.$$

Then

$$T=\bigcup\{T_\alpha|\ \alpha<\lambda\}\,.$$

But then, by (K), there exists a $\alpha<\lambda$ so that the closure of T_α contains an open set relative to T; i.e. there exists a $n<\omega$ and a function f: $n\to\omega$ so that $f(i)\in X_i$ for $i<n$ and if $n<m$ and h: $m\to\omega$ s.t. $h(i)\in X_i$ for $i<m$ and $h\supseteq f$, there exists a $g\in T_\alpha$ with $g\supseteq h$. But this implies that

$$\bigcup\{\mathrm{range}(f)|\ f\in T_\alpha\}\supseteq\ \bigcup\{X_i|\ i>n\}$$

and therefore

$$C_\alpha\cap(\bigcup\{X_i|\ i>n\})=0;$$

a contradiction. ■

References

[1] W. Rudin. *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. 23 (1956), pp. 409–420.

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On a method of constructing ANR-sets. An application of inverse limits

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Abstract. In the present paper we provide a method of constructing ANR-sets from a given ANR-sequence. We establish certain properties of the ANR-sets. Some applications are given. One of them is a simple proof of a theorem of H. Bothe which says that for every natural number n there exists an $(n+1)$-dimensional AR-set containing topologically every separable metric space of dimension $\leqslant n$. We prove that for every n-dimensional compactum X there exists an $(n+1)$-dimensional infinite polyhedron P disjoint from X such that $X\cup P$ is an absolute retract. This result generalizes a theorem of Professor K. Borsuk.

1. A characterization of ANR-sets. By a compactum we mean a compact metric space, and a mapping is understood to mean a continuous function from a topological space to another one. A mapping f from a metric space X into a space Y is called an *ε-mapping* provided that $\operatorname{diam} f^{-1}(y)\leqslant\varepsilon$ for every $y\in f(X)$. If f maps the space X into itself and $\varrho(x,f(x))\leqslant\varepsilon$ for every $x\in X$, where ϱ is a metric in X, then we say that it is an *ε-push* of X. Clearly, an ε-push is an 2ε-mapping. If Y is a subset of X, then we say that X is *ε-deformable* into Y provided there exists a mapping $\varphi\colon X\times I\to X$ such that $\varphi(x,0)=x$, $\varphi(x,1)\in Y$ and $\operatorname{diam}\varphi(\{x\}\times I)\leqslant\varepsilon$ for every $x\in X$. If moreover $\varphi(y,t)=y$ for every $(y,t)\in Y\times I$, then we say that Y is a *strong ε-deformation retract* of X. Note that in this case each mapping $\varphi_t\colon X\to X$ given by the formula $\varphi_t(x)=\varphi(x,t)$ is an ε-push of X.

The aim of this section is to prove the following theorem:

1.1. *Let* X *be a compactum. Then it is an* ANR-*set if and only if for every* $\varepsilon>0$ *there exists an* ANR-*set* $Y\subset X$ *such that* X *is ε-deformable into* Y.

The necessity of the condition is obvious. To prove its sufficiency we need a characterization of ANR-sets due to S. Lefschetz. Recall that a positive number η is said to satisfy the condition of Lefschetz for a space Y and for $\varepsilon>0$ provided that for every polyhedron W, every triangulation T of W, and every subpolyhedron W' of this triangulation containing all vertices of T, every mapping $f'\colon W'\to Y$, such that $\operatorname{diam} f'(\sigma\cap W')\leqslant\eta$ for each simplex $\sigma\in T$, has a continuous extension $f\colon W\to Y$ such that $\operatorname{diam} f(\sigma)\leqslant\varepsilon$ for each simplex $\sigma\in T$.