Some fixed point theorems

by

Barada K. Ray (W. Bengal)

Abstract. A fixed point theorem for set-valued mappings in a complete metric space and some interesting theorems on fixed points in a reflexive Banach space and arbitrary topological spaces have been presented in this paper. The theorems extend and generalize some recent theorems of Sam B. Nadler Jr., R. Kannan, the author and many others.

The famous Banach contraction principle states that if (X, ϱ) be a complete metric space and if T be a contraction mapping (i.e., $\varrho(Tx, Ty) \leqslant \alpha\varrho(x, y)$ for all $x, y \in X$, $0 \leqslant \alpha < 1$) of X into itself, then T has a unique fixed point, i.e., a point z_0 exists such that $Tz_0 = z_0$. Recently S. B. Nadler [6] has proved a similar theorem on multivalued contraction mappings.

THEOREM [6]. *Let (X, ϱ) be a complete metric space. If $F\colon X \to CB(X)$ be a multivalued contraction mapping, then F has a fixed point.*

In one of our recent papers [7] we have established the following theorem:

THEOREM. *If T be a self mapping of a complete metric space (X, ϱ) such that*

$$\varrho(Tx, Ty) \leqslant \alpha\varrho(x, Tx) + \beta\varrho(y, Ty) + \gamma\varrho(x, y)$$

for all $x, y \in X$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\alpha + \beta + \gamma < 1$, then T has a unique fixed point.

The aim of this paper is (i) to extend our result for multivalued mappings in a metric space and (ii) to extend our result in arbitrary topological spaces. A few related theorems have also been presented here.

Preliminaries. We use the following notations and definitions as given in [6].

DEFINITION. Let (X, ϱ) be a metric space, then

(i) $CB(X) = \{C \mid C$ is a nonempty closed and bounded subset of $X\}$,

(ii) $\delta(x, A) = \inf\{\varrho(x, y)\colon y \in A\}$,

(iii) $N(\varepsilon, C) = \{x \in X \mid \varrho(x, c) < \varepsilon$ for some $c \in C\}$, $\varepsilon > 0$ and $C \in CB(X)$,

(iv) $H(A, B) = \inf\{\varepsilon \mid A \subset N(\varepsilon, B)$ and $B \subset N(\varepsilon, A)\}$, $\varepsilon > 0$ and $A, B \in CB(X)$.

The function H is a metric for $CB(X)$ called the *Hausdorff metric* for $CB(X)$.

DEFINITION 2. Let (X, ϱ_1) and (Y, ϱ_2) be two metric spaces. A function $F\colon X \to CB(Y)$ is said to be a *multivalued contraction mapping of X into Y if*

$$H(F(x), F(z)) \leqslant \alpha\varrho_1(x, z) \quad \text{for all } x, z \in X,\ 0 \leqslant \alpha < 1.$$

DEFINITION 3. A point x is said to be a *fixed point of a multivalued mapping* F if $x \in F(x)$.

THEOREM 1. *Let (X, ϱ) be a complete metric space and $F\colon X \to CB(X)$ be a multivalued mapping such that*

$$H(F(x), F(y)) \leqslant \alpha\delta(x, F(x)) + \beta\delta(y, F(y)) + \gamma\varrho(x, y),$$

for all $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\alpha+\beta+\gamma < 1$, $x, y \in X$. If F be continuous on X, then F has a fixed point.

Proof. Let $x_0 \in X$, then $F(x_0) \in CB(X)$. Pick a point $x_1 \in F(x_0)$. Since $F(x_0)$, $F(x_1) \in CB(X)$ and $x_1 \in F(x_0)$, there is a point $x_2 \in F(x_1)$ such that

$$(*) \qquad \varrho(x_1, x_2) \leqslant H(F(x_0), F(x_1)) + \frac{\alpha+\gamma}{1-\beta}.$$

Similarly, since $F(x_1)$, $F(x_2) \in CB(X)$ and $x_2 \in F(x_1)$, there exists a point $x_3 \in F(x_2)$ such that

$$\varrho(x_2, x_3) \leqslant H(F(x_1), F(x_2)) + \left(\frac{\alpha+\gamma}{1-\beta}\right)^2$$

and continuing this way we get a sequence $\{x_i\}_{i=1}^{\infty}$ of points in X such that $x_i \in F(x_{i-1})$ and, that

$$\varrho(x_i, x_{i+1}) \leqslant H(F(x_{i-1}), F(x_i)) + \left(\frac{\alpha+\gamma}{1-\beta}\right)^i.$$

Now

$$\delta(x_i, F(x_i)) = \inf\{\varrho(x_i, y)\colon y \in F(x_i)\} \leqslant \varrho(x_i, x_{i+1}), \quad \text{since} \quad x_{i+1} \in F(x_i).$$

Therefore

$$\varrho(x_i, x_{i+1}) \leqslant H(F(x_{i-1}), F(x_i)) + \left(\frac{\alpha+\gamma}{1-\beta}\right)^i$$
$$\leqslant \alpha\delta(x_{i-1}, F(x_{i-1})) + \beta\delta(x_i, F(x_i)) + \gamma\varrho(x_i, x_{i-1}) + \left(\frac{\alpha+\gamma}{1-\beta}\right)^i.$$

Hence

$$\varrho(x_i, x_{i+1}) \leqslant \left(\frac{\alpha+\gamma}{1-\beta}\right)\varrho(x_{i-1}, x_i) + \frac{(\alpha+\gamma)^i}{(1-\beta)^{i+1}}.$$

Again

$$\varrho(x_{i-1}, x_i) \leqslant \left(\frac{\alpha+\gamma}{1-\beta}\right)\varrho(x_{i-2}, x_{i-1}) + \frac{(\alpha+\gamma)^{i-1}}{(1-\beta)^i}.$$

So

$$\varrho(x_i, x_{i+1}) \leqslant \left(\frac{\alpha+\gamma}{1-\beta}\right)^2 \varrho(x_{i-2}, x_{i-1}) + \frac{2}{\alpha+\gamma}\left(\frac{\alpha+\gamma}{1-\beta}\right)^{i+1}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\leqslant \left(\frac{\alpha+\gamma}{1-\beta}\right)^i \varrho(x_0, x_1) + \frac{i}{\alpha+\gamma}\left(\frac{\alpha+\gamma}{1-\beta}\right)^{i+1}.$$

But then

$$\varrho(x_i, x_{i+k}) \leqslant \varrho(x_i, x_{i+1}) + \ldots + \varrho(x_{i+k-1}, x_{i+k})$$
$$\leqslant \sum_{n=i}^{i+k-1} \lambda^n \varrho(x_0, x_1) + \frac{1}{\alpha+\gamma}\sum_{n=i}^{i+k-1} n\lambda^{n+1}, \qquad k, i > 1,$$

where $\lambda = \dfrac{\alpha+\gamma}{1-\beta}$.

Now from the given condition $\lambda = \dfrac{\alpha+\gamma}{1-\beta} < 1$. So the right hand side becomes sufficiently small as $i \to \infty$. Thus the sequence $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence and by the completeness of X we have $\lim\limits_{i\to\infty} x_i = p_0 \in X$. Since F is continuous on X, $\{F(x_i)\}_{i=1}^{\infty}$ converges to $F(p_0)$. Now $x_i \in F(x_{i-1})$ for all $i = 1, 2, \ldots$ Hence $p_0 \in F(p_0)$. Thus p_0 is a fixed point of F. This completes the proof.

(**) If $A, B \in CB(X)$, $x \in A$, $\eta > 0$, then from the definition of $H(A, B)$, we infer that there is a $y \in B$ such that $\varrho(x, y) \leqslant H(A, B) + \eta$. In the proof of the above theorem $\dfrac{\alpha+\gamma}{1-\beta}$ and consequently $\left(\dfrac{\alpha+\gamma}{1-\beta}\right)^i$ play the role of η, $i = 1, 2, \ldots$

Remark. The conclusion of the above theorem may be obtained by replacing the hypothesis $F\colon X \to CB(X)$ by a weaker one that the diagram of F is closed. I am very much thankful to the referee for pointing out this.

THEOREM 2. *Let E be a topological space, d be a metric on E, and T be a continuous self mapping of E such that*

$$d(Tx, Ty) < \alpha d(x, Tx) + \beta d(y, Ty) + \gamma d(x, y)$$

for each $x \neq y \in E$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\alpha+\beta+\gamma < 1$. Suppose also that there is a point $x_0 \in E$ such that the sequence of iterates $T^n_{x_0}$ has a cluster value u in E. Then u is the unique fixed point of T.

Proof. Suppose $u' \neq u''$ be two fixed points of T. Then by the given condition

$$d(u', u'') = d(Tu', Tu'') < \alpha d(u', Tu') + \beta d(u'', Tu'') + \gamma d(u', u'').$$

So $d(u', u'') < \gamma d(u', u'')$ which is impossible. Hence we infer that T can have at-most one fixed point and we will prove that u is a fixed point of T. Let $v = Tu$ and suppose

$u \neq v$. Now there exists a subsequence $x_r = T^{n_r}_{x_0}$ of $T^n_{x_0}$ which converges to u in E. But

$$d(Tx_r, TTx_r) < \alpha d(x_r, Tx_r) + \beta d(Tx_r, TTx_r) + \gamma d(x_r, Tx_r).$$

So

$$d(Tx_r, TTx_r) < \frac{\alpha+\gamma}{1-\beta} d(x_r, Tx_r),$$

$$d(T^2x_r, T^2Tx_r) < \alpha d(Tx_r, TTx_r) + \beta d(T^2x_r, T^2Tx_r) + \gamma d(Tx_r, TTx_r),$$

or,

$$d(T^2x_r, T^2Tx_r) < \left(\frac{\alpha+\gamma}{1-\beta}\right) d(Tx_r, TTx_r) < \left(\frac{\alpha+\gamma}{1-\beta}\right)^2 d(x_r, Tx_r).$$

$$d(T^3x_r, T^3Tx_r) < \alpha d(T^2x_r, T^3x_r) + \beta d(T^3x_r, T^3Tx_r) + \gamma d(T^2x_r, T^2Tx_r),$$

or,

$$d(T^3x_r, T^3Tx_r) < \left(\frac{\alpha+\gamma}{1-\beta}\right) d(T^2x_r, T^2Tx_r < \left(\frac{\alpha+\gamma}{1-\beta}\right)^3 d(x_r, Tx_r),$$

. . . .

$$d(T^m x_r, T^m Tx_r) < k^m d(x_r, Tx_r), \quad k = \frac{\alpha+\gamma}{1-\beta}$$

for $m = 1, 2, \ldots$ and in particular for $m = n_{r+1} - n_r$ we have

$$d(x_{r+1}, Tx_{r+1}) < Rd(x_r, Tx_r), \quad R = k^{n_{r+1}-n_r} < 1.$$

Now from the triangle inequality

$$d(x_{r+1}, Tx_{r+1}) < Rd(x_r, Tx_r) \leqslant Rd(x_r, u) + Rd(u, v) + Rd(v, Tx_r).$$

Now proceeding to the limit $r \to \infty$ we get

$$d(u, v) \leqslant Rd(u, u) + Rd(u, v) + Rd(u, v) = Rd(u, v).$$

Hence $d(u, v) < d(u, v)$ since $R < 1$ and so it follows from this contradiction that $u = v = Tu$. Thus u is a fixed point of T.

COROLLARY. *If $n \geqslant 1$ be a fixed integer and if the conditions of Theorem 1 hold with T replaced by T^n then u is unique fixed point of T.*

Proof. By Theorem 2 u is a unique fixed point of T^n and since Tu is also a fixed point T^n we have $u = Tu$.

THEOREM 3. *Let X be a Hausdorff space and T be a continuous self-map of X. Let Φ: $X \times X \to [0, \infty)$ be continuous mapping such that for all $x, y \in X$, $x \neq y$*

$$\Phi(Tx, Ty) < \alpha\Phi(x, Tx) + \beta\Phi(y, Ty) + \gamma\Phi(x, y),$$

where $\alpha > 0$, $\beta > 0$, $\gamma > 0$ with $\alpha+\beta+\gamma < 1$. If for some $x_0 \in X$ the sequence of iterates $\{T^n_{x_0}\}$ has a convergent subsequence, then T has a fixed point.

Proof. Let us define the sequence $\{x_n\}$ of elements in X as follows: assume $x_n \neq x_{n+1}$ and let

$$x_1 = Tx_0,\ x_2 = Tx_1 = T^2x_0,\ \ldots,\ x_n = T^n x_0,\ x_{n+1} = Tx_n, \quad n = 0, 1, \ldots$$

Then

$$\Phi(x_1, x_2) < \alpha\Phi(x_0, Tx_0) + \beta\Phi(x_1, Tx_1) + \gamma\Phi(x_0, x_1),$$

i.e.,

$$\Phi(x_1, x_2) < \frac{\alpha+\gamma}{1-\beta}\Phi(x_0, x_1) < \Phi(x_0, x_1).$$

Similarly

$$\Phi(x_2, x_3) = \Phi(Tx_1, Tx_2) < \alpha\Phi(x_1, Tx_1) + \beta\Phi(x_2, Tx_2) + \gamma\Phi(x_1, x_2),$$

or,

$$\Phi(x_2, x_3) < \frac{\alpha+\gamma}{1-\beta}\Phi(x_1, x_2) < \Phi(x_1, x_2) < \Phi(x_0, x_1)$$

and continuing this process we get a monotone sequence of nonnegative real numbers

$$\Phi(x_0, x_1) > \Phi(x_2, x_3) > \ldots > \Phi(x_n, x_{n+1}) > \ldots$$

which must converge along will all its subsequences to some real number λ say

By hypothesis we have a convergent subsequence $\{x_{n_k}\}$ in X which converge to a point $z_0 \in X$, i.e., $\lim_{k\to\infty} x_{n_k} = z_0$, $x_{n_k} = T^{n_k}_{x_0}$ we will show now that $z_0 = Tz_0$ Suppose $z_0 \neq Tz_0$. We define $Z_{n+1} = Tz_n$, $n = 0, 1, 2, \ldots$ Then

$$\Phi(z_0, z_1) > \Phi(z_1, z_2) > \ldots > \Phi(z_n, z_{n+1}) > \ldots \tag{1}$$

But since T is continuous, we get

$$\begin{aligned}\Phi(z_0, Tz_0) &= \Phi(\lim_{k\to\infty} x_{n_k}, T\lim x_{n_k}) = \Phi(\lim_{k\to\infty} x_{n_k}, \lim_{k\to\infty} x_{n_{k+1}}) \\ &= \lim_{k\to\infty}\Phi(x_{n_k}, x_{n_{k+1}}) = \lambda = \lim_{k\to\infty}\Phi(x_{n_{k+n}}, x_{n_{k+n+1}}) = \Phi(z_n, z_{n+1}).\end{aligned} \tag{2}$$

Hence we get a contradiction from (1) and (2) and it follows from this contradiction that z_0 is a fixed point of T. In a similar way we can prove the following theorem.

THEOREM. *Let T_1 and T_2 be two continuous selfmaps of a Hausdorff space X and Φ: $X \times X \to [0, \infty)$ be continuous such that*

$$\Phi(T_1 x, T_2 y) < \alpha\Phi(x, T_1 x) + \beta\Phi(y, T_2 y) + \gamma\Phi(x, y)$$

for all $x \neq y \in X$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\alpha+\beta+\gamma < 1$. If for some $x_0 \in X$ the sequence $\{x_n\}$ where $x_{2n+1} = T_1 x_{2n}$, $x_{2(n+1)} = T_2 x_{2n+1}$, $n = 0, 1, 2, \ldots$ has a convergent subsequence then T_1 and T_2 have a common fixed point.

THEOREM 4. *Let T be a contractive mapping of a complete metric space X into itself such that there exists a subset $E' \subset X$ and a point $x_0 \in E'$ satisfying the following*:

(i) $\varrho(x, x_0) - \varrho(Tx, Tx_0) \geqslant 2\varrho(x_0, Tx_0)$ *for every* $x \in X - E'$,

(ii) $\varrho(Tx, Ty) \leqslant \alpha(\varrho(x, y))\varrho(x, Tx) + \beta(\varrho(x, y))\varrho(y, Ty) + \gamma(\varrho(x, y))\varrho(x, y)$ *for every* $x, y \in E'$ *where* α, β, γ *are monotonic decreasing functions from* $[0, \infty)$ *to* $[0, 1)$ *such that*

$$\alpha(r) + \beta(r) + \gamma(r) < 1 \quad \text{for all } r \in [0, \infty).$$

Then there exists a unique fixed point.

Before we prove the theorem we need the following definition.

DEFINITION. A mapping T of a metric space X into itself is said to be *contractive* if $\varrho(Tx, Ty) < \varrho(x, y)\ \forall x, y \in X,\ x \neq y$.

A contractive mapping is obviously continuous.

Proof of Theorem 4. We first show that $x_n \in E'$ for all n. Suppose $x_0 \neq Tx_0$ and define a sequence $\{x_n\}$ as $x_n = T^n x_0$, $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$ Since T is contractive $\varrho(x_n, x_{n+1})$ is non-increasing and since $x_0 \neq Tx_0$ it follows that

$$\varrho(x_n, x_{n+1}) < \varrho(x_0, x_1), \quad n = 1, 2, \ldots$$

Now from the triangle inequality

$$\begin{aligned} \varrho(x_0, x_n) &\leqslant \varrho(x_0, x_1) + \varrho(x_1, x_{n+1}) + \varrho(x_n, x_{n+1}) \\ &= \varrho(x_0, Tx_0) + \varrho(Tx_0, Tx_n) + \varrho(x_n, x_{n+1}) \\ &> \varrho(Tx_0, x_0) + \varrho(Tx_0, Tx_n) + \varrho(x_0, Tx_0). \end{aligned}$$

Thus

$$\varrho(x_n, x_0) - \varrho(Tx_n, Tx_0) < 2\varrho(x_0, Tx_0).$$

So from the condition (i) of Theorem 4, $x_n \in E'$ for every n. We now prove that the sequence $\{x_n\}$ is bounded. Now

$$\begin{aligned} \varrho(x_1, x_{n+1}) &= \varrho(Tx_0, Tx_n) \\ &\leqslant \alpha(\varrho(x_0, x_n))\varrho(x_0, Tx_0) + \beta(\varrho(x_0, x_n))\varrho(x_n, Tx_n) + \gamma(\varrho(x_0, x_n))\varrho(x_0, x_n). \end{aligned}$$

So from the triangle inequality we get

$$\begin{aligned} \varrho(x_0, x_n) &\leqslant \varrho(x_0, Tx_0) + \varrho(Tx_0, Tx_n) + \varrho(x_n, x_{n+1}) \\ &< 2\varrho(x_0, Tx_0) + \varrho(Tx_0, Tx_n) \\ &\leqslant 2\varrho(x_0, Tx_0) + \alpha\varrho((x_0, x_n))\varrho(x_0, Tx_0) + \\ &\quad + \beta(\varrho(x_0, x_n))\varrho(x_n, Tx_n) + \gamma(\varrho(x_0, x_n))\varrho(x_0, x_n) \\ &< 2\varrho(x_0, Tx_0) + \alpha(\varrho(x_0, x_n))\varrho(x_0, Tx_0) + \\ &\quad + \beta(\varrho(x_0, x_n))\varrho(x_0, Tx_0) + \gamma(\varrho(x_0, x_n))\varrho(x_0, x_n), \end{aligned}$$

i.e.

$$[1 - \gamma(\varrho(x_0, x_n))]\varrho(x_0, x_n) < 2\varrho(x_0, Tx_0) + [\alpha(\varrho(x_0, x_n)) + \beta(\varrho(x_0, x_n))]\varrho(x_0, Tx_0).$$

Or

$$\varrho(x_0, x_n) < \frac{2\varrho(x_0, Tx_0)}{1 - \gamma(\varrho(x_0, x_n))} + \left[\frac{\alpha(\varrho(x_0, x_n)) + \beta(\varrho(x_0, x_n))}{1 - \gamma(\varrho(x_0, x_n))}\right]\varrho(x_0, Tx_0).$$

Now

$$\alpha(\varrho(x_0, x_n)) \leqslant \alpha(0) < 1, \quad \beta(\varrho(x_0, x_n)) < 1, \quad \gamma(\varrho(x_0, x_n)) < 1.$$

Hence

$$\varrho(x_0, x_n) \leqslant 3\varrho(x_0, Tx_0) = R$$

say. Hence the sequence $\{x_n\}$ is bounded. Now let $m > 0$ be an arbitrary integer. We have

$$\begin{aligned} \varrho(x_1, x_2) = \varrho(Tx_0, Tx_1) &\leqslant \alpha(\varrho(x_0, x_1))\varrho(x_0, Tx_0) \\ &\quad + \beta(\varrho(x_0, x_1))\varrho(x_1, Tx_1) + \gamma(\varrho(x_0, x_1))\varrho(x_0, x_1). \end{aligned}$$

Hence

$$\varrho(x_1, x_2) \leqslant \frac{\alpha(\varrho(x_0, x_1)) + \gamma(\varrho(x_0, x_1))}{1 - \beta(\varrho(x_0, x_1))}\varrho(x_0, Tx_0).$$

$$\begin{aligned} \varrho(x_2, x_3) &= \varrho(Tx_1, Tx_2) \\ &\leqslant \alpha(\varrho(x_1, x_2))\varrho(x_1, Tx_1) + \beta(\varrho(x_1, x_2))\varrho(x_2, Tx_2) + \\ &\quad + \gamma(\varrho(x_1, x_2))\varrho(x_1, x_2). \end{aligned}$$

Or,

$$\begin{aligned} \varrho(x_2, x_3) &\leqslant \frac{\alpha(\varrho(x_1, x_2)) + \gamma(\varrho(x_1, x_2))}{1 - \beta(\varrho(x_1, x_2))}\varrho(x_1, x_2) \\ &\leqslant \frac{\alpha(\varrho(x_1, x_2)) + \gamma(\varrho(x_1, x_2))}{1 - \beta(\varrho(x_1, x_2))} \cdot \frac{\alpha(\varrho(x_0, x_1)) + \gamma(\varrho(x_0, x_1))}{1 - \beta(\varrho(x_0, x_1))}\varrho(x_0, Tx_0). \end{aligned}$$

Continuing this way we get

$$\varrho(x_n, x_{n+1}) \leqslant \left[\frac{\alpha(\varrho(x_{n-1}, x_n)) + \gamma(\varrho(x_{n-1}, x_n))}{1 - \beta(\varrho(x_{n-1}, x_n))} \cdots \frac{\alpha(\varrho(x_0, x_1)) + \gamma(\varrho(x_0, x_1))}{1 - \beta(\varrho(x_0, x_1))}\right]\varrho(x_0, Tx_0).$$

Now

$$\varrho(x_n, x_{n+m}) \leqslant \varrho(x_n, x_{n+1}) + \varrho(x_{n+1}, x_{n+2}) + \ldots + \varrho(x_{n+m-1}, x_{n+m}).$$

Hence

$$\varrho(x_n, x_{n+m}) \leqslant [\lambda(0)]^n[1+\lambda(0)+\{\lambda(0)\}^2+\ldots]\varrho(x_0, x_1) \leqslant [\lambda(0)]^n R_0$$

where

$$R_0 = \frac{1}{1-\lambda(0)}\varrho(x_0, x_1), \quad \lambda(0) = \frac{\alpha(0)+\gamma(0)}{1-\beta(0)} \to 0 \quad \text{as} \quad n\to\infty .$$

So there exists an integer N independent of m such that

$$\varrho(x_N, x_{N+m}) < \varepsilon, \quad \forall m>0, \quad \varepsilon>0 .$$

So $\{x_n\}$ is a Cauchy sequence and the completeness of X guarantees the existence of $\xi \in X$ such that $\lim_{n\to\infty} x_n = \xi$. But the continuity of T (since a contractive mapping is continuous) we have

$$T\xi = T\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = \xi .$$

If η, ξ, $\eta \neq \xi$ be two fixed points then

$$\varrho(\xi, \eta) = \varrho(T\xi, T\eta) < \varrho(\xi, \eta)$$

which is a contradiction. So ξ is a unique fixed point of T.

THEOREM 5. *Let (X, ϱ) be a compact metric space and let T be a mapping of X into itself such that*

$$\varrho(Tx, Ty) < \alpha[\varrho(x, Tx)+\varrho(y, Ty)]+\gamma\varrho(x, y)$$

for all $x, y \in X$, $2\alpha+\gamma<1$, $\alpha>0$, $\gamma>0$ with $x \neq y$. Then T has a unique fixed point.

Proof. Let $u = \inf\{\varrho(x, Tx): x \in X\}$ and let us take a sequence $\{x_n\}$ such that $\varrho(x_n, Tx_n) \to u$. Now there exists a subsequence $\{y_k\} \subset \{Tx_n\}$ which converges to certain $v \in X$. So

$$\varrho(v, Tv) \leqslant \varrho(v, y_k)+\varrho(Tx_k, Tv) ,$$

where $y_k = Tx_{n_k}$ and we denote x_{n_k} by x_k. Or,

$$\varrho(v, Tv) \leqslant \varrho(v, y_k)+\alpha\varrho(x_k, Tx_k)+\alpha\varrho(v, Tv)+\gamma\varrho(x_k, v) ,$$

or,

$$\varrho(v, Tv) \leqslant \varrho(v, y_k)+\alpha\varrho(x_k, Tx_k)+\alpha\varrho(v, Tv)+ +\gamma\varrho(x_k, Tx_k)+\gamma\varrho(y_k, Tx_k)+\gamma\varrho(v, y_k) ,$$

or

$$\varrho(v, Tv) \leqslant \frac{1+\gamma}{1-\alpha}\varrho(v, y_k)+\frac{\alpha+\gamma}{1-\alpha}\varrho(x_k, Tx_k) \leqslant 2\varrho(v, y_k)+\varrho(x_k, Tx_k) .$$

Thus $u = \varrho(v, Tv)$. Suppose $v \neq Tv$. Then $\varrho(v, Tv)>0$. Now

$$\varrho(Tv, T^2v) < \alpha\varrho(v, Tv)+\alpha\varrho(Tv, T^2v)+\gamma\varrho(v, Tv) ,$$

or,

$$\varrho(Tv, T^2v) < \frac{\alpha+\gamma}{1-\alpha}\varrho(v, Tv) < \varrho(v, Tv) ,$$

which is a contradiction. This completes the proof.

THEOREM 6. *Let (X, ϱ) be a compact metric space and T be a self-map of X such that*

(i) $\varrho(Tx, Ty) \leqslant \alpha[\varrho(x, Tx)+\varrho(y, Ty)]+\beta\varrho(x, y)$ *for all* $x, y \in X$, $2\alpha+\beta\leqslant 1$, $\beta\geqslant 0$,

(ii) $\varrho(x, Tx)$ *is not constant on any closed subset of X which contains more than one point and is invariant under T. Then T has a fixed point.*

Before we prove Theorem 2, we require the following:

DEFINITION. A family of sets has the *finite intersection property* if every finite sub-family has a non-empty intersection.

LEMMA. *A topological space is compact if and only if every family of closed sets with the finite intersection property has a non-empty intersection.*

Proof. See Dunford and Schwartz [3, p. 17].

Proof of Theorem 2. Let F denote the collection of all non-empty closed subsets of X with finite intersection property, each of which is mapped into itself by T. Since X is a compact metric space, so by Zorn's lemma there exists a minimal in F and let M be the minimal subset of X with respect to being non-empty, closed and invariant under T. If M contains more than one point, then there are p and q with $r = \varrho(p, Tp) < \varrho(q, Tq)$. Let $A = \{x \in M: \varrho(x, Tx) \leqslant r\}$ and let N be the closure of $T(A)$. If $y \in N$ then y is the limit of sequence $\{Tx_n\}$ with $\varrho(x_n, Tx_n) \leqslant r$ for each n. Now,

$$\begin{aligned}\varrho(y, Ty) &\leqslant \varrho(y, Tx_n)+\varrho(Tx_n, Ty)\\ &\leqslant \varrho(y, Tx_n)+\alpha\varrho(x_n, Tx_n)+\alpha\varrho(y, Ty)+\beta\varrho(x_n, y)\\ &\leqslant \varrho(y, Tx_n)+\alpha\varrho(x_n, Tx_n)+\alpha\varrho(y, Ty)+\beta\varrho(y, Tx_n)+\beta\varrho(x_n, Tx_n) ,\end{aligned}$$

or,

$$(1-\alpha)\varrho(y, Ty) \leqslant (1+\beta)\varrho(y, Tx_n)+(\alpha+\beta)\varrho(x_n, Tx_n) .$$

Hence

$$\varrho(y, Ty) \leqslant \frac{1+\beta}{1-\alpha}\varrho(y, Tx_n)+\frac{\alpha+\beta}{1-\alpha}\varrho(x_n, Tx_n) .$$

Thus $\varrho(y, Ty) \leqslant r$. Hence $N \subset A$ and $T(N) \subset T(A) \subset N$ which is a contradiction since N is a proper subset of M. This completes the proof.

THEOREM 7. *Let X be a strictly convex reflexive Banach space and K a bounded closed convex subset of X. Let T be a mapping of K into itself such that*

$$\|Tx-Ty\|\leqslant\alpha\|x-Tx\|+\beta\|y-Ty\|$$

for all $x, y\in K$, $\alpha+\beta\leqslant 1$, $\alpha>0$, $\beta>0$, $(\min(\alpha,\beta)<1$ *and* $\max(\alpha,\beta)\geqslant 0)$. *Then T has a unique fixed point in K.*

Before we prove this theorem we introduce the following definitions and notations etc.

DEFINITION. For a bounded subset M in X let $D(M)$ be the diameter of M:

$$D(M) = \sup\{\|x-y\|:\ x, y\in M\}$$

$\mathrm{co}\,M$ and $\overline{\mathrm{co}}\,M$ will denote the convex hull and the closed convex hull of M.

THEOREM (Smulian [8]). *A necessary and sufficient condition that a Banach space is reflexive is that:*

(C) *Every bounded descending sequence (transfinite) of non-empty closed convex subsets of X have a non-empty intersection.*

Proof of Theorem 7. Let A denote the collection of all non-empty closed and convex subsets of K, each of which is mapped into itself by T. Then by (C) and Zorn's lemma A has a minimal element and let $M\subset K$ be minimal with respect to being non-empty, closed, convex and invariant under T. Then $D(M)=0$ will imply that T has a fixed point. Suppose $D(M)>0$. From now on we shall write D instead of $D(M)$. Let $x\in M$ and assume that $\|x-Tx\| = D$. Take $y=\frac{1}{2}(x+Tx)$ then $y\in M$ and hence

$$\|Ty-x\|\leqslant D \quad\text{and}\quad \|Ty-Tx\|\leqslant D\,.$$

Now from the strict convexity of X we have

$$\tfrac{1}{2}\|Ty-x+(Ty-Tx)\|<D\,.$$

In other words we have $\|Ty-y\|<D$. So there exists a point y belonging to M such that

$$\|Ty-y\| = r<D\,.$$

Take

$$N=\{y\in M:\ \|Ty-y\|\leqslant r\},\qquad P=\overline{\mathrm{co}}(T(N))\,.$$

Then P is non-empty closed and convex subset of M. Now we wish to show that $T\colon P\to P$. So let $y\in P$, then the followings cases are to be considered.

Case 1. $y=Tp$ from some $p\in N$. Then

$$\|Ty-y\| = \|Ty-Tp\|\leqslant\alpha\|y-Ty\|+\beta\|p-Tp\|.$$

Or,

$$\|Ty-y\|\leqslant\frac{\beta}{1-\alpha}\|p-Tp\|\leqslant r$$

and so $y\in N$ and $Ty\in P$.

Case 2.

$$y=\sum_{i=1}^{n}\lambda_i Tp_i,\ p_i\in N,\quad \sum_{i=1}^{n}\lambda_i = 1,\ \lambda_i\geqslant 0\,.$$

Then

$$\begin{aligned}\|Ty-y\| &= \|Ty-\sum_{i=1}^{n}\lambda_i Tp_i\|\\ &\leqslant\sum_{i=1}^{n}\lambda_i\|Ty-Tp_i\|\\ &\leqslant\sum_{i=1}^{n}\lambda_i\{\alpha\|y-Ty\|+\beta\|p_i-Tp_i\|\}\\ &\leqslant\alpha\|y-Ty\|+\sum_{i=1}^{n}\lambda_i\beta r\,.\end{aligned}$$

So $\|Ty-y\|\leqslant r$, $y\in N$, $Ty\in P$.

Case 3. y is the limit of terms of the form $\sum_{i=1}^{n}\lambda_i Tp_i$, $\lambda_i\geqslant 0$, $\sum_{i=1}^{n}\lambda_i = 1$ and $p_i\in N$. Then

$$\begin{aligned}\|Ty-y\| &= \|Ty-\sum_{i=1}^{n}\lambda_i Tp_i\|+\|\sum_{i=1}^{n}\lambda_i Tp_i-y\|\\ &\leqslant r+\|\sum_{i=1}^{n}\lambda_i Tp_i-y\|\,.\end{aligned}$$

Hence $\|Ty-y\|\leqslant r$. Thus $y\in N$ and $Ty\in P$. Since M is minimal we should have $M=P$ and $D(M)=D(P)$. Now it can be easily shown that if A be a subset of a Banach space then $D(A)=D(\overline{\mathrm{co}}(A))$. So

$$\begin{aligned}D(P) &= D(\overline{\mathrm{co}}(T(N))) = D(T(N))\\ &= \sup\{\|Tx-Ty\|:\ x, y\in N\}\\ &\leqslant\sup\{\alpha\|x-Tx\|+\beta\|y-Ty\|:\ x, y\in N\}\\ &\leqslant(\alpha+\beta)r\leqslant r<D(M)\,.\end{aligned}$$

This contradiction shows that $D(M)=0$ and so T has a fixed point z_0 say.

If possible let y_0, $y_0\neq z_0$ be another fixed point of T. Then

$$\begin{aligned}\|z_0-y_0\| &= \|Tz_0-Ty_0\|\\ &\leqslant\alpha\|z_0-Tz_0\|+\beta\|y_0-Ty_0\| = 0\,.\end{aligned}$$

Hence $z_0=y_0$. This completes the proof.

References

[1] D. D. Ang and D. E. Daykin, *Fixed point theorems and convolution equations*, Proc. Amer. Math. Soc. 19 (1968), pp. 1187–1194.

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. 3 (1922), pp. 133–181.

[3] N. Dunford and J. Schwartz, *Linear Operators*, Part 1, New York 1964.

[4] R. Kannan, *Some results on fixed points* IV, Fund. Math. (to appear).

[5] M. G. Maia, *Un osservazione sulle contrazioni metriche*, Rend. Semi. Mat. Univ. Padova 40 (1968), pp. 139–143.

[6] S. B. Nadler, Jr., *Multivalued contraction mappings*, Pacific J. Math. 30 (1969), pp. 475–488.

[7] B. Ray, *Some results on fixed points* (submitted).

[8] — *On nonexpansive mappings in a metric space*, Nanta. Math. 7 (1974), pp. 86–92.

[9] V. Smulian, *On the principle of inclusion in the space of type* (B), Math. Sb. (N. S.) 5 (1939), pp. 327–328.

DEPARTMENT OF MATHEMATICS
REGIONAL ENGINEERING COLLEGE
Durgapur, India

Accepté par la Rédaction le 19. 8. 1974