THEOREM. *There exist non-metric hereditarily indecomposable continua.*

Proof. Let, in the above construction, X and $T_x^{-1}(x)$ for each $x \in X$ be hereditarily indecomposable metric continua; e.g. pseudo-arcs. By Lemma 1, Note 1 and Note 2, we infer that S in this construction is a non-metric hereditarily indecomposable continuum.

References

[1] D. P. Bellamy, *A non-metric indecomposable continuum*, Duke Math. J. 38 (1971), pp. 15–20.
[2] — and L. R. Rubin, *Indecomposable continua in Stone-Čech compactification*, Proc. Amer. Math. Soc. 39 (1973), pp. 427–433.
[3] H. Cook, *Continua which admit only the identity mapping onto non-degenerate subcontinua*, Fund. Math. 60 (1967), pp. 241–249.
[4] A. Emeryk, *An atomic map onto an arbitrary metric continuum*, Fund. Math. 77 (1972), pp. 145–150.

SILESIAN UNIVERSITY, Katowice

Accepté par la Rédaction le 19. 8. 1974

Paracompactness of topological completions

by

Tadashi Ishii (Shizuoka)

Abstract. Let X be a completely regular T_2 space, and $\mu(X)$ a topological completion of X (that is, a completion of X with respect to its finest uniformity agreeing with the topology of X). If $\mu(X)$ is paracompact, then X is said to be *pseudo-paracompact*. In this paper some remarkable properties of pseudo-paracompact spaces are studied.

1. Introduction. The purpose of this paper is to give detailed proofs for the author's abstract [6]. Throughout this paper all spaces are assumed to be completely regular T_2. For every space X, we denote by μ its finest uniformity agreeing with the topology of X, that is, μ is the family of all normal open coverings of X. Concerning pseudo-paracompactness, the following results are known.

THEOREM 1.1 (Morita [13]). *For every M-space X $\mu(X)$ is a paracompact M-space.*

THEOREM 1.2 (Howes [5]). *A space X is pseudo-paracompact if and only if every weakly Cauchy filter in X with respect to μ is contained in some Cauchy filter with respect to μ.*

Let $\{\mathfrak{U}_\lambda|\ \lambda \in \Lambda\}$ be the family of all normal open coverings of a space X. A filter $\mathfrak{F} = \{F_\alpha\}$ in X is weakly Cauchy with respect to μ if for any $\lambda \in \Lambda$ there exists $U \in \mathfrak{U}_\lambda$ such that $U \cap F_\alpha \neq \emptyset$ for every $F_\alpha \in \mathfrak{F}$. In other words, a filter $\mathfrak{F}$ is weakly Cauchy with respect to μ if for any $\lambda \in \Lambda$ there exists a filter $\mathfrak{F}_\lambda$ stronger than $\mathfrak{F}$ such that $L \subset U$ for some $U \in \mathfrak{U}_\lambda$ and $L \in \mathfrak{F}_\lambda$. In this paper we shall study further results related to pseudo-paracompactness. § 2 contains other characterizations of pseudo-paracompact spaces and another proof of Howes's theorem. Furthermore it is shown by an example that there exists a strongly normal (i.e., countably paracompact and collectionwise normal) space which is not pseudo-paracompact. § 3 is concerned with the following:

(1) The sum theorems of pseudo-paracompact spaces.

(2) The sucffient conditions for the preimage X of a paracompact space (or a paracompact q-space [10]) Y under a closed map f to be pseudo-paracompact.

(3) The invariance of strongly normal pseudo-paracompactness under a perfect map.

(4) Characterizations of pseudo-locally-compact and pseudo-paracompact spaces.

A space X is said to be pseudo-locally-compact (pseudo-Lindelöf etc.) if $\mu(X)$ is locally compact (resp. Lindelöf etc.). Concerning (4) other characterizations than Morita's [14] will be given. Finally in § 4 we shall study some properties of pseudo-Lindelöf spaces.

2. Characterizations of pseudo-paracompact spaces. Concerning the topological completion of a space X, we use the terminology and the basic results due to Morita [13]. Let $\{\Phi_\gamma|\ \gamma\in\Gamma\}$ be the family of all normal sequences which consist of normal open coverings of X. Let us put $\Phi_\gamma = \{\mathfrak{U}_{\gamma_i}|\ i = 1, 2, \ldots\}$, where $\mathfrak{U}_{\gamma_{i+1}}$ is a star refinement of U_{γ_i} (that is, $\{\mathrm{St}(U, \mathfrak{U}_{\gamma_{i+1}})|\ U\in\mathfrak{U}_{\gamma_{i+1}}\} > \mathfrak{U}_{\gamma_i}$) for each i. We denote by (X, Φ_γ) the topological space obtained from X by taking

$$\{\mathrm{St}(x, \mathfrak{U}_{\gamma_i})|\ i = 1, 2, \ldots\}$$

as a basis of neighborhoods at each point x of X. Let X/Φ_γ be the quotient space obtained from (X, Φ_γ) by defining those points x and y with $y\in\mathrm{St}(x, \mathfrak{U}_{\gamma_i})$ for each i to be equivalent. Let us denote by i_γ the identity map from X onto (X, Φ_γ) and by $\tilde{\varphi}_\gamma$ the quotient map from (X, Φ_γ) onto X/Φ_γ. If we put

$$\varphi_\gamma = \tilde{\varphi}_\gamma \circ i_\gamma\colon\ X\to X/\Phi_\gamma\,,$$

then φ_γ is a continuous map from X onto a metrizable space X/Φ_γ. Let us now introduce a partial order in $\{\Phi_\gamma|\ \gamma\in\Gamma\}$. If for each i there exists $\mathfrak{U}_{\delta_j}\in\Phi_\delta$ such that $\mathfrak{U}_{\delta_j} > \mathfrak{U}_{\gamma_i}$, we write $\Phi_\gamma < \Phi_\delta$. Suppose that $\Phi_\gamma < \Phi_\delta < \Phi_\varepsilon$. Then it is easy to see that the canonical map $\varphi^\delta_\gamma\colon X/\Phi_\delta\to X/\Phi_\gamma$ is continuous and

$$\varphi^\delta_\gamma\circ\varphi_\delta = \varphi_\gamma\,,\qquad \varphi^\delta_\gamma\circ\varphi^\varepsilon_\delta = \varphi^\varepsilon_\gamma\,.$$

An open covering $\mathfrak{O} = \{O_\alpha\}$ of X is said to be *extendable* to $\mu(X)$ if there exists an open covering $\tilde{\mathfrak{O}} = \{\tilde{O}_\alpha\}$ of $\mu(X)$, say an extension of $\mathfrak{O}$, such that $O_\alpha = \tilde{O}_\alpha\cap X$ for each α. It shoud be noted that every normal open covering of X has an extension to $\mu(X)$ which is a normal open covering of $\mu(X)$ (cf. [11, (I) Lemma 8 and (II) Lemma 1]).

THEOREM 2.1. *For a space X, the following conditions are equivalent.*

(a) *X is pseudo-paracompact.*

(b) *Every open covering of X which is extendable to $\mu(X)$ is a normal covering.*

(c) *The product of X with every compact space is pseudo-normal.*

(d) *Every weakly Cauchy filter in X with respect to μ is contained in some Cauchy filter with respect to μ.*

(e) *If $\mathfrak{F}$ is a filter in X such that the image of $\mathfrak{F}$ has a cluster point in any metric space into which X is continuously mapped, then $\mathfrak{F}$ is contained in some Cauchy filter with respect to μ.*

The equivalence of (a) and (d) is due to Howes [5], but another proof is given below.

Proof of Theorem 2.1. (a)→(b) is obvious.

(a)↔(c). Let K be an arbitrary compact space. Then by [13, Theorem 5.1] we have $\mu(X\times K) = \mu(X)\times K$. As was proved by Tamano [16, Theorem 2], a space Y is paracompact if and only if $Y\times K$ is normal for every compact space K. Hence (a) and (c) are equivalent.

(b)→(d). Suppose that a weakly Cauchy filter $\mathfrak{F} = \{F_\alpha\}$ in X with respect to μ is not contained in any Cauchy filter with respect to μ. Then each point x of $\mu(X)$ has an open neighborhood $N(x)$ such that $N(x)\cap F_{\alpha(x)} = \emptyset$ for some $F_{\alpha(x)}\in\mathfrak{F}$. Let $\mathfrak{W} = \{N(x)\cap X|\ x\in\mu(X)\}$. Then by (b) $\mathfrak{W}$ is a normal open covering of X. Since $\mathfrak{F}$ is a weakly Cauchy filter with respect to μ, we have $(N(x)\cap X)\cap F_\alpha\neq\emptyset$ for some $x\in\mu(X)$ and for any α, which is a contradiction. Thus (d) holds.

(d)↔(e). This immediately follows from the fact that a filter $\mathfrak{F}$ in X is weakly Cauchy with respect to μ if and only if the image of $\mathfrak{F}$ has a cluster point in any metric space into which X is continuously mapped (cf. [1]).

(e)→(a). Let $\mathfrak{F} = \{F_\alpha\}$ be a filter base in $\mu(X)$ such that the image of $\mathfrak{F}$ has a cluster point in any metric space into which $\mu(X)$ is continuously mapped. Since, for any $\varphi_\gamma\colon X\to X/\Phi_\gamma$, $\mu(\varphi_\gamma)$ carries $\mu(X)$ into X/Φ_γ ([13]), $\mu(\varphi_\gamma)(\mathfrak{F})$ has a cluster point in X/Φ_γ. Let us put $\mathfrak{F}_\gamma = \varphi_\gamma^{-1}(\mu(\varphi_\gamma)(\mathfrak{F}))$ for each $\gamma\in\Gamma$ and

$$\mathfrak{G} = \bigcup\{\mathfrak{F}_\gamma|\ \gamma\in\Gamma\}\,.$$

Then $\mathfrak{G}$ is a filter base in X; this follows from the fact that for Φ_γ and Φ_δ there exists Φ_ε such that $\Phi_\gamma < \Phi_\varepsilon$ and $\Phi_\delta < \Phi_\varepsilon$. Now we prove that the image of $\mathfrak{G}$ has a cluster point in any metric space into which X is continuously mapped. To show this, it suffces to prove that $\varphi_\gamma(\mathfrak{G})$ has a cluster point in X/Φ_γ for each $\gamma\in\Gamma$. Suppose that $\Phi_\gamma < \Phi_\delta$. Since $\varphi_\gamma = \varphi^\delta_\gamma\circ\varphi_\delta$, we have $\mu(\varphi_\gamma) = \varphi^\delta_\gamma\circ\mu(\varphi_\delta)$, and hence for each $F\in\mathfrak{F}$

$$\begin{aligned}\varphi_\gamma\big(\varphi_\delta^{-1}(\mu(\varphi_\delta)(F))\big) &= \varphi^\delta_\gamma\circ\varphi_\delta\big(\varphi_\delta^{-1}(\mu(\varphi_\delta)(F))\big)\\ &= \varphi^\delta_\gamma(\mu(\varphi_\delta)(F))\\ &= \mu(\varphi_\gamma)(F)\,,\end{aligned}$$

which shows that $\varphi_\gamma(\mathfrak{F}_\delta) = \varphi_\gamma(\mathfrak{F}_\gamma)$. For Φ_γ and Φ_δ which satisfy neither $\Phi_\gamma < \Phi_\delta$ nor $\Phi_\delta < \Phi_\gamma$, we take Φ_ε such that $\Phi_\gamma < \Phi_\varepsilon$ and $\Phi_\delta < \Phi_\varepsilon$. Then, since for each $F\in\mathfrak{F}$

$$\varphi_\varepsilon^{-1}(\mu(\varphi_\varepsilon)(F))\subset\varphi_\delta^{-1}(\mu(\varphi_\delta)(F))\,,$$

$$\varphi_\gamma\big(\varphi_\varepsilon^{-1}(\mu(\varphi_\varepsilon)(F))\big) = \mu(\varphi_\gamma)(F)\,,$$

we have $\mu(\varphi_\gamma)(F)\subset\varphi_\gamma\big(\varphi_\delta^{-1}(\mu(\varphi_\delta)(F))\big)$ for each $F\in\mathfrak{F}$, which shows that each element of $\varphi_\gamma(\mathfrak{F}_\gamma)$ is contained in the corresponding element of $\varphi_\gamma(\mathfrak{F}_\delta)$ as above. Consequently it follows that $\varphi_\gamma(\mathfrak{G})$ has a cluster point in X/Φ_γ for each γ. Hence by (e) $\mathfrak{G}$ has a cluster point u in $\mu(X)$. To show that $\mathfrak{F}$ has a cluster point u, suppose to be contrary. Then there exists an open neighborhood U of u in $\mu(X)$ such that $U\cap F_\alpha = \emptyset$ for some α. Let $\{G, H\}$ be a normal open covering of $\mu(X)$ such that

$u \in G \subset U$ and $u \in \mu(X) - \mathrm{cl} H$. Then there exists a normal sequence $\{\tilde{\mathfrak{U}}_{\gamma_i} |\ i = 1, 2, \ldots\}$ of open coverings of $\mu(X)$ such that $\tilde{\mathfrak{U}}_{\gamma_1} = \{G, H\}$. If we put

$$\mathfrak{U}_{\gamma_i} = \tilde{\mathfrak{U}}_{\gamma_i} \cap X, \quad \Phi_\gamma = \{\mathfrak{U}_{\gamma_i} |\ i = 1, 2, \ldots\},$$

then $\mu(\varphi_\gamma)(\mathfrak{F})$ does not cluster at $\mu(\varphi_\gamma)(u)$. But this is impossible, since $\varphi_\gamma(\mathfrak{F}_\gamma)(= \mu(\varphi_\gamma)(\mathfrak{F}))$ has a cluster point $\mu(\varphi_\gamma)(u)$. Therefore $\mathfrak{F}$ has a cluster point u, and hence $\mu(X)$ is paracompact by Corson's theorem [1]. Thus we complete the proof.

EXAMPLE 2.2 (A space which is strongly normal but not pseudo-paracompact). Let X be the subspace of the product $\prod_{\alpha \in A} R_\alpha$ which consists of those points which have at most a countable number of non-zero coordinates, where A is an uncountable index set and R_α is the real line for each $\alpha \in A$. In [2], Corson proved that X is strongly normal and that $\upsilon(X) = \prod_{\alpha \in A} R_\alpha$, where $\upsilon(X)$ is the realcompactification of X. Now we prove that

$$\mu(X) = \upsilon(X).$$

For this purpose, it suffices to show that any normal open covering $\mathfrak{U}$ of X admits a countable normal open refinement. Let $\Phi_\gamma = \{\mathfrak{U}_{\gamma_i} |\ i = 1, 2, \ldots\}$ be a normal sequence of open coverings of X, where $\mathfrak{U}_{\gamma_1} = \mathfrak{U}$. Then there exists a continuous map φ_γ from X onto a metric space X/Φ_γ. Since X/Φ_γ is separable by [2, Corollary 4], $\mathfrak{U}$ admits a countable normal open refinement. Hence we have $\mu(X) = \upsilon(X)$. As is well known, $\prod_{\alpha \in A} R_\alpha$ is not normal ([15]). Hence X is not pseudo-normal.

3. Some properties of pseudo-paracompact spaces.

THEOREM 3.1. *If there exists a normal open covering $\mathfrak{U} = \{U_\alpha\}$ of X such that each subspace U_α is pseudo-paracompact, then X is pseudo-paracompact.*

Proof. Let $\mathfrak{D}$ be any open covering of X which is extendable to $\mu(X)$. We prove first that $U_\alpha \cap \mathfrak{D} (= \{U_\alpha \cap O |\ O \in \mathfrak{D}\})$ is a normal open covering of the subspace U_α. Let i: $U_\alpha \to \mathrm{cl}_{\mu(X)} U_\alpha$ be an inclusion map. Since any closed subspace of a topologically complete space is also topologically complete by [13, Theorem 1.5], $\mathrm{cl}_{\mu(X)} U_\alpha$ is topologically complete. Hence $\mu(i)$ carries $\mu(U_\alpha)$ into $\mathrm{cl}_{\mu(X)} U_\alpha$. Let $\tilde{\mathfrak{D}}$ be an extension of $\mathfrak{D}$ to $\mu(X)$. Let us put $\mathfrak{G}_\alpha = \mu(i)^{-1}(\mathrm{cl}_{\mu(X)} U_\alpha \cap \tilde{\mathfrak{D}})$ for each α. Then G_α is an open covering of $\mu(U_\alpha)$, and hence it is a normal covering of $\mu(U_\alpha)$ by paracompactness of $\mu(U_\alpha)$. Since i: $U_\alpha \to \mathrm{cl}_{\mu(X)} U_\alpha$ is an inclusion map, we have $U_\alpha \cap G_\alpha = U_\alpha \cap \mathfrak{D}$, which shows that $U_\alpha \cap \mathfrak{D}$ is a normal open covering of U_α. Therefore by [12, Theorem 1.2], $\mathfrak{D}$ is a normal covering of X. Hence X is pseudo-paracompact by Theorem 2.1. Thus we complete the proof.

THEOREM 3.2. *Let $\{F_\alpha |\ \alpha \in \Omega\}$ be a locally finite closed covering of X such that each subspace F_α is pseudo-paracompact. If X is strongly normal, then X is pseudo-paracompact.*

Proof. Let $\mathfrak{D}$ be any open covering of X which is extendable to $\mu(X)$. By the similar way as in the proof of Theorem 3.1, we can prove that $F_\alpha \cap \mathfrak{D}$ is a normal open covering of F_α. Hence $F_\alpha \cap \mathfrak{D}$ has a locally finite closed refinement $\mathfrak{L}_\alpha = \{L_{\alpha\lambda} |\ \lambda \in \Lambda_\alpha\}$. Let us put $\mathfrak{L} = \bigcup \mathfrak{L}_\alpha$. Then $\mathfrak{L}$ is a locally finite closed refinement of $\mathfrak{D}$. Since X is strongly normal, there exists a locally finite open covering, $\mathfrak{G} = \{G_{\alpha\lambda} |\ \lambda \in \Lambda_\alpha, \alpha \in \Omega\}$ of X such that $L_{\alpha\lambda} \subset G_{\alpha\lambda}$ for any α and λ (Katětov [7]), where we may assume that $\mathfrak{G}$ is a refinement of $\mathfrak{D}$. Since X is normal, $\mathfrak{G}$ is a normal open covering of X. Hence by Theorem 2.1, X is pseudo-paracompact. Thus we complete the proof.

It should be noted that Theorem 3.2 can be also proved by making use of Theorem 3.14.

Now let f: $X \to Y$ be a continuous map. Then there exists its extension $\beta(f)$: $\beta(X) \to \beta(Y)$, where $\beta(S)$ denotes the Stone-Čech compactification of a space S, and it is known that $\beta(f)$ carries $\mu(X)$ into $\mu(Y)$ ([13]). We denote this map by $\mu(f)$. A continuous map f from a space X onto a space Y is called a *WZ-map* ([8]), a *Z-map*, or a *quasi-perfect* (resp. *perfect*) *map* if it satisfies (1), (2), or (3) below respectively:

(1) $\beta(f)^{-1}(y) = \mathrm{cl}_{\beta(X)} f^{-1}(y)$ for each $y \in Y$.

(2) $f(Z)$ is closed in Y for each zero-set Z of X.

(3) f is a closed map such that $f^{-1}(y)$ is countably compact (resp. compact) for each $y \in Y$.

Every closed map is a Z-map, and every Z-map is a WZ-map ([8]).

The following theorem is concerned with a relation between f and $\mu(f)$, and it is used to show that the preimages of paracompact spaces under quasi-perfect maps are pseudo-paracompact.

THEOREM 3.3. *If f is a quasi-perfect map from a space X onto a topologically complete space Y, then $\mu(f)$: $\mu(X) \to Y$ is perfect. More generally, if f is a WZ-map from a space X onto a topologically complete space Y such that $f^{-1}(y)$ is relatively pseudo-compact for each $y \in Y$, then $\mu(f)$: $\mu(X) \to Y$ is perfect.*

A subset A of a space X is said to be *relatively pseudo-compact* if every real-valued continuous function on X is bounded on A. To prove Theorem 3.3, we use the following lemma.

LEMMA 3.4 (Dykes [3]). *If F is a relatively pseudo-compact closed subset of a topologically complete space, then F is compact.*

Proof of Theorem 3.3. Let f be a WZ-map from a space X onto a topologically complete space Y such that $f^{-1}(y)$ is relatively pseudo-compact. Since $\mathrm{cl}_{\mu(X)} f^{-1}(y)$ is compact for each $y \in Y$ by Lemma 3.4, we have

$$\beta(f)^{-1}(y) = \mathrm{cl}_{\beta(X)} f^{-1}(y) = \mathrm{cl}_{\mu(X)} f^{-1}(y).$$

Hence, if we put $X_0 = \beta(f)^{-1}(Y)$, then $X \subset X_0 \subset \mu(X) \subset \beta(X)$, which implies that $\mu(X) = \mu(X_0)$ by [13, Lemma 2.3]. As is easily shown, the preimage of a topologically complete space under a perfect map is also topologically complete. Therefore we have $X_0 = \mu(X)$, which shows that $\mu(f)$: $\mu(X) \to Y$ is perfect. Thus we complete the proof.

COROLLARY 3.5. *If f is a quasi-perfect map from a space X onto a paracompact space Y, then X is pseudo-paracompact. More generally, if f is a WZ-map from a space X onto a paracompact space Y such that $f^{-1}(y)$ is relatively pseudo-compact for each $y \in Y$, then X is pseudo-paracompact.*

In case the fibers $\{f^{-1}(y)\}$ are not necessarily relatively pseudo-compact, we have the following theorem.

THEOREM 3.6. *If there exists a Z-map f from a space X onto a paracompact space Y such that $\mathfrak{B}f^{-1}(y)$ (= the boundary of $f^{-1}(y)$) is relatively pseudo-compact and $f^{-1}(y)$ is pseudo-paracompact for each $y \in Y$, then X is pseudo-paracompact.*

We prove the above theorem by making use of the following lemma.

LEMMA 3.7. *If there exists a Z-map f from a space X onto a paracompact space Y such that $\mathfrak{B}f^{-1}(y)$ is relatively pseudo-compact and for any open covering $\mathfrak{O}$ of X which is extendable to $\mu(X)$, $f^{-1}(y) \cap \mathfrak{O}$ is a normal covering of the subspace $f^{-1}(y)$ for each $y \in Y$, then X is pseudo-paracompact.*

Proof. Let $\mathfrak{O}$ be any open covering of X which is extendable to $\mu(X)$. Then by our assumption, $f^{-1}(y) \cap \mathfrak{O}$ is a normal covering of $f^{-1}(y)$, and hence it has a locally finite cozero refinement $\mathfrak{O}_y$ in $f^{-1}(y)$. Since $\mathfrak{B}f^{-1}(y)$ is relatively pseudo-compact in X, $\mathrm{cl}_{\mu(X)}\mathfrak{B}f^{-1}(y)$ is compact by Lemma 3.4. Therefore we have

$$\mathrm{cl}_{\mu(X)}\mathfrak{B}f^{-1}(y) \subset \eta O_1 \cup \ldots \cup \eta O_n, \quad O_i \in \mathfrak{O} \ (i = 1, \ldots, n),$$

where $\eta O_i = \mu(X) - \mathrm{cl}_{\mu(X)}(X - O_i)$. Let C_i, $i = 1, \ldots, n$, be closed sets in $\mathrm{cl}_{\mu(X)}\mathfrak{B}f^{-1}(y)$ such that $\mathrm{cl}_{\mu(X)}\mathfrak{B}f^{-1}(y) = \bigcup_{i=1}^{n} C_i$ and $C_i \subset \eta O_i$ for each i. Since each C_i is compact, there exists cozero-sets $\tilde{G}_1, \ldots, \tilde{G}_n$ and zero-sets $\tilde{F}_1, \ldots, \tilde{F}_n$ of $\mu(X)$ such that

$$C_i \subset \tilde{F}_i \subset \tilde{G}_i \subset \eta O_i .$$

If we put $G_i = \tilde{G}_i \cap X$ and $F_i = \tilde{F}_i \cap X$, then we have

$$\mathfrak{B}f^{-1}(y) \subset \bigcup_{i=1}^{n} F_i \subset \bigcup_{i=1}^{n} G_i \subset \bigcup_{i=1}^{n} O_i .$$

Let us put

$$\mathfrak{O}'_y = \Big(f^{-1}(y) - \bigcup_{i=1}^{n} F_i\Big) \cap \mathfrak{O}_y .$$

Then it is easily shown that each element of $\mathfrak{O}'_y$ is a cozero-set in X, since each element of $\mathfrak{O}_y$ is a cozero-set in $f^{-1}(y)$. Hence, if we put

$$\mathfrak{U}_y = \mathfrak{O}'_y \cup \{G_i \mid i = 1, \ldots, n\},$$

then $\mathfrak{U}_y$ is a locally finite collection of cozero-sets in X and covers $f^{-1}(y)$. Therefore there exists an open neighborhood $N(y)$ of y such that

$$f^{-1}(N(y)) \subset \bigcup \{O' \mid O' \in \mathfrak{O}'_y\} \cup \Big(\bigcup_{i=1}^{n} G_i\Big),$$

since $\bigcup \{O' \mid O' \in \mathfrak{O}'_y\} \cup \big(\bigcup_{i=1}^{n} G_i\big)$ is a cozero-set containing $f^{-1}(y)$ and f is a Z-map. Let S be a subset of Y such that $\mathfrak{B}f^{-1}(s) = \emptyset$ for each $s \in S$. Then $f^{-1}(s)$ is open and closed in X for $s \in S$, and hence it is a cozero-set in X. This implies that the one-point set $\{s\}$ is open and closed for $s \in S$. Hence if we put

$$\mathfrak{G} = \{N(y) \mid y \in Y - S\} \cup \{s \mid s \in S\},$$

$$\mathfrak{U} = f^{-1}(\mathfrak{G}),$$

then $\mathfrak{G}$ is a normal open covering of Y by paracompactness of Y, and hence $\mathfrak{U}$ is a normal open covering of X. Let $\{H_\alpha \mid \alpha \in A\}$ be a locally finite cozero refinement of $\mathfrak{U}$. As an open covering of H_α, we take $H_\alpha \cap \mathfrak{U}_y$ in case $H_\alpha \subset f^{-1}(N(y))$ $(y \in Y - S)$ and $H_\alpha \cap \mathfrak{O}_y$ in case $H_\alpha \subset f^{-1}(s)$ $(s \in S)$. In this way we can construct a locally finite cozero refinement of $\mathfrak{O}$. As is well known, every locally finite cozero covering of X is normal, and hence $\mathfrak{O}$ is normal. Therefore by Theorem 2.1, X is pseudo-paracompact. Thus we complete the proof.

Proof of Theorem 3.6. Since Y is paracompact, $\mu(f)$ carries $\mu(X)$ onto Y. Let $\mathfrak{O}$ be any open covering of X which is extendable to $\mu(X)$. Then it is proved that $f^{-1}(y) \cap \mathfrak{O}$ is a normal open covering of $f^{-1}(y)$. Indeed, let $i_y\colon f^{-1}(y) \to \mu(f)^{-1}(y)$ be an inclusion map for each $y \in Y$. Then $\mu(i_y)$ carries $\mu(f^{-1}(y))$ into $\mu(f)^{-1}(y)$, and hence $\mu(i_y)^{-1}(\mu(f)^{-1}(y) \cap \tilde{\mathfrak{O}})$ is a normal open covering of $\mu(f^{-1}(y))$ by paracompactness of $\mu(f^{-1}(y))$, where $\tilde{\mathfrak{O}}$ is an extension of $\mathfrak{O}$ to $\mu(X)$. Therefore $f^{-1}(y) \cap \mathfrak{O}$ is a normal covering of $f^{-1}(y)$, since

$$f^{-1}(y) \cap \mu(i_y)^{-1}(\mu(f)^{-1}(y) \cap \tilde{\mathfrak{O}}) = f^{-1}(y) \cap \mathfrak{O} .$$

Consequently, X is pseudo-paracompact by Lemma 3.7. Thus we complete the proof.

As an application of Theorem 3.6, we can prove the following theorem.

THEOREM 3.8. *If there exists a Z-map f from a space X onto a paracompact q-space Y such that $f^{-1}(y)$ is pseudo-paracompact for each $y \in Y$, then X is pseudo-paracompact.*

This theorem is a direct consequence of Theorem 3.6 and the following lemma which is a modification of Michael's theorem [10, Theorem 2.1].

LEMMA 3.9. *Let f be a Z-map from a space X onto a q-space Y. Then $\mathfrak{B}f^{-1}(y)$ is relatively pseudo-compact for each $y \in Y$.*

Proof. Suppose that $\mathfrak{B}f^{-1}(y_0)$ is not relatively pseudo-compact for a point y_0 of Y. Then there exists a real-valued continuous function h on X which is unbounded on $\mathfrak{B}f^{-1}(y_0)$. Let $\{x_i\}$ be a sequence of points of $\mathfrak{B}f^{-1}(y_0)$ such that

$$|h(x_{i+1})| > |h(x_i)| + 1 .$$

If we put

$$V_i = \{x \mid |h(x) - h(x_i)| < \tfrac{1}{2}\},$$

then $x_i \in V_i$, $i = 1, 2, \ldots$, and $\{V_i\}$ is discrete. Since Y is a q-space, there exists a sequence $\{N_i\}$ of open neighborhoods of y_0 such that if $y_i \in N_i$ then $\{y_i\}$ has a cluster point in Y. Hence we can take a sequence $\{z_i\}$ of points of X, a sequence $\{Z_i\}$ of zero-sets of X and a sequence $\{H_i\}$ of cozero-sets of X such that

$$z_1 \in Z_1 \subset H_1 \subset (V_1 \cap f^{-1}(N_1)) - f^{-1}(y_0),$$

$$z_i \in Z_i \subset H_i \subset [V_i \cap f^{-1}(N_i) - f^{-1}f(\bigcup_{j<i} Z_j)] - f^{-1}(y_0), \quad i \geqslant 2.$$

Then we have $f(Z_j) \cap f(Z_k) = \emptyset$ for $j \neq k$. Since $\{H_i\}$ is discrete and $Z_i \subset H_i$ for $i = 1, 2, \ldots$, it is easily proved that $\bigcup_{n=1}^{\infty} Z_{i_n}$ is a zero-set in X for any subsequence $\{Z_{i_n} |\ n = 1, 2, \ldots\}$ of $\{Z_i\}$, which implies that $f(\bigcup_{n=1}^{\infty} Z_{i_n})$ is closed in Y. But $\{f(z_i)\}$ has a cluster point y_1 in Y, since $f(z_i) \in N_i$. By closedness of $f(\bigcup_{i=1}^{\infty} Z_i)$, we have $y_1 \in f(Z_k)$ for some k. This is a contradiction, since $f(\bigcup_{i \neq k} Z_i)$ is closed and $f(\bigcup_{i \neq k} Z_i) \cap f(Z_k) = \emptyset$. Thus we complete the proof.

COROLLARY 3.10. *Let f be a closed (or Z-) map from a space X onto a metric space Y. Then X is pseudo-paracompact in the following cases.*

(a) *$f^{-1}(y)$ is an M-space for each $y \in Y$.*

(b) *$f^{-1}(y)$ is paracompact for each $y \in Y$.*

In Theorem 3.6 (Theorem 3.8), we can not exclude the assumption that $\mathfrak{B}f^{-1}(y)$ is relatively pseudo-compact (resp. Y is a q-space). We can show these facts by the same example below, in which we make use of the closed map from the space Π (cf. [4, 6Q]) onto the quotient space Π/D.

EXAMPLE 3.11. Let φ be a one-one map of N onto Q, where N (resp. Q) is the set of positive integers (resp. rational numbers). For each irrational number r, we select an increasing sequence $\{s_n\}$ of rationals converging to r. For each such sequence, consider the subset $E = \{\varphi^{-1}(s_n) |\ n = 1, 2, \ldots\}$ of N, and let $\mathfrak{E}$ be the family of all such sets E. For each $E \in \mathfrak{E}$, let $E' = \mathrm{cl}_{\beta N} E - N$. We construct a set D selecting one point p_E from each set E' and define Π to be the subspace $N \cup D$ of βN. Then Π is not normal but realcompact ([4, 8H]). By identifying each point of the discrete closed set D, we get the quotient space Π/D. Since Π/D is σ-compact, it is paracompact. Let f be the quotient map from Π onto Π/D. Then f is a closed map such that $f^{-1}(y)$ is a metric space for each $y \in \Pi/D$ and that

$$\mathfrak{B}f^{-1}(y) = D \quad \text{for} \quad y \in \Pi/D - N,$$

$$\mathfrak{B}f^{-1}(y) = \emptyset \quad \text{for} \quad y \in N.$$

But Π is not pseudo-normal, since Π is topologically complete and non-normal.

In Theorem 3.8, we can not replace "Z-map" by "open map".

EXAMPLE 3.12. Let X be a metric space and Y a paracompact space such that the product $X \times Y$ is not normal ([9]). Since X and Y are topologically complete, so is $X \times Y$ by [13, Theorem 1.5]. Hence $X \times Y$ is not pseudo-normal. Let $\varphi: X \times Y \to X$ be the projection map. Then φ is an open map from $X \times Y$ onto a metric space X such that $\varphi^{-1}(x)$ is paracompact for each $x \in X$.

As for Corollary 3.10, we note that if X is the inverse image of a metric space Y under a closed map f such that $f^{-1}(y)$ is an M-space (paracompact), then X is not necessarily an M-space (resp. paracompact). Hoshina proved this for the paracompact case by the following example (cf. [4, 5I]), and the same example shows that this is also true for the case of M-spaces.

EXAMPLE 3.13. Let $\mathfrak{F}$ be an infinite maximal family of infinite subsets of the set N of positive integers such that the intersection of any two is finite. Let $D = \{\alpha_F |\ F \in \mathfrak{F}\}$ be a new set of distinct points, and let $\Psi = N \cup D$ with points of N discrete and neighborhoods of $\alpha_F \in D$ those subsets of Ψ containing α_F and all but finitely many points of F. Then Ψ is completely regular and pseudo-compact but not countably compact. Let Ψ/D be the quotient space obtained from Ψ by identifying each point of D. Then Ψ/D is homeomorphic to the one-point compactification of N, and hence it is metrizable. Let $\varphi: \Psi \to \Psi/D$ be the quotient map. Then it is easily shown that φ is a closed map and $\varphi^{-1}(y)$ is a metric space for each $y \in \Psi/D$. But Ψ is neither an M-space nor a paracompact space.

THEOREM 3.14. *Let $f: X \to Y$ be a quasi-perfect map. If X is strongly normal and pseudo-paracompact, so is Y.*

Proof. Since a normal space is strongly normal if and only if for every locally finite collection $\{F_\lambda\}$ of closed subsets there exists a locally finite collection $\{G_\lambda\}$ of open subsets such that $F_\lambda \subset G_\lambda$ for each λ (Katětov [7]), it is easy to see that the image of a strongly normal space under a quasi-perfect map is also strongly normal. To prove that Y is pseudo-paracompact, let $\mathfrak{O}$ be an open covering of Y which is extendable to $\mu(Y)$. Then $f^{-1}(\mathfrak{O})$ is an open covering of X which is extendable to $\mu(X)$. Hence by paracompactness of $\mu(X)$, $f^{-1}(\mathfrak{O})$ has a locally finite closed refinement $\{K_\alpha\}$ in X. Therefore $\mathfrak{O}$ has a locally finite closed refinement $\{f(K_\alpha)\}$. Since Y is strongly normal, there exists a locally finite open covering $\{H_\alpha\}$ of Y such that $f(K_\alpha) \subset H_\alpha$ for each α, where we may assume that $\{H_\alpha\}$ refines $\mathfrak{O}$. Hence $\mathfrak{O}$ is a normal covering, since $\{H_\alpha\}$ is a normal covering. Therefore by Theorem 2.1, Y is pseudo-paracompact. Thus we complete the proof.

THEOREM 3.15. *For a space X, the following conditions are equivalent.*

(a) *X is pseudo-locally-compact and pseudo-paracompact.*

(b) *There exists a normal open covering $\mathfrak{U} = \{U_\alpha\}$ of X such that each U_α is relatively pseudo-compact in X.*

(c) *There exists a normal sequence $\{\mathfrak{U}_n\}$ of open coverings of X such that for each $x \in X$, $\mathrm{St}(x, \mathfrak{U}_{k(x)})$ is relatively pseudo-compact in X for some $k(x)$.*

(d) *There exists a Z-map f from X onto a locally compact metric space Y such that $f^{-1}(y)$ is relatively pseudo-compact for each $y \in Y$.*

(e) *There exists a Z-map f from X onto a locally compact paracompact space Y such that $f^{-1}(y)$ is relatively pseudo-compact for each $y \in Y$.*

(f) *There exists a WZ-map f from X onto a locally compact paracompact space Y such that $f^{-1}(y)$ is relatively pseudo-compact for each $y \in Y$.*

The equivalence of (a) and (b) is due to K. Morita [14], who proved also the equivalence of (a) and (d) independently.

Proof. (b)→(c) and (d)→(e)→(f) are obvious.

(a)→(b). Since $\mu(X)$ is locally compact and paracompact, each point x of $\mu(X)$ has an open neighborhood $U(x)$ such that $\mathrm{cl}_{\mu(X)}U(x)$ is compact. Let us put $\mathfrak{U} = \{U(x) \cap X \mid x \in \mu(X)\}$. Then it is easy to see that $\mathfrak{U}$ satisfies the required properties, since $\{U(x) \mid x \in \mu(X)\}$ is a normal covering of $\mu(X)$.

(c)→(d). Let $\{K_n\}$ be a decreasing sequence of zero-sets of X such that $K_n \subset \mathrm{St}(x, \mathfrak{U}_n)$ for some $x \in X$ and for each n. Then by our assumption, there is some K_m which is relatively pseudo-compact in X. Since $\mathrm{cl}_{\mu(X)}K_m$ is compact, $\{K_n\}$ has a cluster point in $\mu(X)$. Now we prove that $\bigcap K_n \neq \emptyset$. For this purpose, assume to be contrary, and let $H_n = X - K_n$. Then $\{H_n\}$ is a normal open covering of X, since any countable cozero covering is normal. Hence $\{H_n\}$ is extendable to $\mu(X)$, which implies that $\bigcap \mathrm{cl}_{\mu(X)}K_n = \emptyset$. This is a contradiction. Therefore we have $\bigcap K_n \neq \emptyset$. Consequently by [8] there exists a Z-map f from X onto a metric space Y such that $f^{-1}(y)$ is relatively pseudo-compact for each $y \in Y$. From the construction of Y it follows that Y is locally compact, and hence (d) holds.

(f)→(a). This is a direct consequence of Theorem 3.3.

Thus we complete the proof.

4. Pseudo-Lindelöf property. For a space X we denote by υ the uniformity of X which consists of all countable normal open coverings of X. As for the characterizations of pseudo-Lindelöf spaces, we have the following theorem.

THEOREM 4.1. *For a space X, the following conditions are equivalent.*

(a) *X is pseudo-Lindelöf.*

(b) *X is pseudo-paracompact and any normal open covering of X has a countable subcovering.*

(c) *Every open covering of X which is extendable to $\mu(X)$ has a countable subcovering.*

(d) *Every weakly Cauchy filter in X with respect to υ is contained in some Cauchy filter with respect to μ.*

(e) *If $\mathfrak{F}$ is a filter in X such that the image of $\mathfrak{F}$ has a cluster point in any separable metric space into which X is continuously mapped, then $\mathfrak{F}$ is contained in some Cauchy filter with respect to μ.*

The equivalence of (a) and (b) was proved by Howes [5].

Proof. (a)→(b)→(c) are obvious.

(c)→(d). Suppose that a weakly Cauchy filter $\mathfrak{F} = \{F_\alpha\}$ in X with respect to υ is not contained in any Cauchy filter with respect to μ. Then each point x of $\mu(X)$ has a cozero neighborhood $N(x)$ such that $N(x) \cap F_{\alpha(x)} = \emptyset$ for some $F_{\alpha(x)} \in \mathfrak{F}$. By (c), an open covering $\{N(x) \cap X \mid x \in \mu(X)\}$ of X has a countable subcovering $\{N(x_i) \cap X \mid x_i \in \mu(X),\ i = 1, 2, \ldots\}$, which is a normal covering of X. Since $\mathfrak{F}$ is a weakly Cauchy filter with respect to υ, we have $(N(x_j) \cap X) \cap F_\alpha = N(x_j) \cap F_\alpha \neq \emptyset$ for some j and for each α, which is a contradiction. Thus (d) holds.

(d)↔(e). This follows from the fact that a filter $\mathfrak{F}$ in X is weakly Cauchy with respect to υ if and only if the image of $\mathfrak{F}$ has a cluster point in any separable metric space into which X is continuously mapped.

(e)→(a). Let $\mathfrak{F} = \{F_\alpha\}$ be a filter base in $\mu(X)$ such that the image of $\mathfrak{F}$ has a cluster point in any separable metric space into which $\mu(X)$ is continuously mapped. Let $\{\Phi_\gamma \mid \gamma \in \Gamma\}$ be the family of all normal sequences consisting of countable normal open coverings of X. Then for any map $\varphi_\gamma\colon X \to X/\Phi_\gamma$, $\{\mu(\varphi_\gamma)(F_\alpha)\}$ has a cluster point in X/Φ_γ, since X/Φ_γ is a separable metric space. Let us put $\mathfrak{F}_\gamma = \varphi_\gamma^{-1}(\mu(\varphi_\gamma)(\mathfrak{F}))$ for each γ, and let

$$\mathfrak{G} = \bigcup \{\mathfrak{F}_\gamma \mid \gamma \in \Gamma\}.$$

By the similar way as in the proof of (e)→(a) in Theorem 2.1, it is proved that $\mathfrak{G}$ is a filter base in X such that the image of $\mathfrak{G}$ has a cluster point in any separable metric space into which X is continuously mapped. Therefore by (e) $\mathfrak{G}$ has a cluster point u in $\mu(X)$. Furthermore, it can be easily shown that u is a cluster point of $\mathfrak{F}$, from which it follows that $\mu(X)$ is Lindelöf (cf. [1]). Thus we complete the proof.

As is easily seen from the equivalence of (a) and (c) in Theorem 4.1, the image of a pseudo-Lindelöf space under a continuous map is pseudo-Lindelöf. This result was first pointed out by K. Morita. Therefore it follows that if a space X is the countable union of pseudo-Lindelöf subspaces, then X is also pseudo-Lindelöf.

THEOREM 4.2. *If there exists a Z-map f from a space X onto a Lindelöf space Y such that $f^{-1}(y)$ is pseudo-Lindelöf for each $y \in Y$, then X is pseudo-Lindelöf.*

To prove this theorem, we use the following lemma.

LEMMA 4.3. *If there exists a Z-map f from a space X onto a Lindelöf space Y such that for any open covering $\mathfrak{O}$ of X which is extendable to $\mu(X)$, $f^{-1}(y) \cap \mathfrak{O}$ has a countable subcovering for each $y \in Y$, then X is pseudo-Lindelöf.*

Proof. Let $\mathfrak{O} = \{O_\alpha\}$ be any cozero covering of X which is extendable to $\mu(X)$. Then for each $y \in Y$, $f^{-1}(y)$ is covered by a countable number of elements of $\mathfrak{O}$, that is, $f^{-1}(y) \subset \bigcup_{i=1}^{\infty} O_{\alpha_i}$, $O_{\alpha_i} \in \mathfrak{O}$ $(i = 1, 2, \ldots)$. Since $\bigcup_{i=1}^{\infty} O_{\alpha_i}$ is a cozero-set in X and f is a Z-map, there exists an open neighborhood $N(y)$ of y such that $f^{-1}(N(y)) \subset \bigcup_{i=1}^{\infty} O_{\alpha_i}$. By the Lindelöf property of Y, the covering $\{N(y) \mid y \in Y\}$ of Y admits a countable subcovering $\{N(y_n) \mid y_n \in Y,\ n = 1, 2, \ldots\}$. Since $f^{-1}(N(y_n))$ is covered by a countable number of elements of $\mathfrak{O}$ for each n, $\mathfrak{O}$ has a countable subcovering. Hence by Theorem 4.1, X is pseudo-Lindelöf. Thus we complete the proof.

Proof of Theorem 4.2. By the similar argument as in the proof of Theorem 3.6, it is proved that if $\mathfrak{D}$ is an open covering of X which is extendable to $\mu(X)$, then $f^{-1}(y) \cap \mathfrak{D}$ has a countable subcovering. Hence by Lemma 4.3 the theorem holds. Thus we complete the proof.

Finally we raise a problem concerning pseudo-paracompactness: Is the image of a pseudo-paracompact space under a perfect map pseudo-paracompact?

The referee has kindly pointed out to the author that the following problems has been solved negatively by R. Pol:

(1) Is the preimage of a pseudo-paracompact space under a perfect map pseudo-paracompact?

(2) Is pseudo-paracompactness hereditary to every closed subspace?

(3) Are the problems (1) and (2) affirmative for pseudo-Lindelöf spaces?

The following examples are due to R. Pol.

EXAMPLE 4.4. *There exists a space X and its closed subspace A such that X is pseudo-compact and that A is not pseudo-normal.*

Proof. Let $D = \{0, 1, 2\}$ be a three-point discrete space and $D^{\aleph_1}$ its $\aleph_1$-product. Let $\Sigma = \{x|\ x$ has at most $\aleph_0$ coordinates different from $0\}$ and $K = \{1, 2\}^{\aleph_1}$. Then $K \cap \Sigma = \emptyset$. Take $A \subset K$ which is not pseudo-normal and let $X = A \cup \Sigma$. Then $A = K \cap X$ is closed in X and since by Mazur's theorem Σ is C-embedded in $D^{\aleph_1}$ we have $\mu\Sigma = \mu X = D^{\aleph_1}$. Hence X is pseudo-compact.

EXAMPLE 4.5. *There exists a perfect mapping f: $Y \to X$ such that Y is not pseudo-normal, but X is pseudo-compact.*

Proof. Take $Y = X \oplus A$, where X and A are as in Example 4.4 and $X \oplus A$ denotes the topological union of X and A. Let f: $X \oplus A \to X$ be an identity map on X and A. Then f is perfect, X is pseudo-compact and since $\mu(X \oplus A) = \mu X \oplus \mu A$, the space Y is not pseudo-normal.

The author would like to thank the referee who indicated Examples 4.4 and 4.5.

References

[1] H. H. Corson, *The determination of paracompactness by uniformities*, Amer. J. Math. 80 (1958), pp. 185–190.
[2] — *Normality of subsets of product spaces*, Amer. J. Math. 81 (1959), pp. 785–796.
[3] N. Dykes, *Mappings and realcompact spaces*, Pacific J. Math. 31 (1969), pp. 347–358.
[4] L. Gilman and M. Jerison, *Rings of continuous functions*, Princeton, N. J. 1960.
[5] N. R. Howes, *On completeness*, Pacific J. Math. 38 (1971), pp. 431–440.
[6] T. Ishii, *Paracompactness of topological completions*, Proc. Japan Acad. 50 (1974), pp. 33–38.
[7] M. Katětov, *Extensions of locally finite coverings*, Colloq. Math. 6 (1958), pp. 141–151.
[8] T. Isiwata, *Mappings and spaces*, Pacific J. Math. 20 (1967), pp. 455–480.
[9] E. Michael, *The product of a normal space and a metric space need not be normal*, Bull. Amer. Math. Soc. 69 (1963), pp. 375–376.
[10] — *A note on closed maps and compact sets*, Israel J. Math. 2 (1964), pp. 173–176.
[11] K. Morita, *Simple extension of a space with respect to a uniformity, I, II*, Proc. Japan Acad. 27 (1951), pp. 64–72; 130–137.

[12] K. Morita, *Paracompactness and product spaces*, Fund. Math. 50 (1962), pp. 223–236.
[13] — *Topological completions and M-spaces*, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, 10 (1970), pp. 271–288.
[14] — *On the dimension of the product of Tychonoff spaces*, Gen. Top. and Appl. 3 (1973), pp. 125–133.
[15] A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc. 54 (1948), pp. 977–982.
[16] H. Tamano, *On paracompactness*, Pacific J. Math. 10 (1960), pp. 1043–1047.

SHIZUOKA UNIVERSITY

Accepté par la Rédaction le 19. 8. 1974